GRAVOTURBULENT STAR FORMATION
with Smoothed Particle Hydrodynamics

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Overview

1. **Physics** of star formation
   a) How do stars form?
   b) Where do stars form?
   c) Theory of *gravoturbulent star formation*

2. **Numerical approach** to star formation
   1. Large-eddy simulations (*LES*) with smoothed particle hydrodynamics (*SPH*)
   2. Transition to stellar-dynamics: introducing „sink particles“ to represent protostars (i.e. to describe *subgrid-scale physics*)
Star formation in “typical” spiral:

NGC4622

- Star formation *always* is associated with *clouds of gas and dust*.

- Star formation is essentially a *local phenomenon* (on ~pc scale)

- **HOW** is star formation *influenced* by *global* properties of the galaxy?

(from the Hubble Heritage Team)
The Orion molecular cloud is the birthplace of several young embedded star clusters.

The Trapezium cluster is only visible in the IR and contains about 2000 newly born stars.
stars form in clusters
stars form in molecular clouds
(proto)stellar feedback is important

color composite J,H,K by M. McCaughrean, VLT, Paranal, Chile
Structure and dynamics of young star clusters is coupled to the structure of the molecular cloud in the Taurus cloud. 

(from Hartmann 2002)
Structure and dynamics of young star clusters is coupled to the structure of the Taurus molecular cloud.

Star-forming filaments in *Taurus* cloud (from Hartmann 2002)

- Strukture and dynamics of *molekular cloud* is determined by *supersonic turbulence*.

The observed line width $\sigma_{\text{tot}}$ is much greater than the thermal line width $\sigma_{\text{therm}}$. 

$$\sigma_{\text{tot}} \gg \sigma_{\text{therm}}$$
The star formation process

- How do stars form?
- What determines when and where stars form?
- What regulates the process and determines its efficiency?
- How do global properties of the galaxy influence star formation (a local process)?
- Are there different modes of SF? (Starburst galaxies vs. LSBs, isolated SF vs. clustered SF)

What physical processes initiate and control the formation of stars?
Gravoturbulent star formation

New theory of star formation:

Star formation is controlled by interplay between gravity and supersonic turbulence!

Dual role of turbulence:

- stability on large scales
- initiating collapse on small scales

(full detail in Mac Low & Klessen, 2004, Rev. Mod. Phys., 76, 125-194)
Gravoturbulent star formation

New theory of star formation:

Star formation is controlled by interplay between gravity and supersonic turbulence!

Validity:

This hold on all scales and applies to build-up of stars and star clusters within molecular clouds as well as to the formation of molecular clouds in galactic disk.

(full detail in Mac Low & Klessen, 2004, Rev. Mod. Phys., 76, 125-194)
Gravoturbulent Star Formation

Supersonic turbulence in the galactic disk creates strong density fluctuations (in shocks: $\delta \rho/\rho \approx M^2$)

- chemical phase transition: atomic $\rightarrow$ molecular
- cooling instability
- gravitational instability

Cold molecular clouds form at the high-density peaks.

Turbulence creates density structure, gravity selects for collapse

$\Rightarrow$ GRAVOTUBULENT FRAGMENTATION

Turbulent cascade: Local compression *within* a cloud provokes collapse $\rightarrow$ individual *stars* and *star clusters*

(full detail in Mac Low & Klessen, 2004, Rev. Mod. Phys., 76, 125-194)
Star formation on *global* scales

density fluctuations in warm atomar ISM caused by supersonic turbulence

some are dense enough to form H2 within “reasonable timescale”

→molecular clouds

external perturbuations (i.e. potential changes) increase likelihood
Approaching the problem
Star-forming filaments in the Taurus molecular cloud

Gravoturbulent fragmentation

Gravoturbulent fragmentation in molecular clouds:
- SPH model with 1.6x10^6 particles
- large-scale driven turbulence
- Mach number $M = 6$
- periodic boundaries
- physical scaling:

“Taurus”:
- density $n(H_2) \approx 10^2$ cm$^{-3}$
- $L = 6$ pc, $M = 5000 M_\odot$

(from Ballesteros-Paredes & Klessen, in preparation)
What can we learn from that?

**global properties** (statistical properties)
- SF efficiency
- SF time scale
- IMF
- description of self-gravitating turbulent systems (pdf's, $\Delta$-var.)
- chemical mixing properties

**local properties** (properties of individual objects)
- properties of individual clumps (e.g. shape, radial profile)
- accretion history of individual protostars ($dM/dt$ vs. $t$, $j$ vs. $t$)
- binary (proto)stars (eccentricity, mass ratio, etc.)
- SED's of individual protostars
- dynamic PMS tracks: $T_{\text{bol}}$-$L_{\text{bol}}$ evolution
Turbulent diffusion I

Observations of young star clusters exhibit an enormous degree of chemical homogeneity (e.g. in the Pleiades: Wilden et al. 2002)

Star-forming gas must be well mixed.

How does this constrain models of interstellar turbulence?

