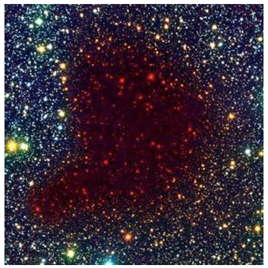
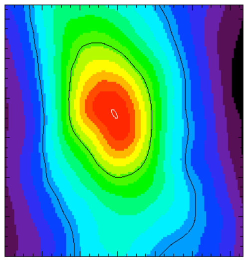
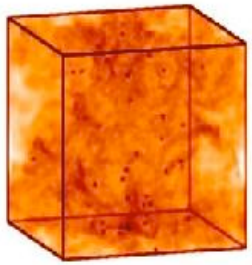
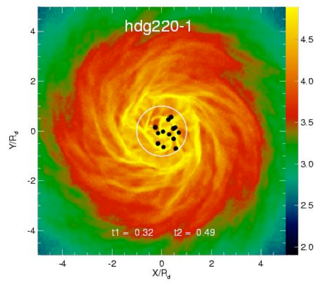
GRAVOTURBULENT
STAR FORMATION
with Smoothed Particle Hydrodynamics





Ralf Klessen

Emmy Noether Research Group (DFG)

Astrophysikalisches Institut Potsdam

AIP

DFG

Los Angeles, IPAM, 20.04.2005

Collaborators

Many thanks to...

- Javier Ballesteros-Paredes (UNAM, Morelia)
- Peter Bodenheimer (UC Santa Cruz)
- Andreas Burkert (Uni. München)
- Fabian Heitsch (Uni. München)
- Dirk Froebrich (Dublin University)
- Simon Glover (AMNH, New York)
- Eva Grebel (Universität Basel)
- Spyros Kitsionas (U. Athens)
- Pavel Kroupa (Universität Bonn)
- Katharina Jappsen (AIP, Potsdam)
- Richard Larson (Yale University)
- Yuexing Li (Columbia University)
- Doug Lin (UC Santa Cruz)
- Mordecai-Mark Mac Low (AMNH, New York)
- Stefan Schmeja (AIP, Potsdam)
- Michael Smith (Armagh University)
- Marco Spaans (RU Groningen)
- Rainer Spurzem (ARI, Heidelberg)
- Enrique Vazquez-Semadeni (UNAM, Morelia)
- HongSheng Zhao (IoA, Cambridge)
- Hans Zinnecker (AIP, Potsdam)

Overview

1. Physics of star formation
   a) How do stars form?
   b) Where do stars form?
   c) Theory of *gravoturbulent star formation*

2. Numerical approach to star formation
   1. Large-eddy simulations *(LES)* with smoothed particle hydrodynamics *(SPH)*
   2. Transition to stellar-dynamics: introducing *„sink particles“* to represent protostars (i.e. to describe *subgrid-scale physics*)

STAR FORMATION

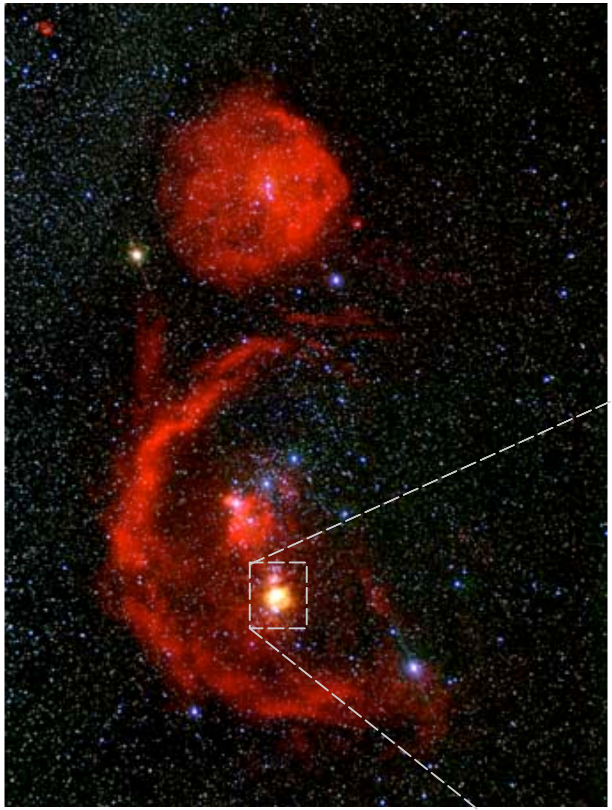
Star formation in "typical" spiral:


(from the Hubble Heritage Team)

NGC4622

- Star formation *always* is associated with *clouds of gas and dust*.
- Star formation is essentially a ***local phenomenon*** (on ~pc scale)
- HOW is star formation is *influenced* by ***global*** properties of the galaxy?

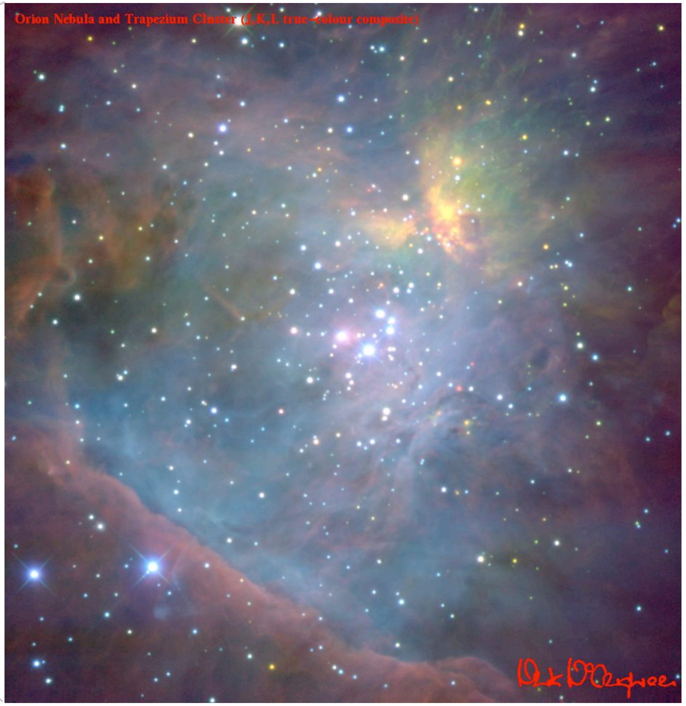
Local star forming region: The Trapezium Cluster in Orion



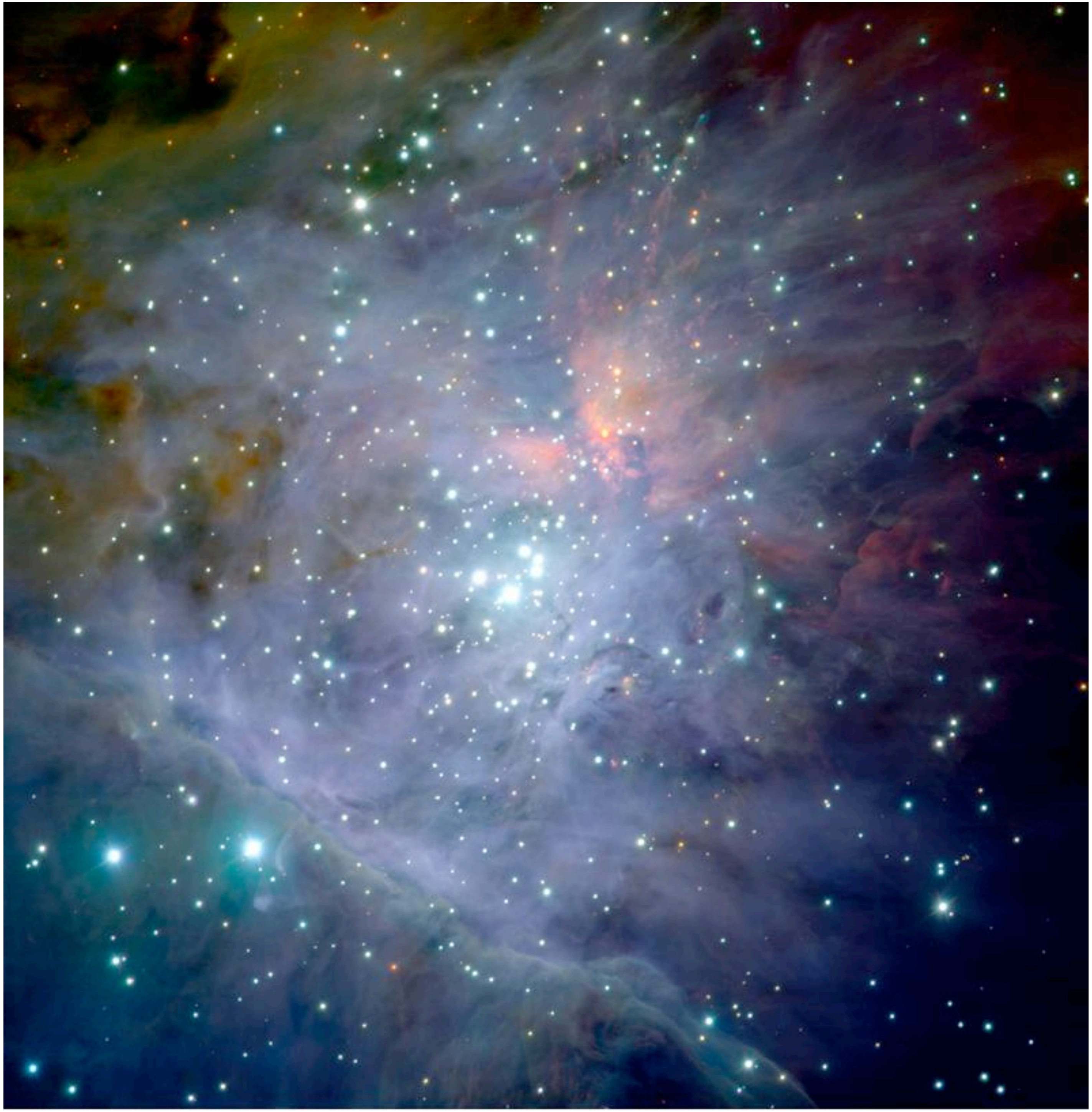
Orion molecular cloud

The Orion molecular cloud is the birth- place of several young embedded star clusters.

