Homework Assignment #8 is due Wednesday, Dec. 9, 2015

Theoretical Astrophysics (MKTP2)

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1. **Parker instability** *25 pt*

   Consider an isothermal gas in the galactic disk which is threaded with a horizontal magnetic field. Assume a constant gravitational field perpendicular to the disk plane in the z direction, i.e. $\vec{g} = -\hat{z}\, g$ and a magnetic field parallel to the disk plane x which varies only with z, i.e. $B = \hat{x}\, B(z)$. For simplicity study the system in two dimensions using cartesian coordinates.

   (a) Assume that the system is in hydrostatic equilibrium with a constant ratio of the magnetic to thermal pressure, i.e.

   $$\alpha \equiv \frac{B^2}{8\pi\, P} = \text{const.} \tag{1}$$

   What is the pressure distribution as a function of z? Use the relation $P = c_\mathrm{s}^2\, \rho$ where c_s is the constant speed of sound and the scale height $H = (1+\alpha)\, c_\mathrm{s}^2/g$ to express the result.

   Now consider this system slightly perturbed out of its equilibrium. Then, from the linear perturbation analysis one gets the following dispersion relation in the xz-plane,

   $$\begin{aligned} n^4 \quad + \quad & c_\mathrm{s}^2 \left[(1+2\alpha) \left(k^2 + \frac{k_0^2}{4} \right) \right] n^2 + \\ & k_x^2\, c_\mathrm{s}^4 \left[2\alpha\, k^2 + k_0^2 \left[\left(1 + \frac{3\alpha}{2} \right) - (1+\alpha)^2 \right] \right] = 0\,, \end{aligned} \tag{2}$$

   (it is a good exercise to derive this relation), where $n = i\omega$, $k_0 = H^{-1}$, and the Fourier modes in the x and z direction for the perturbed quantities are

   $$\exp\left(ik_x\, x - i\omega t\right) \quad , \quad \exp\left(ik_z\, z - i\omega t\right) \tag{3}$$

   with $k^2 = k_x^2 + k_z^2$.

   (b) Show that in the absence of a magnetic field all roots (in terms of n^2) of this dispersion relation are negative, i.e. $n^2 < 0$. What is the physical implication of this result regarding the instability?

   (c) In the case of a non-vanishing magnetic field derive the instability criterion for the Parker instability (magnetic Rayleigh-Taylor instability)

   $$\left(\frac{k}{k_0/2} \right)^2 < 2\alpha + 1\,. \tag{4}$$

   Hint: Use the roots of n^2 to find at least one unstable mode, i.e. $n^2 > 0$.

(d) Show that the instability criterion is equivalent to

$$\lambda_x > \Lambda_x \equiv 4\pi\, H \left[\frac{1}{2\alpha+1}\right]^{1/2} \qquad \text{and} \qquad \lambda_z > \Lambda_z \equiv \frac{\Lambda_x}{\left(1-(\Lambda_x/\lambda_x)^2\right)^{1/2}}\,, \tag{5}$$

with the wavelengths $\lambda_x = 2\pi/k_x$, and $\lambda_z = 2\pi/k_z$.

2. **Plasma Waves: Fluid Treatment** *25 pt*

A common approximation in plasma physics is that the electrons and/or ions can be described by fluid equations. Assume the ions in a plasma can be treated as a smooth, uniform, motionless background charge density that neutralizes the average electron charge density. The electron number density n and velocity $\vec{v}$ obey the continuity equation and the equation of motion, in which the Lorentz force appears in the same way as an external force,

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{v}) = 0\,, \tag{6}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{1}{n m_{\mathrm{e}}}\left(-\vec{\nabla}P + qn\vec{E} + \frac{1}{c}\vec{j}\times\vec{B}\right)\,, \tag{7}$$

where m_{e} is the electron mass, $q = -e$ is the electron charge, P the pressure of the electron gas, $\vec{j}$ the current density, and $\vec{E}$ and $\vec{B}$ are the electric and magnetic field strengths. The electron density and pressure are related by the equation of state,

$$P = Kn^{\gamma}, \tag{8}$$

where K is a constant and where we assume the electrons behave as an adiabatic gas with $\gamma = 5/3$. Electric and magnetic fields, $\vec{E}$ and $\vec{B}$, are related to the charge density qn and the current $\vec{j}$ via Maxwell's equations.

(a) Assume the system originally is in equilibrium and apply a small perturbation of the form,

$$n = n_0 + \delta n\,, \qquad P = P_0 + \delta P\,, \qquad \vec{v} = \delta\vec{v}\,,$$
$$\vec{E} = \delta\vec{E}\,, \qquad \vec{B} = \delta\vec{B}\,, \qquad \vec{j} = \delta\vec{j}\,,$$

where n_0 and P_0 are the homogeneous equilibrium density and pressure. Linearize the equations and find the equation that governs the evolution of the density perturbation δn.

(b) Consider plane wave perturbations of the form $\delta n \propto \exp[i(\vec{k}\cdot\vec{x} - \omega t)]$. Derive their dispersion relation, and find their phase and group velocities as a function of the thermal velocity of the electrons, $v_{\mathrm{th}} = (P_0/m_{\mathrm{e}} n_0)^{1/2}$, and the ratio k/k_{D}, where

$$k_{\mathrm{D}} = \left(\frac{4\pi q^2 n_0}{m_{\mathrm{e}} v_{\mathrm{th}}^2}\right)^{1/2}$$

is the Debye wave-number, which is the inverse of the Debye wavelength that was introduced in the lecture.

(c) Discuss the nature of these waves in the limits of small and large wavelengths.