→ Study mixing in supersonic compressible turbulence....
Turbulent diffusion II

Method:

- second central moment of displacement:

\[
\xi^2(t - t') = \left\langle \left[ \vec{r}_i(t) - \vec{r}_i(t') \right]^2 \right\rangle_i
\]

- classical diffusion equation:

\[
\frac{dn}{dt} = D \nabla^2 n
\]

- relation between \( D \) and \( \xi \):

\[
D(t - t') = \frac{d\xi^2(t - t')}{dt} = 2\left\langle \vec{v}_i(t - t') \cdot \vec{r}_i(t - t') \right\rangle_i
\]
Turbulent diffusion III

Time evolution of diffusion coefficient
(mean motion corrected).

(from Klessen & Lin 2003, PRE, 67, 046311)
Mean-motion corrected diffusion

Simple mixing-length approach works!

\[ D(t) \approx v_{\text{rms}}^2 t \quad t < \tau \]
\[ D(t) \approx v_{\text{rms}}^2 \tau = v_{\text{rms}} \ell \quad t > \tau \]

With \( v_{\text{rms}} \) = rms velocity and \( \ell = L/k = \) shock sep.

(from Klessen & Lin 2003, PRE, 67, 046311)
What can we learn from that?

**global properties** (statistical properties)
- SF efficiency and timescale
- stellar mass function -- IMF
- dynamics of young star clusters
- description of self-gravitating turbulent systems (pdf's, Δ-var.)
- chemical mixing properties

**local properties** (properties of individual objects)
- properties of individual clumps (e.g. shape, radial profile)
- accretion history of individual protostars (dM/dt vs. t, j vs. t)
- binary (proto)stars (eccentricity, mass ratio, etc.)
- SED's of individual protostars
- dynamic PMS tracks: $T_{bol}$-$L_{bol}$ evolution
Most stars form in clusters \( \rightarrow \) \textit{star formation} = \textit{cluster formation}

Trajectories of protostars in a nascent dense cluster created by gravoturbulent fragmentation
Mass accretion rates vary with time and are strongly influenced by the cluster environment.

Influence of EOS

But EOS depends on *chemical state*, on *balance* between *heating* and *cooling*

\[ P \propto \rho^\gamma \]
\[ P \propto \rho T \]
\[ \rightarrow \gamma = 1 + \frac{d\log T}{d\log \rho} \]

\[ n(H_2)_{\text{crit}} \approx 2.5 \times 10^5 \text{ cm}^{-3} \]
\[ \rho_{\text{crit}} \approx 10^{-18} \text{ g cm}^{-3} \]
Influence of EOS

(1) \( p \propto \rho^\gamma \Rightarrow \rho \propto p^{1/\gamma} \)

(2) \( M_{\text{jeans}} \propto \gamma^{3/2} \rho^{(3\gamma-4)/2} \)

\( \gamma < 1: \Rightarrow \text{large} \) density excursion for given pressure
\( \Rightarrow \langle M_{\text{jeans}} \rangle \) becomes small
\( \Rightarrow \) number of fluctuations with \( M > M_{\text{jeans}} \) is large

\( \gamma > 1: \Rightarrow \text{small} \) density excursion for given pressure
\( \Rightarrow \langle M_{\text{jeans}} \rangle \) is large
\( \Rightarrow \) only few and massive clumps exceed \( M_{\text{jeans}} \)
Mass spectrum

with $\rho_{\text{crit}} \approx 2.5 \times 10^5 \text{ cm}^{-3}$
at SFE $\approx 50\%$

“Standard” IMF of single stars
(e.g. Scalo 1998, Kroupa 2002)

sufficient # of brown dwarfs

(Jappsen, Klessen, Larson, Li, Mac Low, 2004, A&A submitted)
Supersonic turbulence is scale free process

\[ \rightarrow \text{POWER LAW BEHAVIOR} \]

\( \text{But also:} \) turbulence and fragmentation are highly stochastic processes \( \rightarrow \) central limit theorem

\[ \rightarrow \text{GAUSSIAN DISTRIBUTION} \]
To get the stellar mass function (IMF) we need to:

- describe **supersonic turbulence** (LES)
- include **self-gravity**
- model **thermodynamic balance** of the gas (heating, cooling, time-dependent chemistry, EOS)
- follow formation of **compact collapsed cores** (transition from hydro to stellar dynamics)
- treat **stellar dynamical processes** (protostellar collisions, ejection by close encounters)
Goal

We want to understand the formation of star clusters in turbulent interstellar gas clouds.

--> We want to describe the transition from a hydrodynamic system (the self-gravitating gas cloud) to one that is dominated by (collisional) stellar dynamics (the final star cluster).

How can we do that?
**Numerical approach I**

Problem of star formation is very complex. It involves many scales ($10^7$ in length, and $10^{20}$ in density) and many physical processes $\rightarrow$ **NO analytic solution**

$\rightarrow$ **NUMERICAL APPROACH**

**BUT**, we need to...

- solve the MHD equations in 3 dimensions
- solve Poisson’s equation (self-gravity)
- follow the full turbulent cascade (in the ISM + in stellar interior)
- include heating and cooling processes (EOS)
- treat radiation transfer
- describe energy production by nuclear burning processes
Numerical approach II

- Simplify!
  Divide problem into little bits and pieces….