The Trapezium cluster is only visible in the IR and contains about 2000 newly born stars.



Trapezium cluster



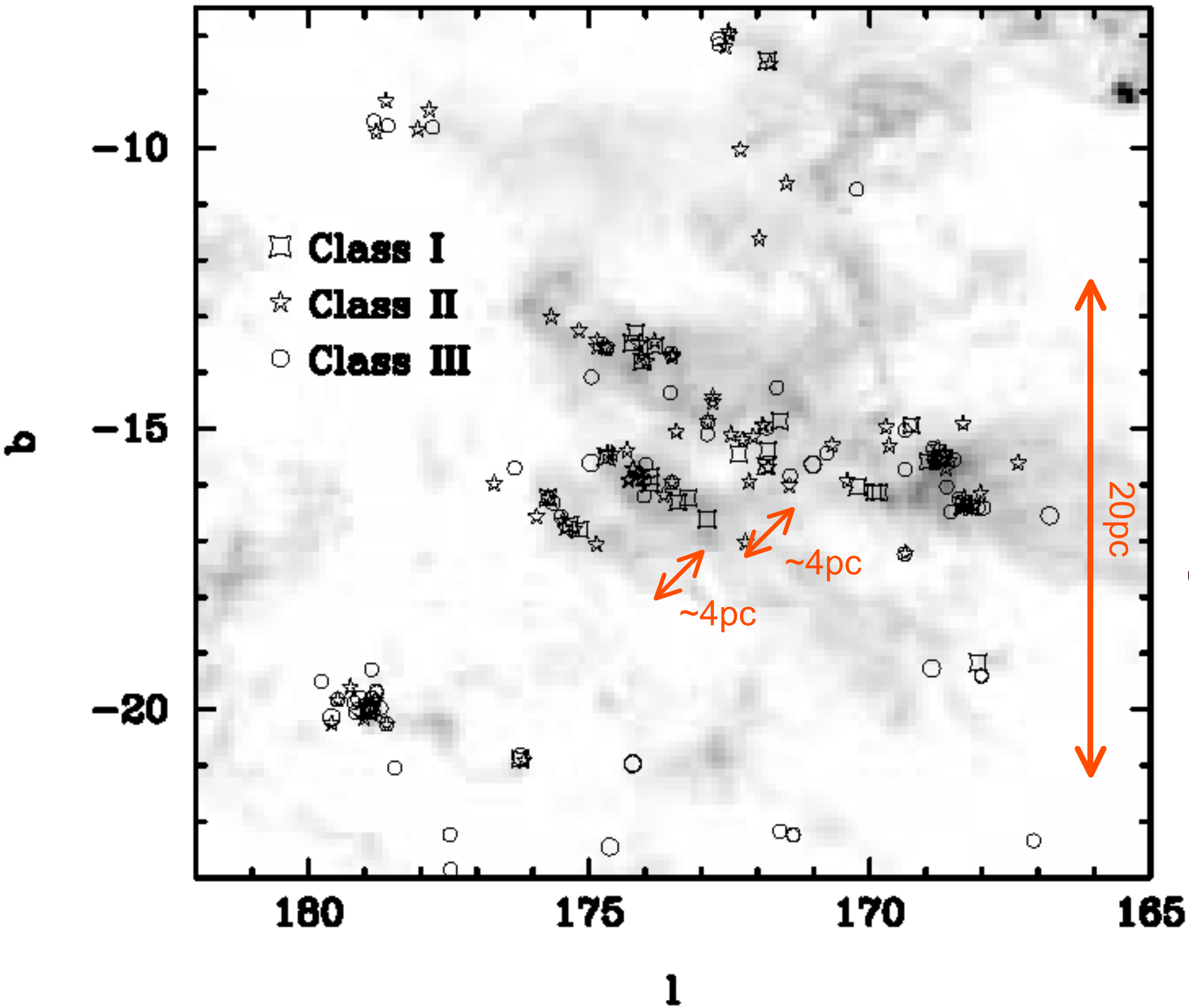
Trapezium Cluster

(detail)

- stars form in clusters
- stars form in molecular clouds
- (proto)stellar feedback is important

(color composite J,H,K by M. McCaughrean, VLT, Paranal, Chile)

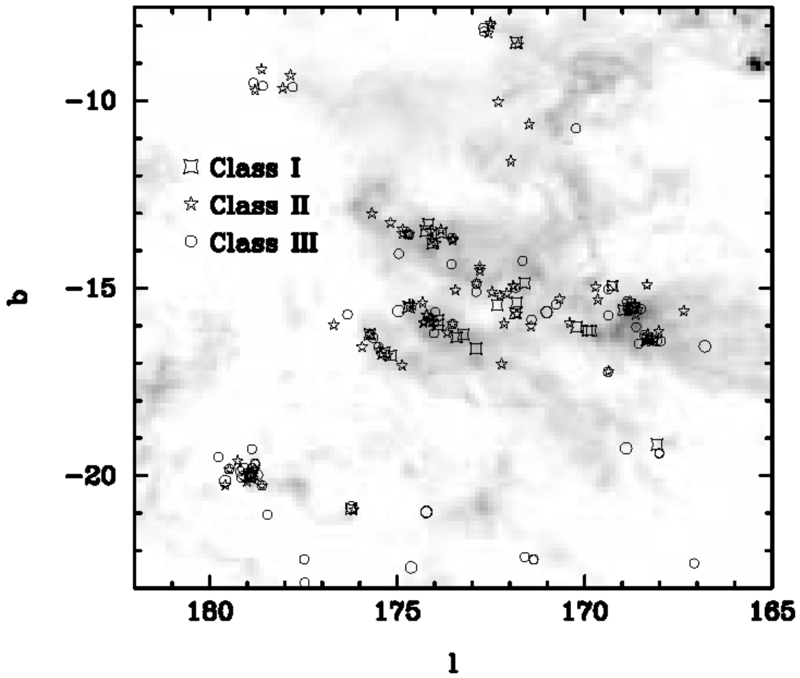
Taurus molecular cloud



star-forming filaments in the *Taurus* cloud

- Structure and dynamics of young star clusters is coupled to ***structure of mol. cloud***

(from Hartmann 2002)



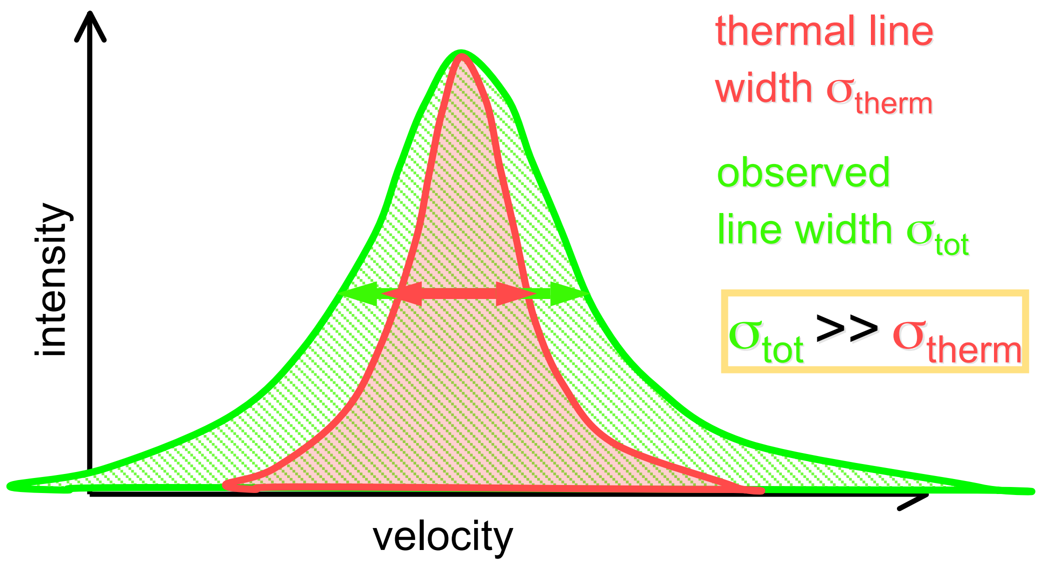
Taurus molecular cloud

Star-forming filaments in *Taurus* cloud

(from Hartmann 2002)

- Strukture and dynamics of ***molekular cloud*** is determined by ***supersonic turbulence***

- Structure and dynamics of young star clusters is coupled to ***structure of molecular cloud***



The star formation process

- *How* do stars form?
- What determines *when* and *where* stars form?
- What *regulates* the process and determines its *efficiency*?
- How do *global* properties of the galaxy influence star formation (a *local* process)?
- Are there different *modes* of SF?
(Starburst galaxies vs. *LSBs*, *isolated* SF vs. *clustered* SF)

→ ***What physical processes initiate and control the formation of stars?***

Gravoturbulent star formation

- New theory of star formation:

Star formation is controlled by interplay between gravity and supersonic turbulence!

- Dual role of turbulence:
  - *stability on large scales*
  - *initiating collapse on small scales*

(full detail in Mac Low & Klessen, 2004, Rev. Mod. Phys., 76, 125-194)

Gravoturbulent star formation

- New theory of star formation:

Star formation is controlled
by interplay between
gravity and
supersonic turbulence!

- Validity:

This hold on *all* scales and applies to build-up of stars and star clusters within molecular clouds as well as to the formation of molecular clouds in galactic disk.

