SPH Basics

Introduction into Smoothed Particle Hydrodynamics

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Overview

- Some words about hydrodynamics
- SPH – The Standard Implementation
- Modifications and Extensions
- Some Caveats
Hydrodynamics:

- text books on hydrodynamics
  - Landau & Lifshitz, Volume VI - Hydrodynamics
  - Landau & Lifshitz, Volume X - Kinetic theory

- derivation
  - gases and fluids are large ensembles of interacting particles
  - state of system is described by location in $6N$ dimensional phase space $f^{(N)}(\vec{q}_1, \ldots, \vec{q}_N, \vec{p}_1, \ldots, \vec{p}_N)d\vec{q}_1 \ldots d\vec{q}_N d\vec{p}_1 \ldots d\vec{p}_N$
  - time evolution governed by ‘equation of motion’ for $6N$-dim probability distribution function $f^{(N)}$
  - $f^{(N)} \rightarrow f^{(n)}$ by integrating over all but $n$ coordinates $\rightarrow$ BBGKY hierarchy of equations of motion (after Born, Bogoliubov, Green, Kirkwood and Yvon)
  - physical observables are typically associated with 1- or 2-body probability density $f^{(1)}$ or $f^{(2)}$
  - at lowest level of hierarchy: 1-body distribution function describes the probability of finding a particle at time $t$ in the volume element $d\vec{q}$ at $\vec{q}$ with momenta in the range $d\vec{p}$ at $\vec{p}$.
  - Boltzmann equation – equation of motion for $f^{(1)}$
    \[
    \frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{q} \cdot \vec{\nabla}_q f + \vec{p} \cdot \vec{\nabla}_p f = \frac{\partial f}{\partial t} + \vec{\nabla} \cdot \vec{\nabla}_q f + \vec{F} \cdot \vec{\nabla}_p f = f_c .
    \]
Hydrodynamics:

- derivation

  - **Boltzmann equation**

    \[
    \frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{q} \cdot \vec{\nabla}_q f + \dot{\vec{p}} \cdot \vec{\nabla}_p f = \frac{df}{dt} + \vec{\nu} \cdot \vec{\nabla}_q f + \vec{F} \cdot \vec{\nabla}_p f = f_c.
    \]

    - first line: transformation from comoving to spatially fixed coordinate system.
    - second line: velocity \( \vec{\nu} = \dot{q} \) and force \( \vec{F} = \dot{\vec{p}} \)
    - all higher order terms are 'hidden' in the collision term \( f_c \)

- observable quantities are typically (velocity) moments of the Boltzmann equation, e.g.
  - **density:**
    \[
    \rho = \int m f(\vec{q}, \vec{p}, t) d\vec{p}
    \]
  - **momentum:**
    \[
    \rho \vec{\nu} = \int m \vec{\nu} f(\vec{q}, \vec{p}, t) d\vec{p}
    \]
  - **kinetic energy density:**
    \[
    \rho \nu^2 = \int m \nu^2 f(\vec{q}, \vec{p}, t) d\vec{p}
    \]
Hydrodynamics:

- derivation
  - in general: the $i$-th velocity moment $\langle \xi_i \rangle$ (of $\xi_i = m\vec{v}^i$) is
    \[
    \langle \xi_i \rangle = \frac{1}{n} \int \xi_i \ f(q, \vec{p}, t) \, d\vec{p}
    \]
    with the mean particle number density $n$ defined as
    \[
    n = \int f(q, \vec{p}, t) \, d\vec{p}
    \]
  - the equation of motion for $\langle \xi_i \rangle$ is
    \[
    \int \xi_i \left\{ \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_q f + \vec{F} \cdot \vec{\nabla}_p f \right\} \, d\vec{p} = \int \xi_i \{f_c\} \, d\vec{p},
    \]
    which after some complicated rearrangement becomes
    \[
    \frac{\partial}{\partial t} n \langle \xi_i \rangle + \vec{\nabla}_a (n \langle \xi_i \vec{v} \rangle) + n \vec{F} \langle \vec{\nabla}_p \xi_i \rangle = \int \xi_i f_c \, d\vec{p}
    \]
    (Maxwell-Boltzmann transport equation for $\langle \xi_i \rangle$)
  - if the RHS is zero, then $\xi_i$ is a conserved quantity. This is only the case for first three moments, mass $\xi_0 = m$, momentum $\vec{\xi}_1 = m\vec{v}$, and kinetic energy $\xi_2 = m\vec{v}^2/2$.
  - MB equations build a hierarchically nested set of equations, as $\langle \xi_i \rangle$ depends on $\langle \xi_{i+1} \rangle$ via $\vec{\nabla}_a (n \langle \xi_i \vec{v} \rangle)$ and because the collision term cannot be reduced to depend on $\xi_i$ only.
    - need for a closure equation
    - in hydrodynamics this is typically the equation of state.
Hydrodynamics:

- **assumptions**
  - **continuum limit:**
    - distribution function \( f \) must be a ‘smoothly’ varying function on the scales of interest → **local average possible**
    - stated differently: the averaging scale (i.e. scale of interest) must be larger than the mean free path of individual particles
    - stated differently: microscopic behavior of particles can be neglected
    - concept of fluid element must be meaningful
  - **only ‘short range forces’:**
    - forces between particles are short range or saturate → **collective effects can be neglected**
    - stated differently: correlation length of particles in the system is finite (and smaller than the scales of interest)

- **limitations**
  - shocks (scales of interest become smaller than mean free path)
  - phase transitions (correlation length may become infinite)
  - description of self-gravitating systems
  - description of fully fractal systems
Hydrodynamics:

- the equations of hydrodynamics
  - hydrodynamics $\equiv$ book keeping problem
    One must keep track of the ‘change’ of a fluid element due to various physical processes acting on it. How do its ‘properties’ evolve under the influence of compression, heat sources, cooling, etc.?
  - Eulerian vs. Lagrangian point of view

![Eulerian vs. Lagrangian](image)

- consider spatially fixed volume element  
- following motion of fluid element

- hydrodynamic equations $=$ set of equations for the five conserved quantities ($\rho$, $\rho \mathbf{v}$, $\rho \mathbf{v}^2/2$) plus closure equation (plus transport equations for ‘external’ forces if present, e.g. gravity, magnetic field, heat sources, etc.)
Hydrodynamics:

- the equations of hydrodynamics
  - equations of hydrodynamics

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho = -\rho \vec{\nabla} \cdot \vec{v} \\
\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \eta \vec{\nabla}^2 \vec{v} + \left( \zeta + \frac{\eta}{3} \right) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \\
\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \vec{\nabla} \epsilon = T \frac{ds}{dt} - \frac{p}{\rho} \vec{\nabla} \cdot \vec{v} \\
\vec{\nabla}^2 \phi = 4\pi G \rho \\
p = \mathcal{R} \rho T
\]

- Poisson’s equation
- equation of state

\[
\vec{F}_B = -\vec{\nabla} \frac{\vec{B}^2}{8\pi} + \frac{1}{4\pi} (\vec{B} \cdot \vec{\nabla}) \vec{B} \\
\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B})
\]

- magnetic force
- Lorentz equation

\(\rho = \text{density}, \ \vec{v} = \text{velocity}, \ p = \text{pressure}, \ \phi = \text{gravitational potential}, \ \zeta \text{ and } \eta = \text{viscosity coefficients}, \ \epsilon = \rho \vec{v}^2 /2 = \text{kinetic energy density}, \ T = \text{temperature}, \ s = \text{entropy}, \ \mathcal{R} = \text{gas constant}, \ \vec{B} = \text{magnetic field (cgs units)}\)
Hydrodynamics:

- the equations of hydrodynamics
  - mass transport – continuity equation
    \[
    \frac{dp}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = -\rho \nabla \cdot \vec{v}
    \]
    (conservation of mass)
  - transport equation for momentum – Navier Stokes equation
    \[
    \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p - \nabla \phi + \eta \nabla^2 \vec{v} + \left( \zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \vec{v})
    \]
    momentum change due to
    \(\rightarrow\) pressure gradients: \(-\rho^{-1} \nabla p\)
    \(\rightarrow\) (self) gravity: \(-\nabla \phi\)
    \(\rightarrow\) viscous forces (internal friction, contains \(\text{div} (\partial v_i / \partial x_j)\) terms):
    \[
    \eta \nabla^2 \vec{v} + \left( \zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \vec{v})
    \]
    (conservation of momentum, general form of momentum transport: \(\partial_t (\rho v_i) = -\partial_j \Pi_{ij}\))
  - transport equation for internal energy
    \[
    \frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \nabla \epsilon = T \frac{ds}{dt} - \frac{p}{\rho} \nabla \cdot \vec{v}
    \]
    \(\rightarrow\) follows from the thermodynamic relation \(d\epsilon = T \, ds - p \, dV = T \, ds + p / \rho^2 \, d\rho\) which described changes in \(\epsilon\) due to entropy changed and to volume changes (compression, expansion)
    \(\rightarrow\) for adiabatic gas the first term vanishes \((s = \text{constant})\)
    \(\rightarrow\) heating sources, cooling processes can be incorporated in \(ds\)
    (conservation of energy)
Hydrodynamics:

- the equations of hydrodynamics
  - closure equation – equation of state
    - general form of equation of state $p = p(T, \rho, ...)$
    - ideal gas: $p = \mathcal{R} \rho T$
    - special case – isothermal gas: $p = c_s^2 T$ (as $\mathcal{R} T = \frac{c_s^2}{\rho}$)
SPH Basics:

- **literature**

- **concept of SPH**
  - ‘invented’ independently by Lucy (1977) and Gingold & Monaghan (1977)
  - originally proposed as Monte Carlo approach to calculate the time evolution of gaseous systems
  - more intuitively understood as interpolation scheme:

    The fluid is represented by an ensemble of particles $i$, each carrying mass $m_i$, momentum $m_i \tilde{v}_i$, and hydrodynamic properties (like pressure $p_i$, temperature $T_i$, internal energy $\epsilon_i$, entropy $s_i$, etc.). The time evolution is governed by the equation of motion plus additional equations to modify the hydrodynamic properties of the particles. Hydrodynamic observables are obtained by a local averaging process.
SPH Basics:

- properties of local averaging processes
  - local averages \( \langle f(\vec{r}) \rangle \) for any quantity \( f(\vec{r}) \) can be obtained by convolution with an appropriate smoothing function \( W(\vec{r}, \vec{h}) \):
    \[
    \langle f(\vec{r}) \rangle \equiv \int f(\vec{r}^\prime)W(\vec{r} - \vec{r}^\prime, \vec{h})\,d^3r^\prime.
    \]
  - the function \( W(\vec{r}, \vec{h}) \) is called smoothing kernel
  - the kernel must satisfy the following two conditions:
    \[
    \int W(\vec{r}, \vec{h})\,d^3r = 1 \quad \text{and} \quad \langle f(\vec{r}) \rangle \rightarrow f(\vec{r}) \quad \text{for} \quad \vec{h} \rightarrow 0
    \]
  - the kernel \( W \) therefore follows the same definitions as Dirac’s delta function \( \delta(\vec{r}) \):
    \[
    \lim_{\vec{h} \rightarrow 0} W(\vec{r}, \vec{h}) = \delta(\vec{r}).
    \]
  - most SPH implementations use spherical kernel functions
    \[
    W(\vec{r}, \vec{h}) \equiv W(r, h) \quad \text{with} \quad r = |\vec{r}| \quad \text{and} \quad h = |\vec{h}|.
    \]
    (one could also use triaxial kernels, e.g. Martel et al. 1995)
  - as the kernel function \( W \) can be seen as approximation to the \( \delta \)-function for small but finite \( h \) we can expand the averaged function \( \langle f(\vec{r}) \rangle \) into a Taylor series for \( h \) to obtain an estimate for \( f(\vec{r}) \); if \( W \) is an even function, the first order term vanishes and the errors are second order in \( h \)
    \[
    \langle f(\vec{r}) \rangle = f(\vec{r}) + \mathcal{O}(h^2)
    \]
    this holds for functions \( f \) that are smooth and do not exhibit steep gradients over the size of \( W \) (\( \rightarrow \) problems in shocks).
    (more specifically the expansion is \( \langle f(\vec{r}) \rangle = f(\vec{r}) + \kappa h^2 \nabla^2 f(\vec{r}) + \mathcal{O}(h^3) \))
SPH Basics:

- properties of local averaging processes
  - within its intrinsic accuracy, the smoothing process therefore is a linear function with respect to summation and multiplication:
    \[
    \langle f(\vec{r}) + g(\vec{r}) \rangle = \langle f(\vec{r}) \rangle + \langle g(\vec{r}) \rangle \\
    \langle f(\vec{r}) \cdot g(\vec{r}) \rangle = \langle f(\vec{r}) \rangle \cdot \langle g(\vec{r}) \rangle 
    \]
    (one follows from the linearity of integration with respect to summation, and two is true to $O(h^2)$)
  - derivatives can be ‘drawn into’ the averaging process:
    \[
    \frac{d}{dt} \langle f(\vec{r}) \rangle = \langle \frac{d}{dt} f(\vec{r}) \rangle \\
    \vec{\nabla} \langle f(\vec{r}) \rangle = \langle \vec{\nabla} f(\vec{r}) \rangle
    \]
    Furthermore, the spatial derivative of $f$ can be transformed into a spatial derivative of $W$ (no need for finite differences or grid):
    \[
    \vec{\nabla} \langle f(\vec{r}) \rangle = \langle \vec{\nabla} f(\vec{r}) \rangle = \int f(\vec{r}') \vec{\nabla} W(|\vec{r} - \vec{r}'|, h) \, d^3r' 
    \]
    (shown by integrating by parts and assuming that the surface term vanishes; if the solution space is extended far enough, either the function $f$ itself or the kernel approach zero)
  - basic concept of SPH is a particle representation of the fluid $\rightarrow$ integration transforms into summation over discrete set of particles; example density $\rho$:
    \[
    \langle \rho(\vec{r}_i) \rangle = \sum_j m_j W(|\vec{r}_i - \vec{r}_j|, h) 
    \]
    in this picture, the mass of each particle is smeared out over its kernel region; the density at each location is obtained by summing over the contributions of the various particles $\rightarrow$ smoothed particle hydrodynamics!
SPH Basics:

- properties of local averaging processes
  - 'scatter' versus 'gather' approach:
    \[
    \langle \rho(\vec{r}_i) \rangle = \sum_j m_j W(|\vec{r}_i - \vec{r}_j|, h).
    \]
    allows for two different interpretations...

1. particle \( i \) collects the contributions from all other particles \( j \) which smoothing volumes \( h_j \) scatter onto location \( \vec{r}_i \)
   \( h \rightarrow h_j \), i.e. use \( W(|\vec{r}_i - \vec{r}_j|, h_j) \) in the summation
2. particle \( i \) gathers the contributions from all particles which centers fall within the smoothing volume of \( i \)
   \( h \rightarrow h_i \), i.e. use \( W(|\vec{r}_i - \vec{r}_j|, h_i) \) in the summation

if all particles have the same smoothing length \( h = h_i = h_j \)
both approaches are equivalent; otherwise different \( j \) contribute
to the sum —→ violation of Newton’s 3. law!!
therefore, enforce force anti-symmetry by using the (arithmetic) average of the smoothing lengths for all particle pairs
\[
h \rightarrow h_{ij} = \frac{h_i + h_j}{2}.
\]
SPH Basics:

- the kernel function
  - different functions meet the requirement \( \int W(|\vec{r}|, h) \, d^3r = 1 \) and \( \lim_{h \to 0} \int W(|\vec{r} - \vec{r}'|, h) f(\vec{r}') \, d^3r' = f(\vec{r}) \):
    - \textbf{Gaussian kernel:}
      \[
      W(r, h) = \frac{1}{\pi^{3/2} h^3} \exp \left( -\frac{r^2}{h^2} \right)
      \]
      - \textit{pro:} mathematically sound
      - \textit{pro:} derivatives exist to all orders and are smooth
      - \textit{contra:} all particles contribute to a location
    - \textbf{spline functions with compact support}
    - \textbf{the standard kernel: cubic spline}
      with \( \xi = r/h \) it is defined as
      \[
      W(r, h) \equiv \frac{1}{\pi h^3} \begin{cases}
        1 - \frac{3}{2} \xi^2 + \frac{3}{4} \xi^3, & \text{for } 0 \leq \xi \leq 1; \\
        \frac{1}{4} (2 - \xi)^3, & \text{for } 1 \leq \xi \leq 2; \\
        0, & \text{otherwise}.
      \end{cases}
      \]
      - \textit{pro:} compact support \( \implies \) all interactions are zero for \( r > 2h \)
      - \textit{pro:} number of particles involved in the average remains small (typically between 30 and 80)
      - \textit{pro:} second derivative is continuous
      - \textit{pro:} dominant error term is second order in \( h \)
    - \textbf{in principle different kernel functions could be used for different equations} (but it brings no obvious advantage, except maybe in the case of XSPH)
    - \textbf{specialized kernels can be constructed for different types of problems}
SPH Basics:

- variable smoothing length $h$
  - spatial resolution of SPH is limited by $h$, the scale over which forces and physical properties are smeared out
  - to make optimum use of the Lagrangian nature of SPH one has to allow for variations of $h$: in high-density regions $h$ should be small, in regions of low density $h$ should be large
  - the optimum value of $h$ is such that every particle has $\sim 50$ neighbors within the smoothing volume
  - caveats:
    - introduction of additional errors (the Taylor series now contains contributions from $\nabla h$, furthermore time derivatives $\partial h/\partial t$ occur); however, these errors are of second or higher order and thus the same as the one inherent to SPH anyway
    - modification of the kernel gradient
      \[
      \nabla W(|\vec{r} - \vec{r}'|, h) = \nabla W(|\vec{r} - \vec{r}'|, h) + \left. \frac{\partial}{\partial h} W(|\vec{r} - \vec{r}'|, h) \nabla h \right|_r
      \]
      the new term is $\propto \nabla h$ and becomes important only if the smoothing length varies on scales less than the smoothing lengths itself $\rightarrow$ it is generally neglected (see Nelson & Papaloizou 1994).

- equation of ‘motion’ for $h$ can be coupled to the density: from $h = h_0 (\rho_0/\rho)^3$ it follows from using the continuity equation
  \[
  \frac{dh}{dt} = \frac{1}{3} \frac{\rho}{\rho dt} = \frac{1}{3} h \nabla \cdot \vec{v} \tag{1}
  \]
  alternative methods exist (see e.g. Steinmetz & Müller 1993)
SPH Basics:

- the fluid equations in SPH
  - there is an infinite number of possible SPH implementations of the hydrodynamic equations!
  - some notation: \( h_{ij} = (h_i + h_j) / 2 \), \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \), \( \vec{v}_{ij} = \vec{v}_i - \vec{v}_j \), and \( \vec{V}_i \) is the gradient with respect to the coordinates of particle \( i \); all measurements are taken at particle positions (e.g. \( \rho_i = \rho(\vec{r}_i) \))
  - general form of SPH equations:
    \[
    \langle f_i \rangle = \sum_{j=1}^{N_i} \frac{m_j}{\rho_j} f_j W(r_{ij}, h_{ij})
    \]
  - density — continuity equation (conservation of mass)
    \[
    \rho_i = \sum_{j=1}^{N_i} m_j W(r_{ij}, h_{ij})
    \]
    or
    \[
    \frac{d\rho_i}{dt} = \sum_{j=1}^{N_i} m_j \vec{v}_{ij} \cdot \vec{V}_i W(r_{ij}, h_{ij})
    \]
    (the second implementation is almost never used, see however Monaghan 1991 for an application to water waves)
    
    important
    density is needed for ALL particles BEFORE computing other averaged quantities —— at each timestep, SPH computations consist of TWO loops, first the density is obtained for each particle, and then in a second round, all other particle properties are updated.

- pressure is defined via the equation of state (for example for isothermal gas \( p_i = c_s^2 \rho_i \))
SPH Basics:

- the fluid equations in SPH
  - velocity — Navier Stokes equation (conservation of momentum)
    → consider for now only pressure contributions: Euler’s equation
    \[
    \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho} \vec{\nabla}p = -\vec{\nabla} \left(\frac{p}{\rho}\right) - \frac{p}{\rho^2} \vec{\nabla} \rho \quad (\ast)
    \]
    here, the identity \(\vec{\nabla}(p\rho^{-1}) = \rho^{-1} \vec{\nabla}p - p\rho^{-2} \vec{\nabla} \rho\) is used
    → in the SPH formalism this reads as
    \[
    \frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right) \vec{\nabla}W(r_{ij}, h_{ij})
    \]
    where the first term in \((\ast)\) is neglected because it leads to surface terms in the averaging procedure; it is assumed that either the pressure or the kernel becomes zero at the integration border; if this is not the case correction terms need to be added above.
    the equation is anti-symmetric in \(i\) and \(j\) and conserves momentum locally and globally.
    → if self-gravity is taken into account, the gravitational force needs to be added on the RHS
    \[
    \vec{F}_G = -\vec{\nabla}\phi_i = -G \sum_{j=1}^{N} \frac{m_j r_{ij}}{r_{ij}^2}
    \]
    note that the sum needs to be taken over ALL particles ← computationally expensive
SPH Basics:

- the fluid equations in SPH
  - *velocity* — Navier Stokes equation (conservation of momentum)
  - the contribution of *viscosity*:
    - converts ordered kinetic energy into random kinetic energy (heat)
    - molecular viscosity in most astrophysical problems is small (except maybe in shocks) $\rightarrow$ SPH normally has NO explicit treatment of physical viscosity.
    - however, *artificial viscosity* is needed to prevent particle interpenetration.
    - this is achieved by smearing out shocks and by introducing dissipation in regions with strong velocity divergence
    - there are MANY ways to formulate artificial viscosity!!
  - the standard formulation of viscous pressures is
    \[ p_\alpha = \Pi_\alpha \rho^2 = -\alpha \rho \ell c_s (\nabla \cdot \vec{v}) \quad , \]
    \[ p_\beta = \Pi_\beta \rho^2 = -\beta \rho \ell^2 (\nabla \cdot \vec{v})^2 . \]
  - $\alpha$ and $\beta$ are free parameters and control the strength of the viscous terms (typical values are $\alpha = 1$ and $\beta = 2$); $\ell$ is the scale over which shocks are smeared out (typically $\ell \approx 2h$).
    - $p_\alpha$ is a combined *shear* and *bulk* viscosity — it dampens post-shock oscillations
    - $p_\beta$ is a von Neumann-Richtmyer viscosity — necessary to prevent interpenetration in high Mach number shocks
SPH Basics:

- the fluid equations in SPH
  - **velocity** — **Navier Stokes equation** (conservation of momentum)
    - the SPH implementation of the standard artificial viscosity is
      \[
      F_i^{\text{visc}} = -\sum_{j=1}^{N_i} m_j \Pi_{ij} \nabla_i W(r_{ij}, h_{ij}),
      \]
      where the viscosity tensor \( \Pi_{ij} \) is defined by
      \[
      \Pi_{ij} = \begin{cases} 
      (-\alpha c_i \mu_{ij} + \beta \mu_{ij}^2) / \rho_{ij} & \text{for } \vec{v}_{ij} \cdot \vec{r}_{ij} \leq 0, \\
      0 & \text{for } \vec{v}_{ij} \cdot \vec{r}_{ij} > 0,
      \end{cases}
      \]
      where \( \mu_{ij} = \frac{h \vec{v}_{ij} \cdot \vec{r}_{ij}}{\rho_{ij}^2 + 0.01 h^2} \).
      with \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \), \( \vec{v}_{ij} = \vec{v}_i - \vec{v}_j \), mean density \( \rho_{ij} = (\rho_i + \rho_j) / 2 \), and mean sound speed \( c_{ij} = (c_i + c_j) / 2 \).

- Advantages of the standard artificial viscosity
  - Galilean invariant
  - vanishes for rigid body rotation (but not for differential rotation!!)
  - conserved linear and angular momenta

- Disadvantages of the standard formula
  - generates entropy in shear flows — **Balsara viscosity**
  - leads to strong dissipation (one simulates ‘honey’ instead of interstellar gas) — **time-dependent viscosity & XSPH**
  - arbitrariness (no physical motivation) — **Flebbe-type viscocities**

- many alternative formulations exist

- set together, the momentum equation is
  \[
  \frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij} \right) \nabla_i W(r_{ij}, h_{ij}) - \nabla \phi_i
  \]
SPH Basics:

- the fluid equations in SPH
  - *energy equation* (conservation of momentum)
    - recall the hydrodynamic energy equation:
      \[
      \frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \bar{v} \cdot \nabla \epsilon = \frac{ds}{dt} - \frac{p}{\rho} \nabla \cdot \bar{v}
      \]
    - for *adiabatic* systems \((c = \text{const})\) the SPH form follows as
      \[
      \frac{d\epsilon_i}{dt} = \frac{p_i}{\rho_i^2} \sum_{j=1}^{N_i} m_j \bar{v}_{ij} \cdot \nabla_i W(r_{ij}, h_{ij})
      \]
      (note that the alternative form
      \[
      \frac{d\epsilon_i}{dt} = \frac{1}{2} \sum_{j=1}^{N_i} m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \bar{v}_{ij} \cdot \nabla_i W(r_{ij}, h_{ij})
      \]
      can lead to unphysical solutions, like negative internal energy)
    - *dissipation* due to (artificial) viscosity leads to a term
      \[
      \frac{d\epsilon_i}{dt} = \frac{1}{2} \sum_{j=1}^{N_i} m_j \Pi_{ij} \bar{v}_{ij} \cdot \nabla_i W(r_{ij}, h_{ij})
      \]
    - the presence of *heating sources* or *cooling* processes can be incorporated into a function \(\Gamma_i\).
  - altogether:
    \[
    \frac{d\epsilon_i}{dt} = \frac{p_i}{\rho_i^2} \sum_{j=1}^{N_i} m_j \bar{v}_{ij} \cdot \nabla_i W_{ij} + \frac{1}{2} \sum_{j=1}^{N_i} m_j \Pi_{ij} \bar{v}_{ij} \cdot \nabla_i W_{ij} + \Gamma_i
    \]
SPH Basics:

- the fluid equations in SPH
  - *entropy equation*
    - alternatively to the energy equation one can integrate an equation for the ‘entropy’
    - the *entropic function* $A(s)$ is defined by
      $$ p = A(s) \rho^\gamma. $$
    - the internal energy follows as
      $$ \epsilon = \frac{A(s)}{\gamma - 1} \rho^\gamma \gamma^{-1} \quad (\ast) $$
    - the time evolution of $A(s)$ depends on the emissivity per unit volume $\Gamma$ (heat sources and sinks) and on the viscosity; one possible SPH implementation is
      $$ \frac{dA_i}{dt} = -\frac{\gamma - 1}{\rho_i} \Gamma_i + \frac{1}{2} \frac{\gamma - 1}{\rho_i^{\gamma - 1}} \sum_{j=1}^{N_i} m_j \Pi_{ij} \vec{v}_{ij} \cdot \vec{\nabla}W(r_{ij}, h_{ij}). $$
    - the time evolution of $\epsilon_i$ is then derived from this equation via $(\ast)$, the temperature $T_i$ of particle $i$ is directly proportional to $\epsilon_i$. 
SPH Basics:

- time integration
  - time integration is done similar to $N$-body methods
  - there are two main schemes: leap-frog and predictor-corrector methods
  - variable timesteps
    - efficient use of CPU power in strongly inhomogeneous systems
    - typically, the lengths of timestep bins differ by factor 2
    - criteria for choosing the timestep
      - Courant-Friedrichs-Lewy plus viscosity criterion
        \[
        \delta t_{cv} = \frac{0.3 h}{c_s + h|\vec{\nabla} \cdot \vec{v}| + 1.2(\alpha c_s + \beta h|\vec{\nabla} \cdot \vec{v}|)}.
        \]
      - force criterion
        \[
        \delta t_f = 0.3 \sqrt[3]{\frac{h}{|\vec{F}|}},
        \]
      - global error tolerance criteria are possible in Runge-Kutta schemes
- boundary conditions
  - closed (or periodic) boundaries can be handled by introducing ‘ghost’ particles
  - open boundaries are difficult, because of large pressure gradients (e.g. water surface on air)
Modifications of SPH:

- alternative ways to force anti-symmetry
  - instead of using one kernel and take a mean value for $h_i$ average of the kernel contributions of each particle:
    \[
    W\left(|\vec{r}_i - \vec{r}_j|, \frac{h_i + h_j}{2}\right) \rightarrow \frac{1}{2} \left\{ W(|\vec{r}_i - \vec{r}_j|, h_i) + W(|\vec{r}_i - \vec{r}_j|, h_j) \right\}
    \]
  - instead of the arithmetic mean for the quantity $p/\rho^2$ use the geometric one:
    \[
    \frac{1}{2} \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \quad \longrightarrow \quad \sqrt{\frac{p_i p_j}{\rho_i \rho_j}}
    \]

- XSPH (Monaghan 1989)
  - in the standard formulation the particle is advanced by integrating
    \[
    \frac{d\vec{r}_i}{dt} = \vec{v}_i
    \]
  - it may be more physical (and in the spirit of SPH) when moving the particle with the smoothed flow velocity
    \[
    \frac{d\vec{r}_i}{dt} = \hat{\vec{v}}_i \quad \text{with} \quad \hat{\vec{v}}_i = \vec{v}_i + \eta \sum_{j=1}^{N_i} \frac{m_j}{\rho_{ij}} (\vec{v}_i - \vec{v}_j) W(|\vec{r}_i - \vec{r}_j|, h_{ij})
    \]
    where $\eta \approx 0.5$.
  - this allows for a strongly reduced artificial viscosity term \(\longrightarrow\) reach higher Reynolds numbers when modeling interstellar turbulence
  - XSPH also allows for the introduction of the \emph{Cassama-Holm subgrid model of turbulence} (Monaghan 2002 – astro-ph/0204118)
Modifications of SPH:

- alternative formulations of viscosity
  - Balsara viscosity:
    - standard viscosity generates entropy in shear flows (Balsara 1989)
    - add a correction term \(\nabla \times \vec{v}\)
    - new viscosity:
      \[
      \Pi_{ij} = \begin{cases} 
        \left( -\alpha c_{ij} \mu_{ij} + \beta \mu_{ij}^2 \right)/\rho_{ij} & \text{for } \vec{v}_{ij} \cdot \vec{r}_{ij} \leq 0, \\
        0 & \text{for } \vec{v}_{ij} \cdot \vec{r}_{ij} > 0
      \end{cases}
      \]
      where now
      \[
      \mu_{ij} = \frac{h\vec{v}_{ij} \cdot \vec{r}_{ij} f_i + f_j}{\vec{r}_{ij}^2 + 0.01h^2} \frac{f_i + f_j}{2}
      \]
    - this representation vanishes in pure shear flows, but is identical to the standard version in purely compressional flows
  - for more physically motivated viscosity see
    - etc
Modifications of SPH:

- alternative formulations of viscosity
  - switch to reduce viscosity (Morris & Monaghan 1997)
    - artificial viscosity is a strongly undesired quantity, as it leads to dissipation that is much higher than in astrophysical gases
    - for realistic models one wants as little artificial viscosity as possible (e.g. important for turbulence simulations — we model ‘honey’ instead of interstellar gas)
  - in principle, artificial viscosity is only needed in regions of strong compression (shocks)
  - introduce a switch which leads to high $\Pi_{ij}$ when $\vec{\nabla} \cdot \vec{v}$ becomes strongly negative and then let $\Pi_{ij}$ ‘decay’ to zero afterwards
  - implementation:
    - each particle $i$ carries its own value $\alpha_i$ (and $\beta_i$, e.g. with $\beta_i = 2\alpha_i$)
    - time evolution
      \[ \alpha_i = \alpha_{\min} + A \exp(-t/\tau) \]
      with decay time $\tau \approx 10h/c_s$.\]
Modifications of SPH:

- fully conservative formulation using Lagrange multipliers

  - the Lagrangian for compressible flows which are generated by the thermal energy $\epsilon(\rho, s)$ acts as effective potential is
    \[ \mathcal{L} = \int \rho \left\{ \frac{1}{2} v^2 - u(\rho, s) \right\} d^3r. \]
equations of motion follow with $s = \text{const}$ from
    \[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{v}} - \frac{\partial \mathcal{L}}{\partial r} = 0 \]

- after some SPH arithmetics, one can derive the following acceleration equation for particle $i$
  \[ \frac{d\vec{v}_i}{dt} = - \sum_{j=1}^{N_i} m_j \left\{ \frac{1}{f_i \rho_i^2} \vec{\nabla}_i W(r_{ij}, h_i) + \frac{1}{f_j \rho_j^2} \vec{\nabla}_i W(r_{ij}, h_j) \right\} \]
where
    \[ f_i = \left[ 1 + \frac{h_i}{3\rho_i} \frac{\partial \rho_i}{\partial \rho_i} \right] \]

- the Lagrange multiplier used here is the constraint that $h_i$ is adjusted such that each smoothing volume contains a fixed amount of mass

- under this constraint, the formulation conserves energy, entropy, linear and angular momentum ($\vec{\nabla} h$ terms are taken into account implicitly)
Properties of SPH:

- no clear mathematical convergence study \(\implies\) reliability of method needs to rely on comparison with analytic solutions and on empirical tests (e.g. comparing results obtained with different particle numbers)

- SPH is more dissipative than most grid-based methods

- SPH is Lagrangian, it can resolve large density contrasts wherever needed (regions of interest need not to be defined in advance)

- SPH provides good resolution in high-density regions, however, only poorly resolves low-density regions

- SPH generally performs poorly when handling shocks (but see GPH)

- SPH is a particle scheme \(\implies\) good for describing the transition from gaseous to stellar-dynamical systems (i.e. good for describing the formation of stellar clusters)

- SPH cannot (yet) handle magnetic field satisfactory (problems with stability and with \(\nabla \cdot \vec{B} = 0\) requirement)

- SPH can be combined with the special purpose hardware GRAPE