**GRAVOTURBULENT CLOUD FRAGMENTATION**

- We try to…
  - solve the HD equations in 3 dimensions
  - solve Poisson’s equation (self-gravity)
  - include a (humble) approach to supersonic turbulence
  - describe perfect gas (with polytropic EOS)
  - follow collapse: include “sink particles”
    (this will “handle” our subgrid-scale physics)
Intermezzo: HD & SPH

derivation of equations of hydrodynamics

Boltzmann equation for 1D distribution function:

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{q} \cdot \nabla_q f + \dot{p} \cdot \nabla_p f
\]

moments of distribution function:
- density $\rho$
- momentum $\vec{p}$
- energy $\varepsilon$

SPH: smoothed particle hydrodynamics

particle-based scheme to solve eqn.’s of hydrodynamics

thermodynamic behavior --> equation of state (EOS)
Hydrodynamics
gases and fluids are large ensembles of interacting particles

state of system is described by location in $6N$ dimensional phase space $f^{(N)}(\vec{q}_1...\vec{q}_N, \vec{p}_1...\vec{p}_N)d\vec{q}_1...d\vec{q}_N d\vec{p}_1...d\vec{p}_N$

time evolution governed by ‘equation of motion’ for $6N$-dim probability distribution function $f^{(N)}$

$f^{(N)} \rightarrow f^{(n)}$ by integrating over all but $n$ coordinates

BBGKY hierarchy of equations of motion (after Born, Bogoliubov, Green, Kirkwood and Yvon)

physical observables are typically associated with 1- or 2-body probability density $f^{(1)}$ or $f^{(2)}$

at lowest level of hierarchy: 1-body distribution function describes the probability of finding a particle at time $t$ in the volume element $d\vec{q}$ at $\vec{q}$ with momenta in the range $d\vec{p}$ at $\vec{p}$.

Boltzmann equation – equation of motion for $f^{(1)}$

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \vec{\nabla}_q f + \dot{\vec{p}} \cdot \vec{\nabla}_p f$$

$$= \frac{\partial f}{\partial t} + \ddot{\vec{v}} \cdot \vec{\nabla}_q f + \vec{F} \cdot \vec{\nabla}_p f = f_c.$$
Boltzmann equation

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{q} \cdot \nabla_q f + \dot{p} \cdot \nabla_p f = \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_q f + \vec{F} \cdot \nabla_p f = f_c. \]

→ first line: transformation from comoving to spatially fixed coordinate system.
→ second line: velocity \( \vec{v} = \dot{q} \) and force \( \vec{F} = \dot{p} \)
→ all higher order terms are 'hidden' in the collision term \( f_c \)

observable quantities are typically (velocity) moments of the Boltzmann equation, e.g.

→ density:
\[ \rho = \int m f(\vec{q}, \vec{p}, t) d\vec{p} \]

→ momentum:
\[ \rho \vec{v} = \int m \vec{v} f(\vec{q}, \vec{p}, t) d\vec{p} \]

→ kinetic energy density:
\[ \rho \vec{v}^2 = \int m \vec{v}^2 f(\vec{q}, \vec{p}, t) d\vec{p} \]
- In general: the $i$-th velocity moment $\langle \xi_i \rangle$ (of $\xi_i = m \vec{v}^i$) is

$$
\langle \xi_i \rangle = \frac{1}{n} \int \xi_i \ f(\vec{q}, \vec{p}, t) d\vec{p}
$$

with the mean particle number density $n$ defined as

$$
n = \int f(\vec{q}, \vec{p}, t) d\vec{p}
$$

- The equation of motion for $\langle \xi_i \rangle$ is

$$
\int \xi_i \left\{ \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_q f + \vec{F} \cdot \vec{\nabla}_p f \right\} d\vec{p} = \int \xi_i \ \{ f_c \} \ d\vec{p},
$$

which after some complicated rearrangement becomes

$$
\frac{\partial}{\partial t} n \langle \xi_i \rangle + \vec{\nabla}_q \left( n \langle \xi_i \vec{v} \rangle \right) + n \vec{F} \langle \vec{\nabla}_p \xi_i \rangle = \int \xi_i \ f_c \ d\vec{p}
$$

(Maxwell-Boltzmann transport equation for $\langle \xi_i \rangle$)
• If the RHS is zero, then $\xi_i$ is a conserved quantity. This is only the case for first three moments, mass $\xi_0 = m$, momentum $\vec{\xi}_1 = m\vec{v}$, and kinetic energy $\xi_2 = m\vec{v}^2/2$.

• MB equations build a hierarchically nested set of equations, as $\langle \xi_i \rangle$ depends on $\langle \xi_{i+1} \rangle$ via $\vec{\nabla}_q (n\langle \xi_i \vec{v} \rangle)$ and because the collision term cannot be reduced to depend on $\xi_i$ only.
  $\rightarrow$ Need for a closure equation
  $\rightarrow$ In hydrodynamics this is typically the equation of state.
assumptions

○ **continuum limit:**
  → distribution function $f$ must be a ‘smoothly’ varying function on the scales of interest $\rightarrow$ local average possible
  → stated differently: the averaging scale (i.e. scale of interest) must be larger than the mean free path of individual particles
  → stated differently: microscopic behavior of particles can be neglected
  → **concept of fluid element must be meaningful**

○ **only ‘short range forces’:**
  → forces between particles are short range or saturate $\rightarrow$ collective effects can be neglected
  → stated differently: correlation length of particles in the system is finite (and smaller than the scales of interest)
limitations

- shocks (scales of interest become smaller than mean free path)
- phase transitions (correlation length may become infinite)
- description of self-gravitating systems
- description of fully fractal systems
the equations of hydrodynamics

- hydrodynamics ≡ book keeping problem
  One must keep track of the ‘change’ of a fluid element due to various physical processes acting on it. How do its ‘properties’ evolve under the influence of compression, heat sources, cooling, etc.?