(full detail in Mac Low & Klessen, 2004, Rev. Mod. Phys., 76, 125-194)

Gravoturbulent Star Formation

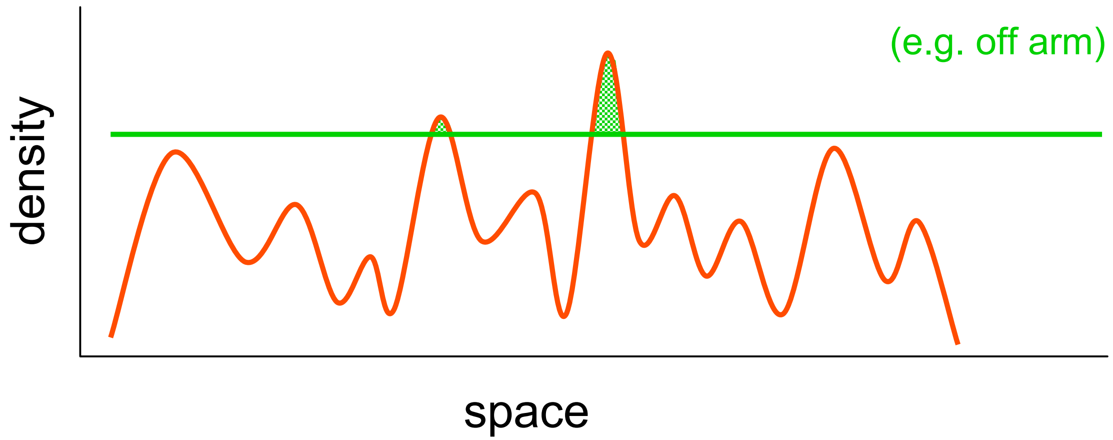
- *Supersonic turbulence* in the galactic disk creates strong density fluctuations (in shocks: $\delta\rho/\rho \approx M^2$)
  - chemical phase transition: atomic → molecular
  - cooling instability
  - gravitational instability
- Cold *molecular clouds* form at the high-density peaks.
- *Turbulence* creates density structure, *gravity* selects for collapse

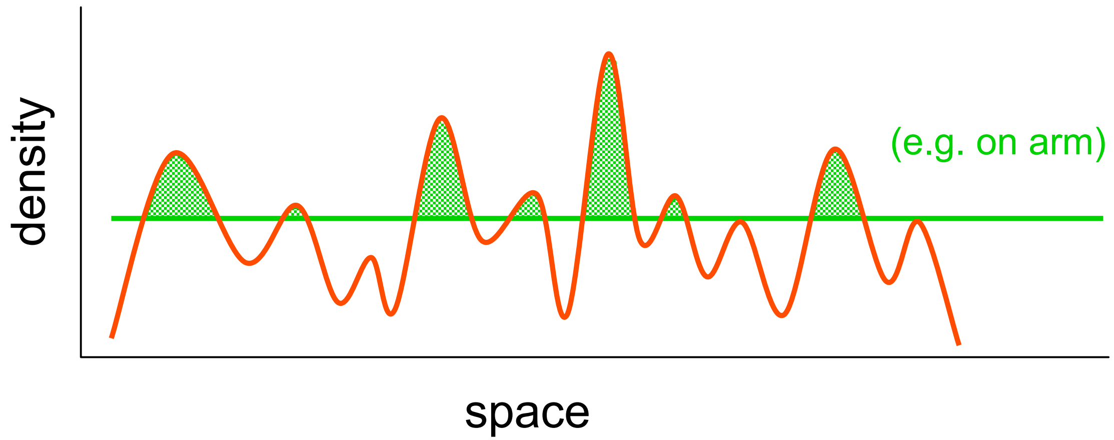
  ⟶ ***GRAVOTUBULENT FRAGMENTATION***

- *Turbulent cascade:* Local compression *within* a cloud provokes collapse → individual *stars* and *star clusters*

(full detail in Mac Low & Klessen, 2004, Rev. Mod. Phys., 76, 125-194)

Star formation on *global* scales





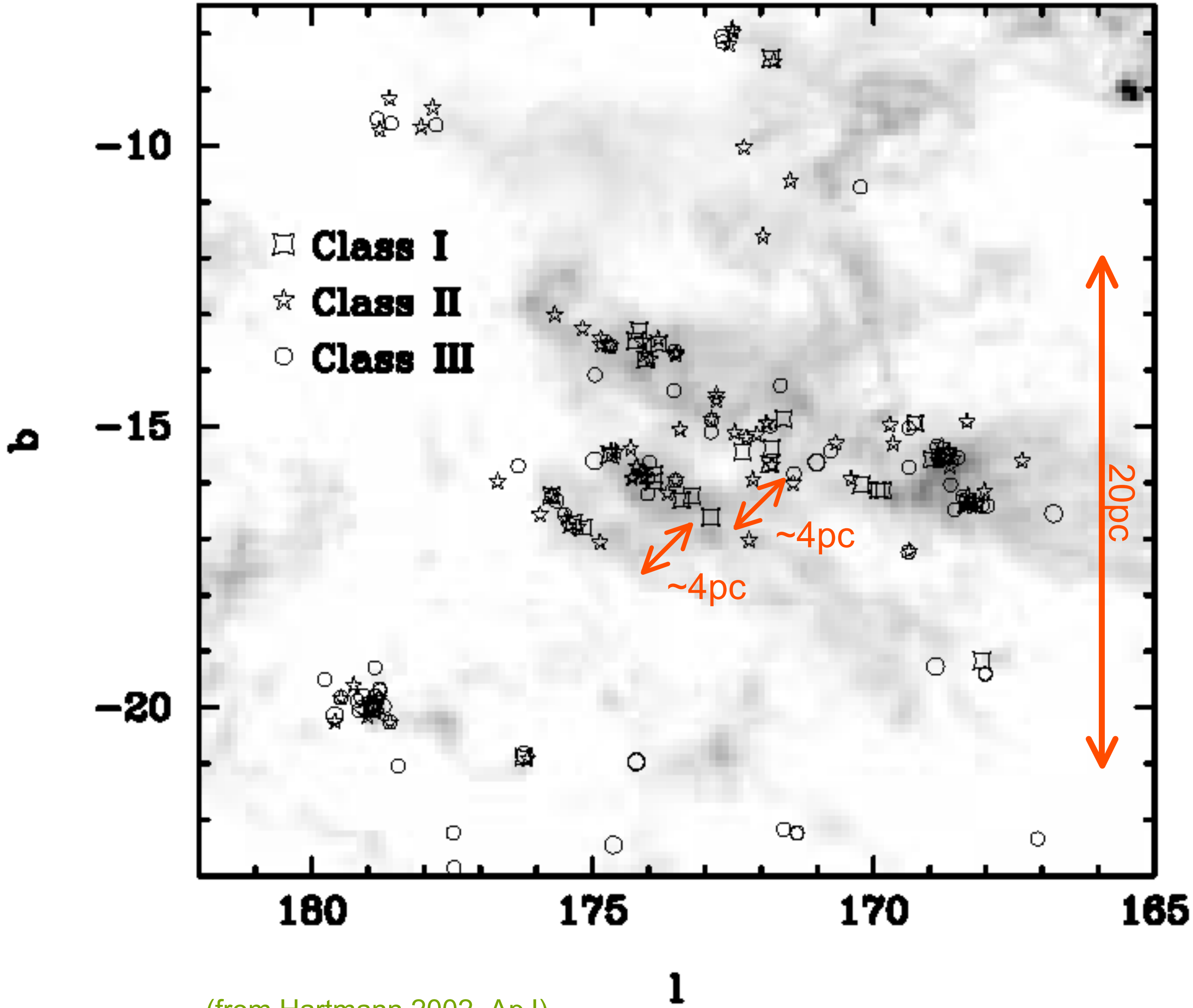
density fluctuations in warm atomar ISM caused by supersonic turbulence

some are dense enough to form H2 within "reasonable timescale" →molecular clouds

external perturbuations (i.e. potential changes) increase likelihood

Approaching the problem

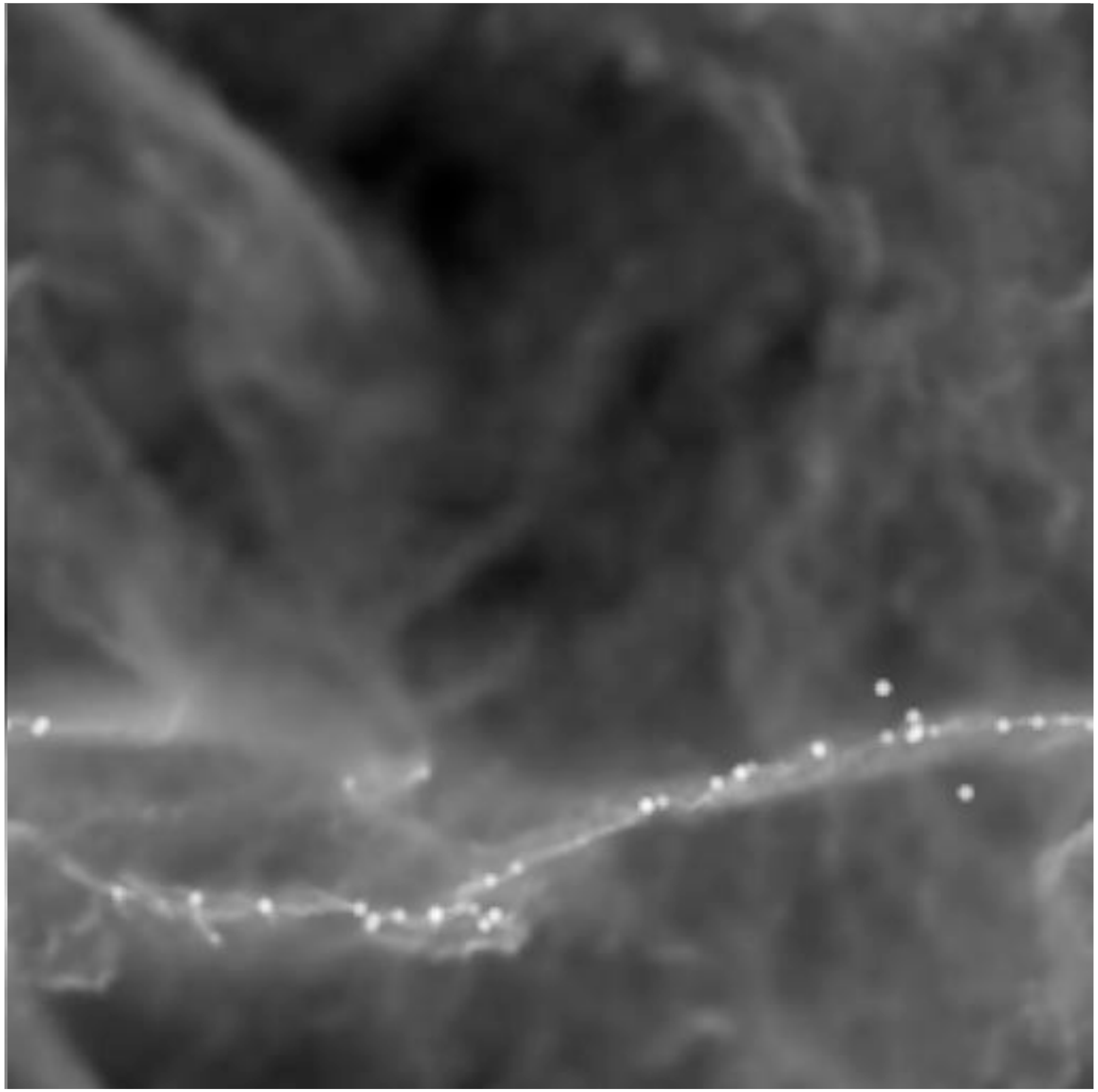
Taurus SF cloud



(from Hartmann 2002, ApJ)