- Eulerian vs. Lagrangian point of view

Consider spatially fixed volume element following motion of fluid element
- hydrodynamic equations = set of equations for the five conserved quantities \((\rho, \rho \vec{v}, \rho \vec{v}^2/2)\) plus closure equation (plus transport equations for ‘external’ forces if present, e.g. gravity, magnetic field, heat sources, etc.)
equations of hydrodynamics

\[
\frac{d \rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho = -\rho \vec{\nabla} \cdot \vec{v} \quad \text{(continuity equation)}
\]

\[
\frac{d \vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \eta \vec{\nabla}^2 \vec{v} + \left(\zeta + \frac{\eta}{3}\right) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \quad \text{(Navier-Stokes equation)}
\]

\[
\frac{d \epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \vec{\nabla} \epsilon = T \frac{ds}{dt} - \frac{p}{\rho} \vec{\nabla} \cdot \vec{v} \quad \text{(energy equation)}
\]

\[
\vec{\nabla}^2 \phi = 4\pi G \rho \quad \text{(Poisson’s equation)}
\]

\[
p = \mathcal{R} \rho T \quad \text{(equation of state)}
\]
\[ \vec{F}_B = -\nabla \frac{\vec{B}^2}{8\pi} + \frac{1}{4\pi} (\vec{B} \cdot \nabla) \vec{B} \]  
(magnetic force)

\[ \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \]  
(Lorentz equation)

\( \rho = \) density, \( \vec{v} = \) velocity, \( p = \) pressure, \( \phi = \) gravitational potential, \( \zeta \) and \( \eta \) viscosity coefficients, \( \epsilon = \rho \vec{v}^2/2 = \) kinetic energy density, \( T = \) temperature, \( s = \) entropy, \( R = \) gas constant, \( \vec{B} = \) magnetic field (cgs units)
mass transport – continuity equation

\[ \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho = -\rho \vec{\nabla} \cdot \vec{v} \]

(conservation of mass)
○ transport equation for momentum – Navier Stokes equation
\[
\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \phi + \eta \nabla^2 \mathbf{v} + \left( \zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{v})
\]

momentum change due to
→ pressure gradients: \((-\rho^{-1} \nabla p)\)
→ (self) gravity: \(-\nabla \phi\)
→ viscous forces (internal friction, contains \(\text{div}(\partial v_i/\partial x_j)\) terms):
\[
\eta \nabla^2 \mathbf{v} + \left( \zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{v})
\]

(conservation of momentum, general form of momentum transport: \(\partial_t (\rho v_i) = -\partial_j \Pi_{ij}\))
transport equation for internal energy

\[
\frac{d \epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \nabla \epsilon = T \frac{d s}{d t} - \frac{p}{\rho} \nabla \cdot \vec{v}
\]

→ follows from the thermodynamic relation

\[d \epsilon = T \, d s - p \, d V = T \, d s + \frac{p}{\rho^2} \, d \rho\]

which described changes in \(\epsilon\) due to entropy changed and to volume changes (compression, expansion)

→ for adiabatic gas the first term vanishes (\(s = \text{constant}\))

→ heating sources, cooling processes can be incorporated in \(d s\)

(conservation of energy)
- closure equation – equation of state
  - general form of equation of state $p = p(T, \rho, \ldots)$
  - ideal gas: $p = \mathcal{R}\rho T$
  - special case – isothermal gas: $p = c_s^2T$ (as $\mathcal{R}T = c_s^2$)

Note:
- in reality, computing the EOS is VERY complex!
- depends on detailed balance between heating and cooling
- these depend on chemical composition (which atomic and molecular species, dust)
- and on the ability to radiate away "cooling lines" and black body radiation
- --> problem of radiation transfer (see, e.g., IPAM III)
In general:

- the „standard way“ of solving the equations of (magneto) hydrodynamics is using finite differences on a grid

- alternative use particle-based scheme: \textit{SPH}

- see IPAM workshop I
concept of SPH

- ‘invented’ independently by Lucy (1977) and Gingold & Monaghan (1977)
- originally proposed as Monte Carlo approach to calculate the time evolution of gaseous systems
- more intuitively understood as interpolation scheme:

The fluid is represented by an ensemble of particles $i$, each carrying mass $m_i$, momentum $m_i \vec{v}_i$, and hydrodynamic properties (like pressure $p_i$, temperature $T_i$, internal energy $\epsilon_i$, entropy $s_i$, etc.). The time evolution is governed by the equation of motion plus additional equations to modify the hydrodynamic properties of the particles. Hydrodynamic observables are obtained by a local averaging process.
properties of local averaging processes

○ local averages $\langle f(\vec{r}) \rangle$ for any quantity $f(\vec{r})$ can be obtained by convolution with an appropriate smoothing function $W(\vec{r}, \vec{h})$:

$$\langle f(\vec{r}) \rangle \equiv \int f(\vec{r}') W(\vec{r} - \vec{r}', \vec{h}) \, d^3r'.$$

the function $W(\vec{r}, \vec{h})$ is called smoothing kernel

○ the kernel must satisfy the following two conditions:

$$\int W(\vec{r}, \vec{h}) \, d^3r = 1 \quad \text{and} \quad \langle f(\vec{r}) \rangle \longrightarrow f(\vec{r}) \quad \text{for} \quad \vec{h} \to 0$$

the kernel $W$ therefore follows the same definitions as Dirac’s delta function $\delta(\vec{r})$:

$$\lim_{h \to 0} W(\vec{r}, h) = \delta(\vec{r}).$$

○ most SPH implementations use spherical kernel functions

$$W(\vec{r}, \vec{h}) \equiv W(r, h) \quad \text{with} \quad r = |\vec{r}| \quad \text{and} \quad h = |\vec{h}|.$$ 

(one could also use triaxial kernels, e.g. Martel et al. 1995)
properties of local averaging processes

- as the kernel function $W$ can be seen as approximation to the $\delta$-function for small but finite $h$ we can expand the averaged function $\langle f(\vec{r}') \rangle$ into a Taylor series for $h$ to obtain an estimate for $f(\vec{r}')$; if $W$ is an even function, the first order term vanishes and the errors are second order in $h$

\[
\langle f(\vec{r}') \rangle = f(\vec{r}') + \mathcal{O}(h^2)
\]

this holds for functions $f$ that are smooth and do not exhibit steep gradients over the size of $W$ (→ problems in shocks).

(more specifically the expansion is $\langle f(\vec{r}) \rangle = f(\vec{r}) + \kappa h^2 \nabla^2 f(\vec{r}) + \mathcal{O}(h^3)$)
properties of local averaging processes

- within its intrinsic accuracy, the smoothing process therefore is a linear function with respect to summation and multiplication:

\[
\langle f(\vec{r}) + g(\vec{r}) \rangle = \langle f(\vec{r}) \rangle + \langle g(\vec{r}) \rangle \\
\langle f(\vec{r}) \cdot g(\vec{r}) \rangle = \langle f(\vec{r}) \rangle \cdot \langle g(\vec{r}) \rangle
\]

(one follows from the linearity of integration with respect to summation, and two is true to \( O(h^2) \))

- derivatives can be ‘drawn into’ the averaging process:

\[
\frac{d}{d t} \langle f(\vec{r}) \rangle = \left\langle \frac{d}{d t} f(\vec{r}) \right\rangle \\
\vec{\nabla} \langle f(\vec{r}) \rangle = \left\langle \vec{\nabla} f(\vec{r}) \right\rangle
\]

Furthermore, the spatial derivative of \( f \) can be transformed into a spatial derivative of \( W \) (no need for finite differences or grid):

\[
\vec{\nabla} \langle f(\vec{r}) \rangle = \left\langle \vec{\nabla} f(\vec{r}) \right\rangle = \int f(\vec{r}') \vec{\nabla} W(|\vec{r} - \vec{r}'|, h) \, d^3 r'.
\]

(shown by integrating by parts and assuming that the surface term vanishes; if the solution space is extended far enough, either the function \( f \) itself or the kernel approach zero)
properties of local averaging processes

- basic concept of SPH is a particle representation of the fluid $\longrightarrow$ integration transforms into summation over discrete set of particles; example density $\rho$:

$$\langle \rho(\vec{r}_i) \rangle = \sum_j m_j W(|\vec{r}_i - \vec{r}_j|, h).$$

in this picture, the mass of each particle is smeared out over its kernel region; the density at each location is obtained by summing over the contributions of the various particles $\longrightarrow$ smoothed particle hydrodynamics!
the kernel function

- different functions meet the requirement $\int W(|\vec{r}|, h) \, d^3r = 1$
  and $\lim_{h \to 0} \int W(|\vec{r} - \vec{r}'|, h) f(\vec{r}') \, d^3r' = f(\vec{r})$:

  $\rightarrow$ Gaussian kernel:

  $$W(r, h) = \frac{1}{\pi^{3/2} h^3} \exp \left( -\frac{r^2}{h^2} \right)$$

  - *pro*: mathematically sound
  - *pro*: derivatives exist to all orders and are smooth
  - *contra*: all particles contribute to a location

  $\rightarrow$ spline functions with compact support
the kernel function

- different functions meet the requirement \( \int W(|\vec{r}|, h) \, d^3r = 1 \) and \( \lim_{h \to 0} \int W(|\vec{r} - \vec{r}'|, h) \, f(\vec{r}') \, d^3r' = f(\vec{r}) \):

  → the standard kernel: cubic spline

  with \( \xi = r/h \) it is defined as

  \[
  W(r, h) \equiv \frac{1}{\pi h^3} \begin{cases} 
  1 - \frac{3}{2} \xi^2 + \frac{3}{4} \xi^3, & \text{for } 0 \leq \xi \leq 1; \\
  \frac{1}{4}(2 - \xi)^3, & \text{for } 1 \leq \xi \leq 2; \\
  0, & \text{otherwise}. 
  \end{cases}
  \]

- \textit{pro}: compact support → all interactions are zero for \( r > 2h \) → number of particles involved in the average remains small (typically between 30 and 80)