Star-forming filaments in the *Taurus* molecular cloud

Gravoturbulent fragmentation



(from Ballesteros-Paredes & Klessen, in preparation)

Gravoturbulent fragmentation in molecular clouds:

- SPH model with 1.6×10^6 particles
- large-scale driven turbulence
- Mach number $\mathcal{M} = 6$
- periodic boundaries
- physical scaling:

"Taurus":

→ density $n(H_2) \approx 10^2\,cm^{-3}$:

→ $L = 6\,pc$, $M = 5000\,M_\odot$

What can we learn from that?

- *global properties* (statistical properties)
  - SF efficiency
  - SF time scale
  - IMF
  - description of self-gravitating turbulent systems (pdf's, Δ-*var.)*
  - chemical mixing properties
- *local properties* (properties of individual objects)
  - properties of individual clumps (e.g. shape, radial profile)
  - accretion history of individual protostars (dM/dt vs. t, j vs. t)
  - binary (proto)stars (eccentricity, mass ratio, etc.)
  - SED's of individual protostars
  - dynamic PMS tracks: T_{bol}-L_{bol} evolution

(details in review by Mac Low & Klessen, 2004, Rev. Mod. Phys., 76, 125-194)

Turbulent diffusion I

- Observations of young star clusters exhibit an enormous degree of chemical homogeity
(e.g. in the Pleiades: Wilden et al. 2002)
- Star-forming gas must be well mixed.
- How does this constrain models of interstellar turbulence?
- → Study mixing in supersonic compressible turbulence….

Turbulent diffusion II

- Method:
  - second central moment of displacement:

$$\xi^2_{\vec{r}}(t-t') = \left\langle [\vec{r}_i(t) - \vec{r}_i(t')]^2 \right\rangle_i$$

  - classical diffusion equation:

$$\frac{dn}{dt} = D\vec{\nabla}^2 n$$

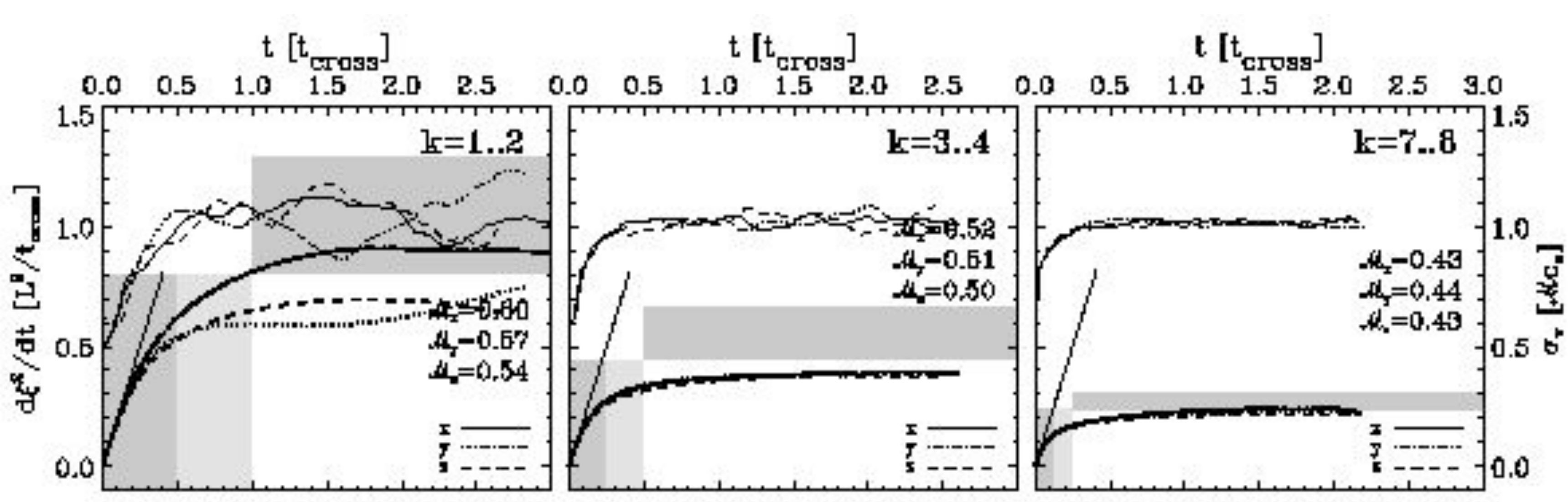
  - relation between D and ξ:

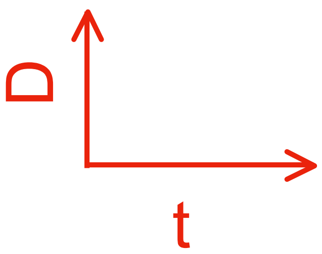
$$D(t-t') = \frac{d\xi^2_{\vec{r}}(t-t')}{dt} = 2\left\langle \vec{v}_i(t-t') \cdot \vec{r}_i(t-t') \right\rangle_i$$

Turbulent diffusion III

- Time evolution of diffusion coefficient

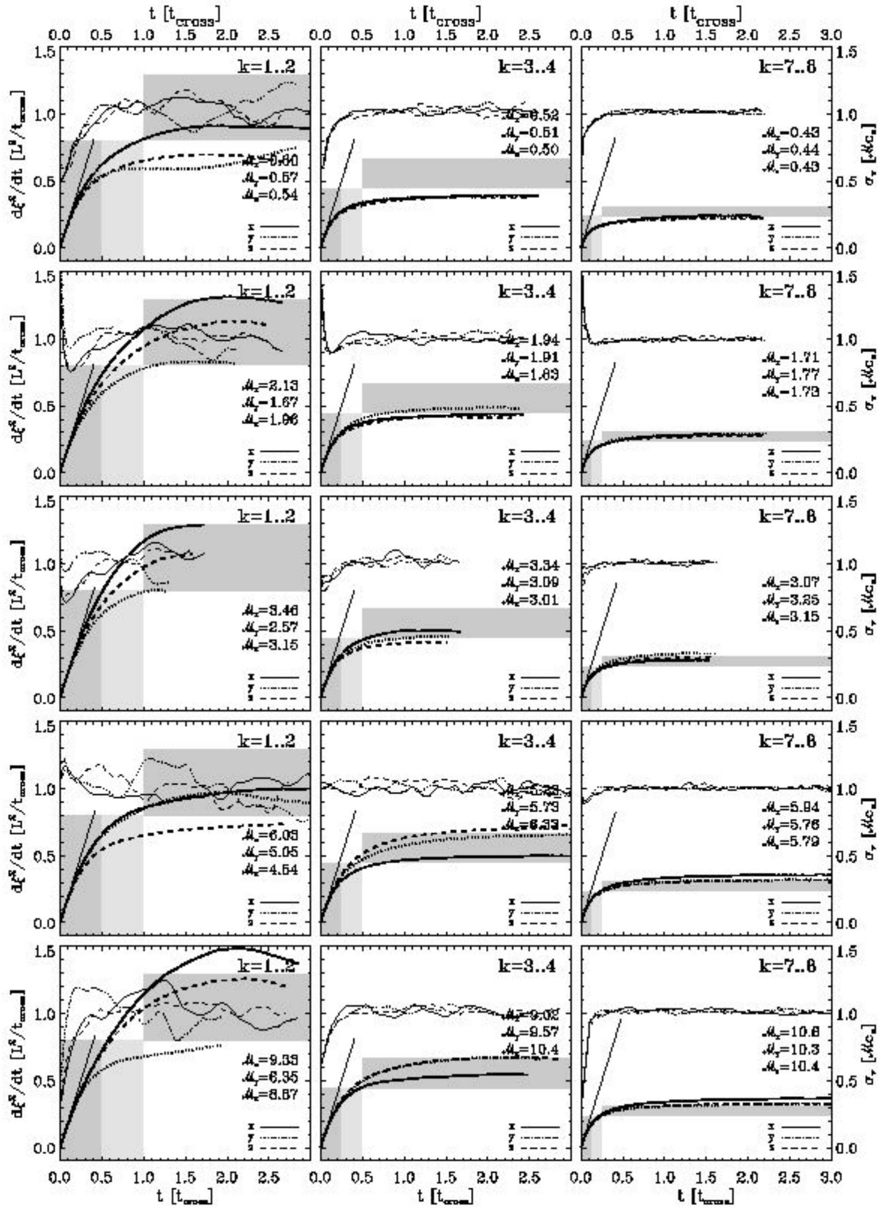
  (mean motion corrected).