- \textit{pro}: second derivative is continuous

- \textit{pro}: dominant error term is second order in \( h \)
the fluid equations in SPH

- there is an infinite number of possible SPH implementations of the hydrodynamic equations!
- some notation: $h_{ij} = (h_i + h_j)/2$, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, and $\vec{\nabla}_i$ is the gradient with respect to the coordinates of particle $i$; all measurements are taken at particle positions (e.g. $\rho_i = \rho(\vec{r}_i)$)
- *general form of SPH equations:*

\[
\langle f_i \rangle = \sum_{j=1}^{N_i} \frac{m_j}{\rho_j} f_j W(r_{ij}, h_{ij})
\]
the fluid equations in SPH

- **density** — continuity equation (conservation of mass)

\[ \rho_i = \sum_{j=1}^{N_i} m_j W(r_{ij}, h_{ij}) \]

or

\[ \frac{d\rho_i}{dt} = \sum_{j=1}^{N_i} m_j \vec{v}_{ij} \cdot \nabla_i W(r_{ij}, h_{ij}) \]

(the second implementation is almost never used, see however Monaghan 1991 for an application to water waves)

**important**

density is needed for **ALL** particles **BEFORE** computing other averaged quantities — at each timestep, SPH computations consist of **TWO** loops, first the **density** is obtained for each particle, and then in a second round, all **other** particle properties are updated.
the fluid equations in SPH

- *pressure* is defined via the *equation of state* (for example for isothermal gas $p_i = c_s^2 \rho_i$)
the fluid equations in SPH

- **velocity** — Navier Stokes equation (conservation of momentum)

\[
\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \sum_i F_i = F_{\text{pressure}} + F_{\text{viscosity}} + F_{\text{gravity}}
\]

rate of change of momentum of fluid element depends on sum of all forces acting on it.
the fluid equations in SPH

- **velocity** — Navier Stokes equation (conservation of momentum)
  
  → consider for now only pressure contributions: Euler’s equation

\[
\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p = -\nabla \left( \frac{p}{\rho} \right) - \frac{p}{\rho^2} \nabla \rho \quad (*)
\]

here, the identity \( \nabla (p\rho^{-1}) = \rho^{-1} \nabla p - p\rho^{-2} \nabla \rho \) is used

→ in the SPH formalism this reads as

\[
\frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla_i W(r_{ij}, h_{ij})
\]

where the first term in (*) is neglected because it leads to surface terms in the averaging procedure; it is assumed that either the pressure or the kernel becomes zero at the integration border; if this is not the case correction terms need to be added above.
the fluid equations in SPH

- **velocity** — Navier Stokes equation (conservation of momentum)

  → the SPH implementation of the standard artificial viscosity is

  \[
  \vec{F}_i^{\text{visc}} = - \sum_{j=1}^{N_i} m_j \Pi_{ij} \vec{\nabla}_i W(r_{ij}, h_{ij}) ,
  \]

  where the viscosity tensor \( \Pi_{ij} \) is defined by

  \[
  \Pi_{ij} = \begin{cases} 
  \frac{(-\alpha c_{ij} \mu_{ij} + \beta \mu_{ij}^2)}{\rho_{ij}} & \text{for } \vec{v}_{ij} \cdot \vec{r}_{ij} \leq 0 , \\
  0 & \text{for } \vec{v}_{ij} \cdot \vec{r}_{ij} > 0 ,
  \end{cases}
  \]

  where

  \[
  \mu_{ij} = \frac{h \vec{v}_{ij} \cdot \vec{r}_{ij}}{\vec{r}_{ij}^2 + 0.01h^2} .
  \]

  with \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \), \( \vec{v}_{ij} = \vec{v}_i - \vec{v}_j \), mean density \( \rho_{ij} = (\rho_i + \rho_j)/2 \), and mean sound speed \( c_{ij} = (c_i + c_j)/2 \).
the fluid equations in SPH

- velocity — Navier Stokes equation (conservation of momentum)

  → if self-gravity is taken into account, the gravitational force needs to be added on the RHS

\[
\vec{F}_G = -\vec{\nabla} \phi_i = -G \sum_{j=1}^{N} \frac{m_j r_{ij}}{r_{ij}^2 r_{ij}}
\]

note that the sum needs to be taken over ALL particles ← computationally expensive

→ set together, the momentum equation is

\[
\frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij} \right) \vec{v}_j W(r_{ij}, h_{ij}) - \nabla \phi_i
\]
the fluid equations in SPH

- **energy equation** (conservation of momentum)
  
  → recall the hydrodynamic energy equation:
  \[
  \frac{d\varepsilon}{dt} = \frac{\partial \varepsilon}{\partial t} + \vec{v} \cdot \vec{\nabla} \varepsilon = \frac{ds}{dt} - \frac{p}{\rho} \vec{\nabla} \cdot \vec{v}
  \]

  → for **adiabatic** systems \((c = \text{const})\) the SPH form follows as
  \[
  \frac{d\varepsilon_i}{dt} = \frac{p_i}{\rho_i^2} \sum_{j=1}^{N_i} m_j \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}, h_{ij}),
  \]

  (note that the alternative form
  \[
  \frac{d\varepsilon_i}{dt} = \frac{1}{2} \sum_{j=1}^{N_i} m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}, h_{ij})
  \]

  can lead to unphysical solutions, like negative internal energy)
the fluid equations in SPH

- **energy equation** (conservation of momentum)
  
  → *dissipation* due to (artificial) viscosity leads to a term
  
  \[
  \frac{d\varepsilon_i}{dt} = \frac{1}{2} \sum_{j=1}^{N_i} m_j \Pi_{ij} \mathbf{v}_{ij} \cdot \nabla_i W(r_{ij}, h_{ij})
  \]

  → the presence of *heating* sources or *cooling* processes can be incorporated into a function \( \Gamma_i \).