Turbulent diffusion IV

- Mean-motion corrected diffusion
- Simple mixing-length approach works!
  - $D(t) \approx v_{rms}^2\, t \qquad t<\tau$
  - $D(t) \approx v_{rms}^2\, \tau$
    $= v_{rms}\, \ell \qquad t>\tau$
  - With v_{rms} = rms velocity and $\ell = L/k$ = shock sep.



(from Klessen & Lin 2003, PRE, **67**, 046311)

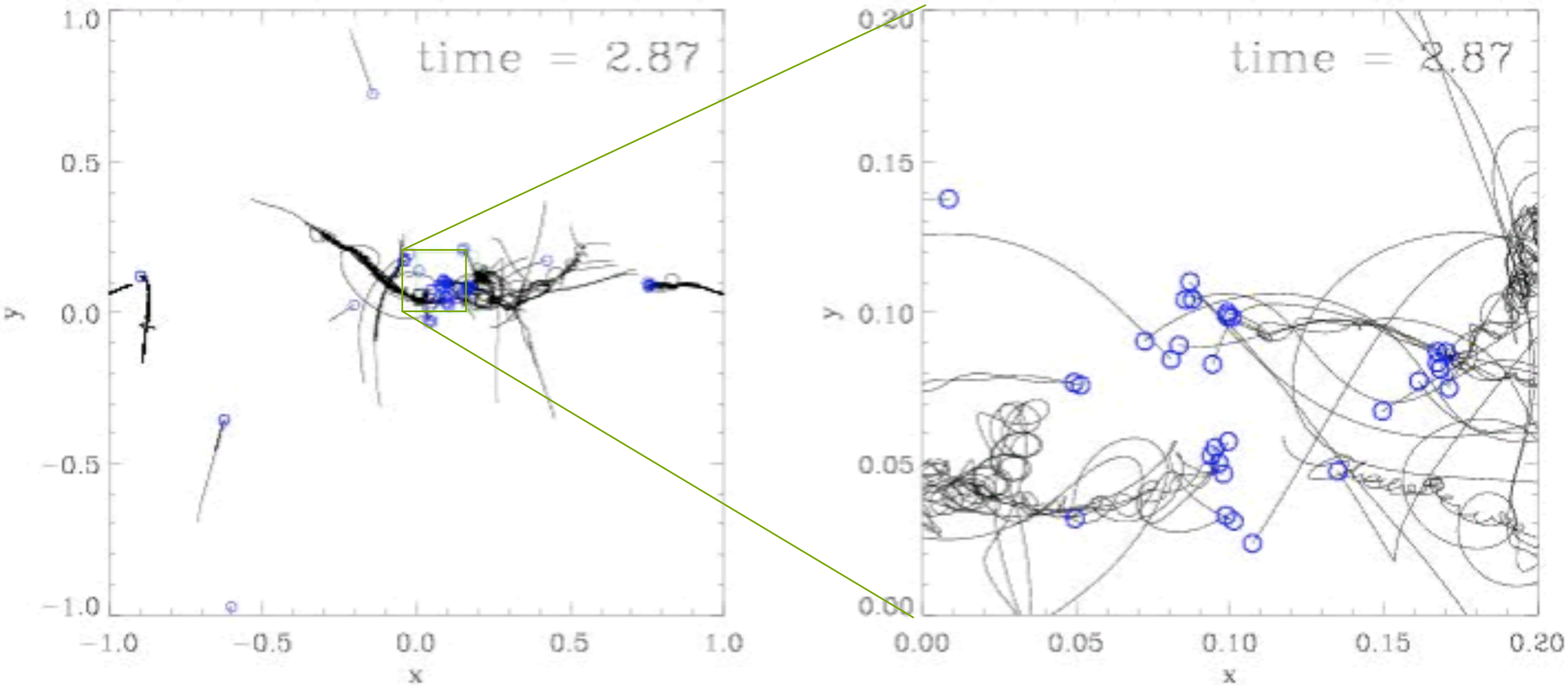
(details in review by Mac Low & Klessen, 2004, Rev. Mod. Phys., 76, 125-194)

What can we learn from that?

- *global properties* (statistical properties)
  - SF efficiency and timescale
  - stellar mass function -- IMF
  - dynamics of young star clusters
  - description of self-gravitating turbulent systems (pdf's, Δ-*var.)*
  - chemical mixing properties
- *local properties* (properties of individual objects)
  - properties of individual clumps (e.g. shape, radial profile)
  - accretion history of individual protostars (dM/dt vs. t, j vs. t)
  - binary (proto)stars (eccentricity, mass ratio, etc.)
  - SED's of individual protostars
  - dynamic PMS tracks: T_{bol}-L_{bol} evolution

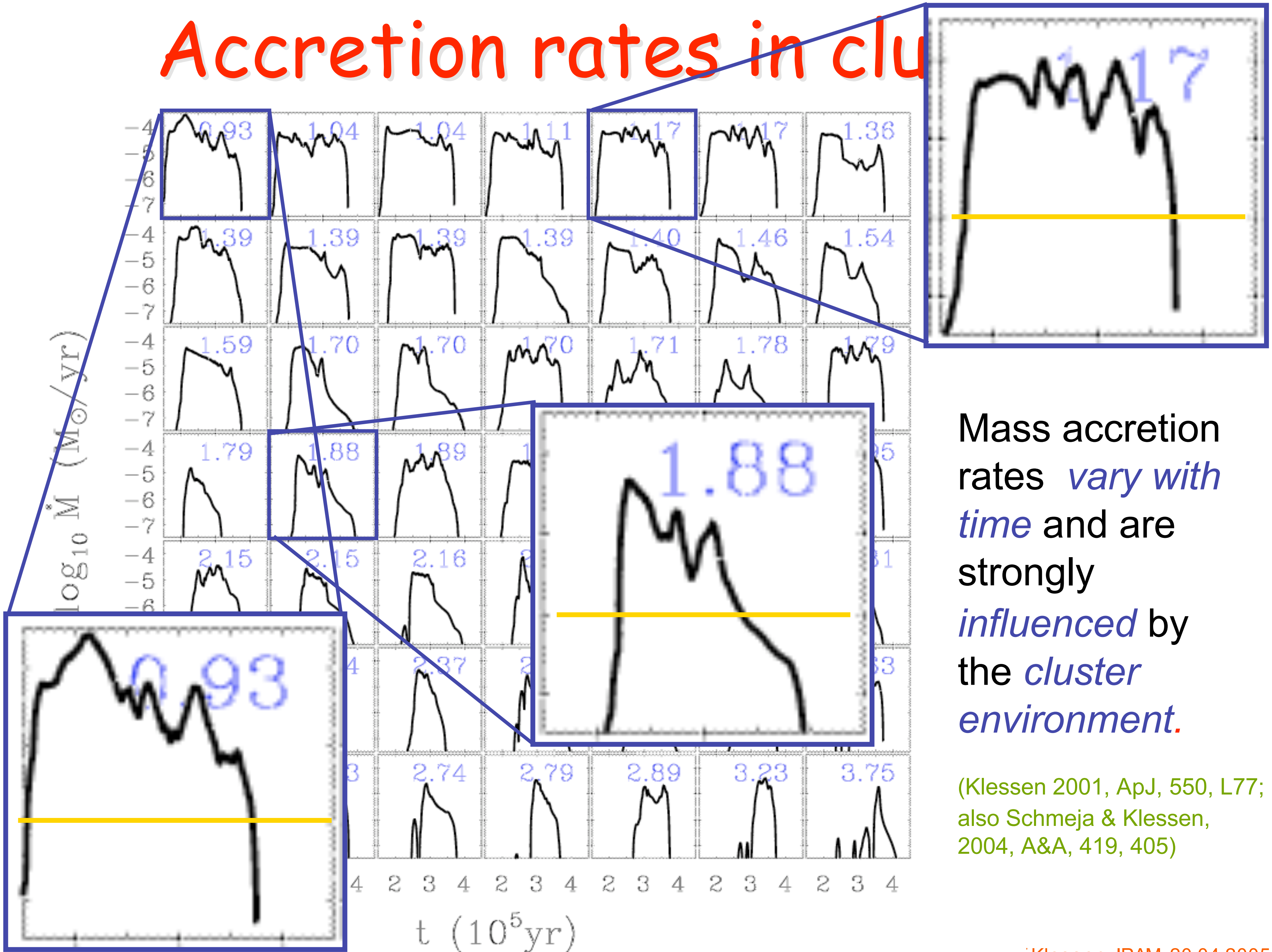
Star cluster formation

Most stars form in clusters → *star formation = cluster formation*



Trajectories of protostars in a nascent dense cluster created by gravoturbulent fragmentation
(from Klessen & Burkert 2000, ApJS, 128, 287)

Accretion rates in clu

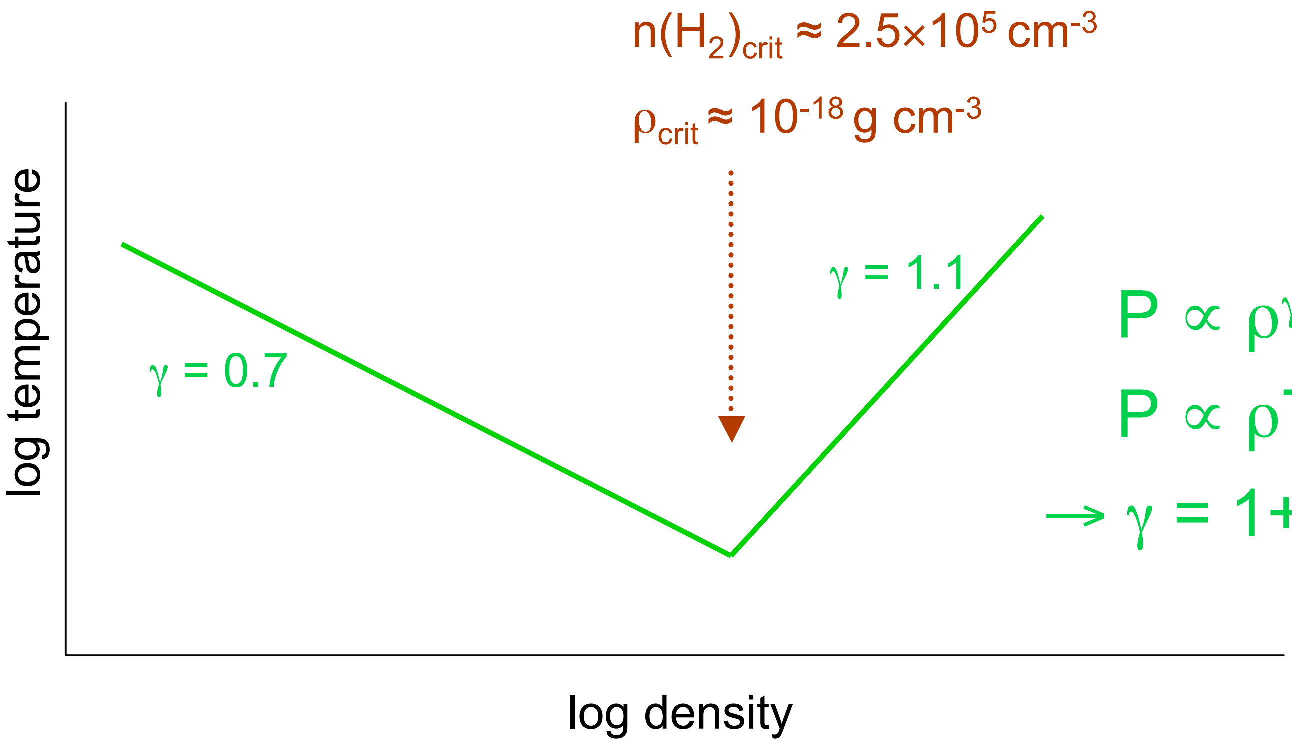


Mass accretion rates *vary with time* and are strongly *influenced* by the *cluster environment*.

(Klessen 2001, ApJ, 550, L77; also Schmeja & Klessen, 2004, A&A, 419, 405)

Influence of EOS

- But EOS depends on *chemical state*, on *balance* between *heating* and *cooling*



$P \propto \rho^\gamma$

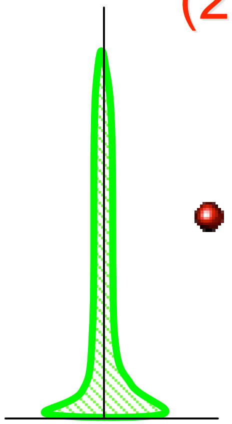
$P \propto \rho T$

$\rightarrow \gamma = 1 + d\log T / d\log\rho$

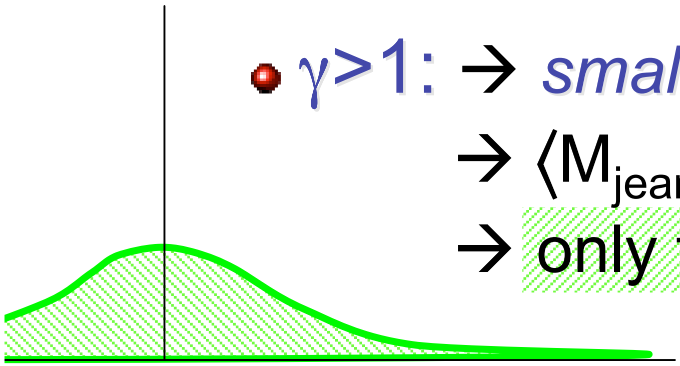
Influence of EOS

(1) $p \propto \rho^{\gamma} \rightarrow \rho \propto p^{1/\gamma}$

(2) $M_{jeans} \propto \gamma^{3/2} \rho^{(3\gamma-4)/2}$

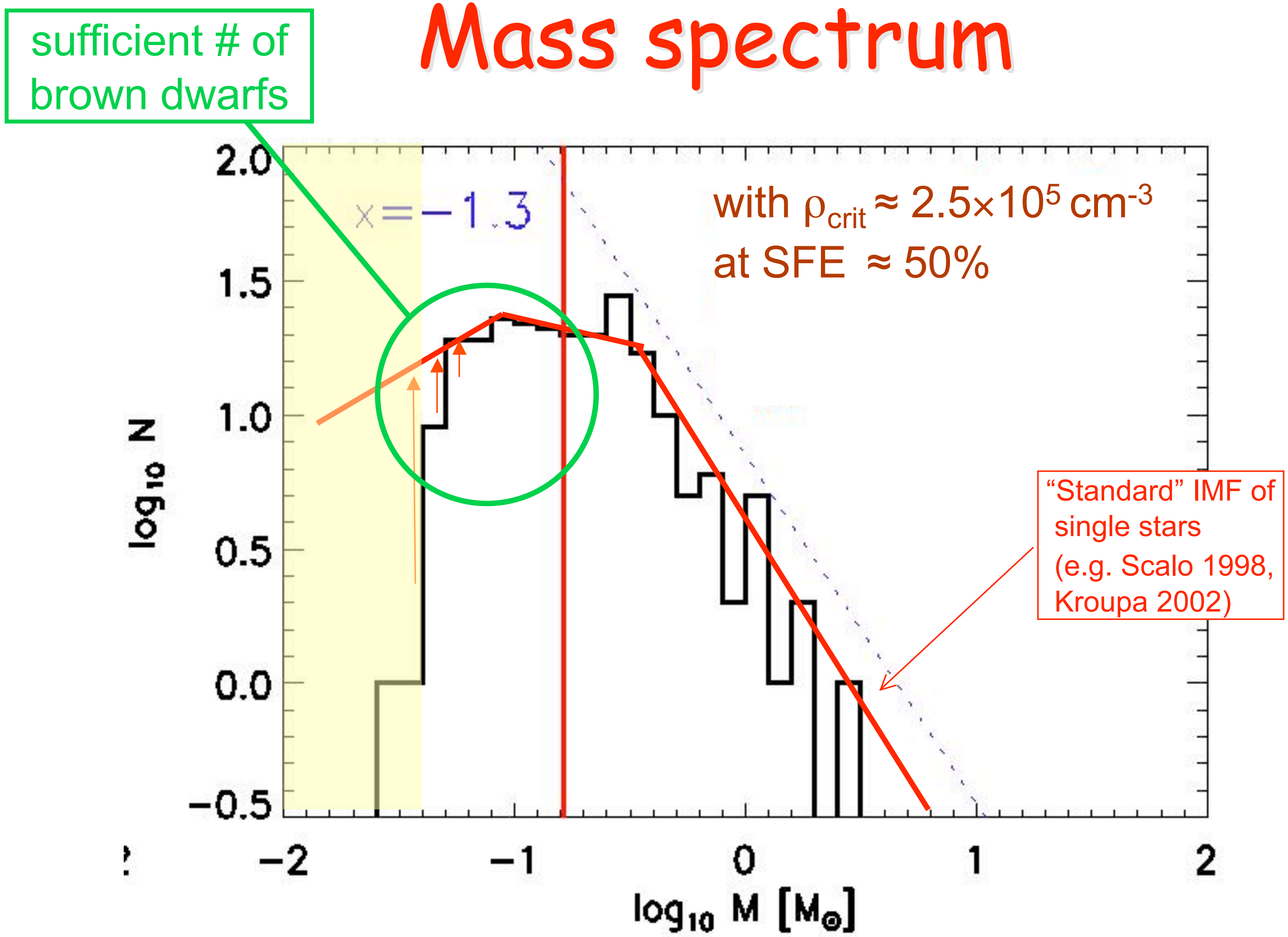


- $\gamma<1$: → *large* density excursion for given pressure
  → $\langle M_{jeans} \rangle$ becomes small
  → number of fluctuations with $M > M_{jeans}$ is large



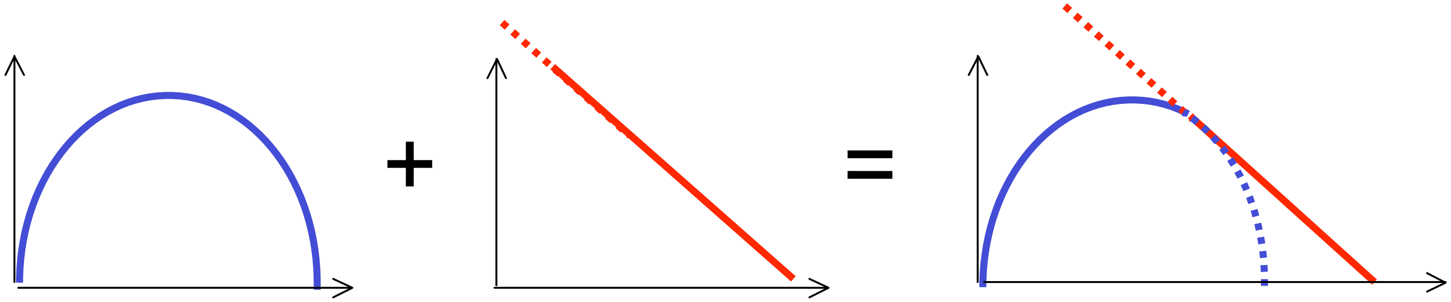
- $\gamma>1$: → *small* density excursion for given pressure
  → $\langle M_{jeans} \rangle$ is large
  → only few and massive clumps exceed M_{jeans}

Mass spectrum



(Jappsen, Klessen, Larson, Li, Mac Low, 2004, A&A submitted)

Plausibility argument for shape



- Supersonic turbulence is scale free process

  → *POWER LAW BEHAVIOR*

- *But also:* turbulence and fragmentation are highly stochastic processes → central limit theorem

  → *GAUSSIAN DISTRIBUTION*

IMF: Summary

- To get the stellar mass function (IMF) we need to:
  - describe supersonic turbulence (LES)
  - include self-gravity
  - model thermodynamic balance of the gas (heating, cooling, time-dependent chemistry, EOS)
  - follow formation of compact collapsed cores (transition from hydro to stellar dynamcis)
  - treat stellar dynamical processes (protostellar collisions, ejection by close encounters)

HYDRODYNAMICS

STELLAR DYNAMICS

NUMERICS

Goal

- We want to understand the formation of star clusters in turbulent interstellar gas clouds.