→ altogether:

\[
\frac{d\varepsilon_i}{dt} = \frac{p_i}{\rho_i^2} \sum_{j=1}^{N_i} m_j \mathbf{v}_{ij} \cdot \nabla_i W_{ij} + \frac{1}{2} \sum_{j=1}^{N_i} m_j \Pi_{ij} \mathbf{v}_{ij} \cdot \nabla_i W_{ij} + \Gamma_i
\]

(can lead to unphysical solutions, like negative internal energy)
fully conservative formulation using Lagrange multipliers

- the Lagrangian for compressible flows which are generated by the thermal energy $\epsilon(\rho, s)$ acts as effective potential is

$$\mathcal{L} = \int \rho \left\{ \frac{1}{2} v^2 - u(\rho, s) \right\} d^3 r.$$  

- equations of motion follow with $s = \text{const}$ from

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \vec{v}} - \frac{\partial \mathcal{L}}{\partial \vec{r}} = 0$$

- after some SPH arithmetics, one can derive the following acceleration equation for particle $i$

$$\frac{d \vec{v}_i}{dt} = - \sum_{j=1}^{N_i} m_j \left\{ \frac{1}{f_i \rho_i^2} \nabla_i W(r_{ij}, h_i) + \frac{1}{f_j \rho_j^2} \nabla_i W(r_{ij}, h_j) \right\}$$

where

$$f_i = \left[ 1 + \frac{h_i}{3 \rho_i \partial h_i} \right]$$
Large-eddy simulations

- We use LES to model the large-scale dynamics.
- Principal problem: only large scale flow properties.
  - Reynolds number: \( \text{Re} = \frac{LV}{\nu} \) \( (\text{Re}_{\text{nature}} \gg \text{Re}_{\text{model}}) \)
  - Dynamic range much smaller than true physical one.
- Need subgrid model (in our case simple: only dissipation).
  - More complex when processes (chemical reactions, nuclear burning, etc.) on subgrid scale determine large-scale dynamics.
- Also: stochasticity of the flow \( \Rightarrow \) unpredictable when and where “interesting things” happen.
  - Occurrence of localized collapse.
  - Location and strength of shock fronts.
  - Etc.
LES with SPH

For self-gravitating gases SPH is probably okay …

- fully Lagrangian (particles are free to move where needed)
- good resolution in high-density regions (in collapse)
- particle based --> good for transition from hydrodynamics to stellar dynamics

BUT:

- low resolution in low-density region
- difficult to reach very high levels of refinement
  (however, particle splitting may be promising path)
- dissipative and need for artificial viscosity
- how to handle subgrid scales?
Gravoturbulent SF with SPH

Comparison between particle-based and grid-based methods: **SPH vs. ZEUS**

- Klessen, Heitsch, Mac Low (2000)
- Heitsch, Mac Low, Klessen (2001)
- Ossenkopf, Klessen, Heitsch (2001)

Both methods are complementary…

→ Bracketing reality!

As a crude estimate:

**SPH is better in high-density regions**

**ZEUS is better in low-density regions**
lower resolution in low-density regions in SPH, better resolution in high-density regions compared to ZEUS

→ less power on large scales
→ more power on small scales

(Ossenkopf, Klessen, Heitsch 2001)
SPH vs. ZEUS

contracting regions in are larger in ZEUS
→ more easily destroyed by ambient turbulence
→ „star formation“ proceeds slower and with less efficiency

(Ossenkopf, Klessen, Heitsch 2001)
sink particles *replace* high-density regions (collapsed) by "point masses" which interact only gravitationally and can *accrete* ambient gas particles which come closer than some critical radius $R$. 

**SPH with sink particles I**
SPH with sink particles I

- sink particle diameter
- sink separation at formation
- sink density threshold
SPH with sink particles II

- sink particle diameter
- sink separation at formation
- sink density threshold

- length scale
- density

- length scale
- density
SPH with sink particles III

- sink particle diameter
- sink separation at formation
- sink density threshold
SPH with sink particles IV

- sink particle diameter
- sink separation at formation
- sink density threshold
SPH with sink particles V

High-resolution runs vs. low-resolution calculations
Some final remarks...

**GRAVOTURBULENT STAR FORMATION:**
This dynamic theory can explain and reproduce many features of star-forming regions on small as well as on large galactic scales.

Some open questions:
- role of magnetic fields?
- role of thermodynamic state of the gas?
- what drives turbulence?
- how are small scales (local molecular clouds) connected to large-scale dynamics?
- what terminates star formation locally?
Some final remarks...

**NUMERICS:**
SPH appears able to describe gravoturbulent fragmentation and star formation in molecular clouds.

**Pro:**
- *Lagrangian* character of method.
- can resolve *large density contrasts*.
- good for transition from hydro- to stellar dynamics
  --> accreting sink particles describe protostars

**Con:**
- low resolution in low-density regions.
- difficulties with shock-capturing and treating B-fields.

**Next steps:**
- particle-splitting to locally increase resolution,
- GPM, XSPH with “physical” viscosity
Outlook & First Examples

**WHAT WE REALLY NEED:** *more physics!!*

We need good *subgrid-scale models* for unresolved scales in our calculations.

**2 Examples:**

- **LOCAL SCALES:** so far, we use dumb sink particles to protostellar collapse --> we do not know how the star “inside” forms and how it backreacts onto the ambient environment
  --> combine 3D hydro with 1D/2D PMS models

- **GALACTIC SCALES:** can gravoturbulent models give us some handle on star-formation efficiency?
  --> some thoughts...
Example 1
Towards a complete picture...

**COMBINE:**
- **3D** hydrodynamic simulations of the *turbulent fragmentation of entire molecular cloud regions.*

**WITH:**
- detailed **2D** hydrodynamic modelling of *protostellar accretion disks.* (e.g. Yorke & Bodenheimer 1999)
  → with rad. transfer → *SED, T_{bol}, L_{bol},* etc.

**AND/OR:**
- implicit **1D** radiation-hydrodynamic scheme with time-dependent convection and D network following the *collapse of individual cores towards the MS* (Wuchterl & Tscharnuter 2002)
  → *PMS tracks, absolute stellar ages for cluster stars*
Formation of a $1\text{M}_\odot$ star

Dynamical evolution of a molecular cloud region of size $(0.32\text{pc})^3$ containing $200\text{ M}_\odot$ of gas
(from Klessen & Burkert 2000)

Within two free-fall times the system builds up a cluster of deeply embedded accreting protostellar cores. We select the protostellar core with mass closest to $1\text{M}_\odot$ and use its mass accretion rate as *input* for the detailed 1D-RHD calculation (see Wuchterl & Tscharnuter 2002)

The system is shown initially, and at a stage when the $1\text{M}_\odot$-fragment reaches zero age (i.e. when it becomes optically thick for the first time).

(from Wuchterl & Klessen 2001)
For ages less than $10^6$ years, different collapse conditions lead to different evolutionary tracks. Later the dynamical tracks converge.

There are large differences to the hydrostatic track, due to different stellar structure.

(From Wuchterl & Klessen 2001)
Dynamical evolution of a molecular cloud region of size $(0.32 \text{pc})^3$ containing $200 \, M_\odot$ of gas (from Klessen & Burkert 2000)

Within one to two free-fall times the system builds up a cluster of deeply embedded accreting protostellar cores.

We select a protostellar core which will form a single star and use its mass accretion rate and angular momentum gain into a control volume containing the core as input for a detailed 2D calculation (Bodenheimer & Klessen, in preparation)
SED's of a 1 $M_\odot$ star

$\tau \approx 15000$ yr  
$M_{\text{tot}} \approx 0.75 M_\odot$

$\tau \approx 30000$ yr  
$M_{\text{tot}} \approx 0.89 M_\odot$

(Bodenheimer & Klessen, in preparation)
Star formation on global scales

- **SF on global scales** = formation of molecular clouds

- MC’s form at *stagnation points* of convergent large-scale flows (need ~0.5kpc$^3$ of gas) $\rightarrow$ high density $\rightarrow$ enhanced cooling $\rightarrow$ fast H$_2$ formation & gravitational instability $\rightarrow$ local collapse and star formation

- External perturbations *increase* the local likelihood of MC formation (e.g. in spiral density waves, galaxy interactions, etc.)
Star formation on *global* scales

- Density fluctuations in warm atomic ISM caused by supersonic turbulence.
- Some are dense enough to form H2 within a "reasonable timescale" and form molecular clouds.
- External perturbations (i.e., potential changes) increase the likelihood of star formation.
Correlation between \( \text{H}_2 \) and \( \text{HI} \)

Compare \( \text{H}_2 \) - \( \text{HI} \) in M33:
- \( \text{H}_2 \): BIMA-SONG Survey, see Blitz et al.
- \( \text{HI} \): Observations with Westerbork Radio T.

\( \text{H}_2 \) clouds are seen in regions of high \( \text{HI} \) density (in spiral arms and filaments)

(Deul & van der Hulst 1987, Blitz et al. 2004)
Star formation on global scales

probability distribution function of density ($\rho$-pdf) for decaying supersonic turbulence

$varying$ $rms$ $Mach$ $numbers$:

$M1 > M2 > M3 > M4 > 0$

mass weighted $\rho$-pdf, each shifted by $\Delta \log N = 1$

(from Klessen, 2001)
Star formation on global scales

$H_2$ formation rate:

$$\tau_{H_2} \approx \frac{1.5 \text{ Gyr}}{n_H / 1 \text{ cm}^{-3}}$$

For $n_H \geq 100 \text{ cm}^{-3}$, $H_2$ forms within 10 Myr, this is about the lifetime of typical MC’s.

What fraction of the galactic ISM reaches such densities?

mass weighted $\rho$-pdf, each shifted by $\Delta \log N = 1$

(from Klessen, 2001; rate from Hollenback, Werner, & Salpeter 1971)