--> We want to describe the transition from a hydro-dynamic system (the self-gravitating gas cloud) to one that is dominated by (collisional) stellar dynamics (the final star cluster).

- How can we do that?

Numerical approach I

- Problem of star formation is very complex. It involves many scales (10^7 in length, and 10^{20} in density) and many physical processes → *NO* analytic solution
  → NUMERICAL APPROACH

- BUT, we need to…
  - solve the MHD equations in 3 dimensions
  - solve Poisson's equation (self-gravity)
  - follow the full turbulent cascade (in the ISM + in stellar interior)
  - include heating and cooling processes (EOS)
  - treat radiation transfer
  - describe energy production by nuclear burning processes

IMPOSSIBLE

Numerical approach II

- Simplify!
  Divide problem into little bits and pieces….
- **GRAVOTURBULENT CLOUD FRAGMENTATION**
- We try to…
  - solve the HD equations in 3 dimensions
  - solve Poisson's equation (self-gravity)
  - include a (humble) approach to supersonic turbulence
  - describe perfect gas (with polytropic EOS)
  - follow collapse: include "sink particles"
    (this will "handle" our subgrid-scale physics)

Intermezzo: HD & SPH

- derivation of equations of hydrodynamics
  - Boltzmann equation for 1D distribution function:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \vec{\nabla}_{\mathrm{q}} f_{\vec{\mathrm{p}}} + \dot{\vec{p}} \cdot \vec{\nabla}_{\mathrm{p}} f$$

  - moments of distribution function:
    density ρ, momentum $\vec{\mathrm{p}}$, energy ε
- SPH: smoothed particle hydrodynamics
  - particle-based scheme to solve eqn.'s of hydrodynamics
  - thermodynamic behavior --> equation of state (EOS)

Hydrodynamics

- gases and fluids are *large* ensembles of interacting particles
- $\longrightarrow$ state of system is described by location in $6N$ dimensional phase space $f^{(N)}(\vec{q}_1...\vec{q}_N, \vec{p}_1...\vec{p}_N) d\vec{q}_1...d\vec{q}_N d\vec{p}_1...d\vec{p}_N$
- time evolution governed by 'equation of motion' for $6N$-dim probability distribution function $f^{(N)}$
- $f^{(N)} \rightarrow f^{(n)}$ by integrating over all but n coordinates $\longrightarrow$ BBGKY hierarchy of equations of motion (after Born, Bogoliubov, Green, Kirkwood and Yvon)
- physical observables are typically associated with 1- or 2-body probability density $f^{(1)}$ or $f^{(2)}$
- at lowest level of hierarchy: 1-body distribution function describes the probability of finding a particle at time t in the volume element $d\vec{q}$ at $\vec{q}$ with momenta in the range $d\vec{p}$ at $\vec{p}$.
- **Boltzmann equation** – equation of motion for $f^{(1)}$

$$
\begin{aligned}
\frac{df}{dt} &\equiv \frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \vec{\nabla}_{\mathrm{q}} f + \dot{\vec{p}} \cdot \vec{\nabla}_{\mathrm{p}} f \\
&= \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\mathrm{q}} f + \vec{F} \cdot \vec{\nabla}_{\mathrm{p}} f = f_{\mathbf{c}}
\end{aligned}
$$

- **Boltzmann equation**

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \vec{\nabla}_q f + \dot{\vec{p}} \cdot \vec{\nabla}_p f$$
$$= \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_q f + \vec{F} \cdot \vec{\nabla}_p f = f_c .$$

→ first line: transformation from comoving to spatially fixed coordinate system.

→ second line: velocity $\vec{v} = \dot{\vec{q}}$ and force $\vec{F} = \dot{\vec{p}}$

→ all higher order terms are 'hidden' in the collision term f_c

- observable quantities are typically (velocity) moments of the Boltzmann equation, e.g.

→ density:

$$\rho = \int m\, f(\vec{q}, \vec{p}, t) d\vec{p}$$

→ momentum:

$$\rho\vec{v} = \int m\vec{v}\, f(\vec{q}, \vec{p}, t) d\vec{p}$$

→ kinetic energy density:

$$\rho\vec{v}^{\,2} = \int m\vec{v}^{\,2}\, f(\vec{q}, \vec{p}, t) d\vec{p}$$

- in general: the i-th velocity moment $\langle \xi_i \rangle$ (of $\xi_i = m\vec{v}^{\,i}$) is

$$\langle \xi_i \rangle = \frac{1}{n} \int \xi_i \, f(\vec{q}, \vec{p}, t) d\vec{p}$$

  with the mean particle number density n defined as

$$n = \int f(\vec{q}, \vec{p}, t) \, d\vec{p}$$

- the equation of motion for $\langle \xi_i \rangle$ is

$$\int \xi_i \left\{ \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\mathrm{q}} f + \vec{F} \cdot \vec{\nabla}_{\mathrm{p}} f \right\} d\vec{p} = \int \xi_i \left\{ f_{\mathbf{c}} \right\} \, d\vec{p} \,,$$

  which after some complicated rearrangement becomes

$$\frac{\partial}{\partial t} n \langle \xi_i \rangle + \vec{\nabla}_{\mathrm{q}} \left(n \langle \xi_i \vec{v} \rangle \right) + n\vec{F} \langle \vec{\nabla}_{\mathrm{p}} \xi_i \rangle = \int \xi_i f_{\mathbf{c}} \, d\vec{p}$$

  (Maxwell-Boltzmann transport equation for $\langle \xi_i \rangle$)

- if the RHS is zero, then ξ_i is a conserved quantity. This is only the case for first three moments, mass $\xi_0 = m$, momentum $\vec{\xi}_1 = m\vec{v}$, and kinetic energy $\xi_2 = m\vec{v}^{\,2}/2$.
- MB equations build a hierarically nested set of equations, as $\langle\xi_i\rangle$ depends on $\langle\xi_{i+1}\rangle$ via $\vec{\nabla}_{\mathrm{q}}\left(n\langle\xi_i\vec{v}\rangle\right)$ and because the collision term cannot be reduced to depend on ξ_i only.

  $\longrightarrow$ need for a closure equation

  $\longrightarrow$ in hydrodynamics this is typically the equation of state.

assumptions

- **continuum limit:**
  - $\rightarrow$ distribution function f must be a 'smoothly' varying function on the scales of interest $\longrightarrow$ local average possible
  - $\rightarrow$ stated differently: the averaging scale (i.e. scale of interest) must be larger than the mean free path of individual particles
  - $\rightarrow$ stated differently: microscopic behavior of particles can be neglected
  - $\rightarrow$ concept of fluid element must be meaningful
- **only 'short range forces':**
  - $\rightarrow$ forces between particles are short range or saturate $\longrightarrow$ collective effects can be neglected
  - $\rightarrow$ stated differently: correlation length of particles in the system is finite (and smaller than the scales of interest)

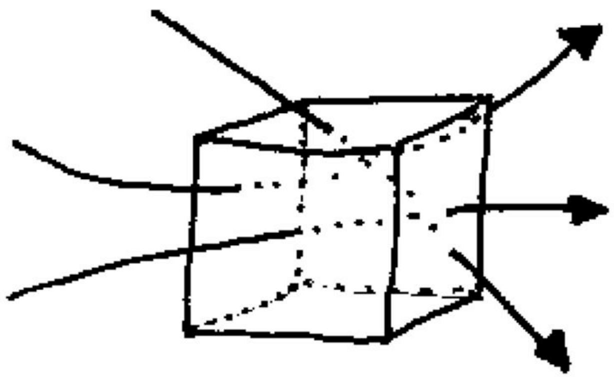
limitations

- shocks (scales of interest become smaller than mean free path)
- phase transitions (correlation length may become infinite)
- description of self-gravitating systems
- description of fully fractal systems

the equations of hydrodynamics

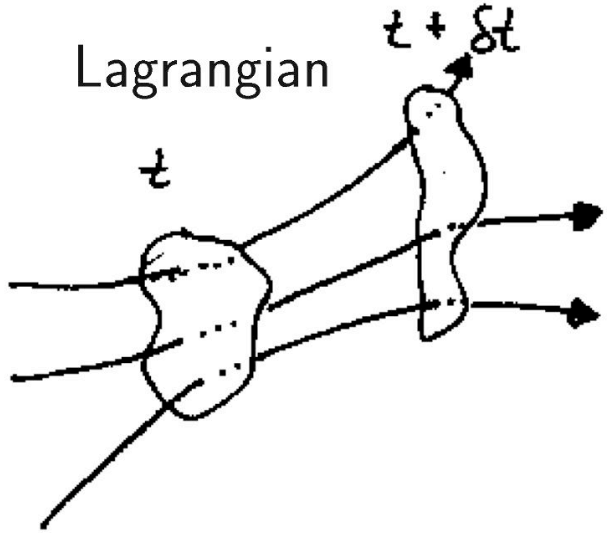
- hydrodynamics $\equiv$ book keeping problem
  One must keep track of the 'change' of a fluid element due to various physical processes acting on it. How do its 'properties' evolve under the influence of compression, heat sources, cooling, etc.?
- Eulerian vs. Lagrangian point of view

Eulerian



consider spatially fixed volume element

Lagrangian



following motion of fluid element

- hydrodynamic equations = set of equations for the five conserved quantities $(\rho, \rho\vec{v}, \rho\vec{v}^{\,2}/2)$ plus closure equation (plus transport equations for 'external' forces if present, e.g. gravity, magnetic field, heat sources, etc.)

○ equations of hydrodynamics

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}\rho = -\rho \vec{\nabla} \cdot \vec{v}$$ (continuity equation)

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} p - \vec{\nabla}\phi + \eta \vec{\nabla}^2 \vec{v} + \left(\zeta + \frac{\eta}{3}\right) \vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$

(Navier-Stokes equation)

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \vec{\nabla}\epsilon = T\frac{ds}{dt} - \frac{p}{\rho}\vec{\nabla} \cdot \vec{v}$$ (energy equation)

$$\vec{\nabla}^2 \phi = 4\pi G \rho$$ (Poisson's equation)

$$p = \mathcal{R} \rho T$$ (equation of state)

$$\vec{F}_B = -\vec{\nabla}\,\frac{\vec{B}^2}{8\pi} + \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B} \quad \text{(magnetic force)}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) \quad \text{(Lorentz equation)}$$

ρ = density, $\vec{v}$ = velocity, p = pressure, ϕ = gravitational potential, ζ and η viscosity coefficients, $\epsilon = \rho\vec{v}^{\,2}/2$ = kinetic energy density, T = temperature, s = entropy, $\mathcal{R}$ = gas constant, $\vec{B}$ = magnetic field (cgs units)

- mass transport – continuity equation

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}\rho = -\rho \vec{\nabla} \cdot \vec{v}$$

(conservation of mass)

- transport equation for momentum – Navier Stokes equation

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\phi + \eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

momentum change due to

→ pressure gradients: $(-\rho^{-1}\,\vec{\nabla}p)$

→ (self) gravity: $-\vec{\nabla}\phi$

→ viscous forces (internal friction, contains $\mathrm{div}(\partial v_i/\partial x_j)$ terms): $\eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$

(conservation of momentum, general form of momentum transport: $\partial_t(\rho v_i) = -\partial_j \Pi_{ij}$)

◦ transport equation for internal energy

$$\frac{d\epsilon}{dt} = \frac{\partial\epsilon}{\partial t} + \vec{v}\cdot\vec{\nabla}\epsilon = T\,\frac{ds}{dt} - \frac{p}{\rho}\,\vec{\nabla}\cdot\vec{v}$$

→ follows from the thermodynamic relation $d\epsilon = T\,ds - p\,dV = T\,ds + p/\rho^2 d\rho$ which described changes in ϵ due to entropy changed and to volume changes (compression, expansion)

→ for adiabatic gas the first term vanishes ($s =$constant)

→ heating sources, cooling processes can be incorporated in ds

(conservation of energy)

○ closure equation – equation of state

→ general form of equation of state $p = p(T, \rho, \ldots)$

→ ideal gas: $p = \mathcal{R}\rho T$

→ special case – isothermal gas: $p = c_\mathrm{s}^2 T$ (as $\mathcal{R}T = c_\mathrm{s}^2$)

Note:

- in reality, computing the EOS is VERY complex!
- depends on detailed *balance* between *heating* and *cooling*
- these depend on *chemical composition* (which atomic and molecular species, dust)
- and on the ability to radiate away „cooling lines“ and black body radiation
  --> problem of *radiation transfer* (see, e.g., IPAM III)

In general:

- the „standard way“ of solving the equations of (magneto) hydrodynamics is using finite differences on a grid
- alternative use particle-based scheme: *SPH*
- see IPAM workshop I

SPH

concept of SPH

- ‘invented’ independently by Lucy (1977) and Gingold & Monaghan (1977)
- originally proposed as Monte Carlo approach to calculate the time evolution of gaseous systems
- more intuitively understood as interpolation scheme:

> The fluid is represented by an ensemble of particles i, each carrying mass m_i, momentum $m_i\vec{v}_i$, and hydrodynamic properties (like pressure p_i, temperature T_i, internal energy ϵ_i, entropy s_i, etc.). The time evolution is governed by the equation of motion plus additional equations to modify the hydrodynamic properties of the particles. Hydrodynamic observables are obtained by a local averaging process.

properties of local averaging processes

- local averages $\langle f(\vec{r})\rangle$ for any quantity $f(\vec{r})$ can be obtained by convolution with an appropriate smoothing function $W(\vec{r},\vec{h})$:

$$\langle f(\vec{r})\rangle \equiv \int f(\vec{r}\,')W(\vec{r}-\vec{r}\,',\vec{h})\,d^3r'\,.$$

  the function $W(\vec{r},\vec{h})$ is called smoothing kernel

- the kernel must satisfy the following two conditions:

$$\int W(\vec{r},\vec{h})\,d^3r = 1 \quad \text{and} \quad \langle f(\vec{r})\rangle \longrightarrow f(\vec{r}) \ \text{ for } \ \vec{h}\to 0$$

  the kernel W therefore follows the same definitions as Dirac's delta function $\delta(\vec{r})$: $\lim_{h\to 0} W(\vec{r},h) = \delta(\vec{r})$.

- most SPH implementations use spherical kernel functions

$$W(\vec{r},\vec{h}) \equiv W(r,h) \quad \text{with} \quad r=|\vec{r}| \ \text{ and } \ h=|\vec{h}|.$$

  (one could also use triaxial kernels, e.g. Martel et al. 1995)

properties of local averaging processes

- as the kernel function W can be seen as approximation to the δ-function for small but finite h we can expand the averaged function $\langle f(\vec{r})\rangle$ into a Taylor series for h to obtain an estimate for $f(\vec{r})$; if W is an even function, the first order term vanishes and the errors are second order in h

  $$\langle f(\vec{r})\rangle = f(\vec{r}) + \mathcal{O}(h^2)$$

  this holds for functions f that are smooth and do not exhibit steep gradients over the size of W ($\rightarrow$ problems in shocks).

  (more specifically the expansion is $\langle f(\vec{r})\rangle = f(\vec{r}) + \kappa h^2 \vec{\nabla}^2 f(\vec{r}) + \mathcal{O}(h^3)$)

properties of local averaging processes

- within its intrinsic accuracy, the smoothing process therefore is a linear function with respect to summation and multiplication:

$$\langle f(\vec{r}) + g(\vec{r}) \rangle = \langle f(\vec{r}) \rangle + \langle g(\vec{r}) \rangle$$
$$\langle f(\vec{r}) \cdot g(\vec{r}) \rangle = \langle f(\vec{r}) \rangle \cdot \langle g(\vec{r}) \rangle$$

(one follows from the linearity of integration with respect to summation, and two is true to $\mathcal{O}(h^2)$)

- derivatives can be 'drawn into' the averaging process:

$$\frac{d}{dt}\langle f(\vec{r}) \rangle = \left\langle \frac{d}{dt} f(\vec{r}) \right\rangle$$
$$\vec{\nabla}\langle f(\vec{r}) \rangle = \langle \vec{\nabla} f(\vec{r}) \rangle$$

Furthermore, the spatial derivative of f can be transformed into a spatial derivative of W (no need for finite differences or grid):

$$\vec{\nabla}\,\langle f(\vec{r}) \rangle = \left\langle \vec{\nabla} f(\vec{r}) \right\rangle = \int f(\vec{r}\,')\,\vec{\nabla} W(|\vec{r} - \vec{r}\,'|, h)\, d^3r'\,.$$

(shown by integrating by parts and assuming that the surface term vanishes; if the solution space is extended far enough, either the function f itself or the kernel approach zero)

properties of local averaging processes

- basic concept of SPH is a particle representation of the fluid $\longrightarrow$ *integration* transforms into *summation* over discrete set of particles; example density ρ:

$$\langle\rho(\vec{r}_i)\rangle = \sum_j m_j W(|\vec{r}_i - \vec{r}_j|, h) \ .$$

in this picture, the mass of each particle is smeared out over its kernel region; the density at each location is obtained by summing over the contributions of the various particles $\longrightarrow$ *smoothed particle hydrodynamics!*

the kernel function

- different functions meet the requirement $\int W(|\vec{r}|, h)\, d^3r \;=\; 1$ and $\lim_{h\to 0} \int W(|\vec{r} - \vec{r}\,'|, h) f(\vec{r}\,')\, d^3r' = f(\vec{r})$:

  → Gaussian kernel:

$$W(r,h) = \frac{1}{\pi^{3/2}h^3} \exp\left(-\frac{r^2}{h^2}\right)$$

  · *pro:* mathematically sound

  · *pro:* derivatives exist to all orders and are smooth

  · *contra:* all particles contribute to a location

  → spline functions with compact support

the kernel function

- different functions meet the requirement $\int W(|\vec{r}|, h)\, d^3r = 1$ and $\lim_{h\to 0} \int W(|\vec{r} - \vec{r}\,'|, h) f(\vec{r}\,')\, d^3r' = f(\vec{r})$:

  $\rightarrow$ the standard kernel: cubic spline
  with $\xi = r/h$ it is defined as

$$W(r,h) \equiv \frac{1}{\pi h^3} \begin{cases} 1 - \frac{3}{2}\xi^2 + \frac{3}{4}\xi^3, & \text{for } \; 0 \le \xi \le 1; \\ \frac{1}{4}(2-\xi)^3, & \text{for } 1 \le \xi \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

  · *pro:* compact support $\longrightarrow$ all interactions are zero for $r > 2h$ $\longrightarrow$ number of particles involved in the average remains small (typically between 30 and 80)

  · *pro:* second derivative is continuous

  · *pro:* dominant error term is second order in h

the fluid equations in SPH

- there is an infinite number of possible SPH implementations of the hydrodynamic equations!
- some notation: $h_{ij} = (h_i + h_j)/2$, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, and $\vec{\nabla}_i$ is the gradient with respect to the coordinates of particle i; all measurements are taken at particle positions (e.g. $\rho_i = \rho(\vec{r}_i)$)
- *general form of SPH equations:*

$$\langle f_i \rangle = \sum_{j=1}^{N_i} \frac{m_j}{\rho_j} f_j W(r_{ij}, h_{ij})$$

the fluid equations in SPH

- *density* — continuity equation (conservation of mass)

$$\rho_i = \sum_{j=1}^{N_i} m_j W(r_{ij}, h_{ij})$$

or

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \sum_{j=1}^{N_i} m_j \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}, h_{ij})$$

(the second implementation is almost never used, see however Monaghan 1991 for an application to water waves)

important

density is needed for *ALL* particles *BEFORE* computing other averaged quantities $\longrightarrow$ at each timestep, SPH computations consist of *TWO* loops, first the *density* is obtained for each particle, and then in a second round, all *other* particle properties are updated.

the fluid equations in SPH

- *pressure* is defined via the equation of state (for example for isothermal gas $p_i = c_{\rm s}^2 \rho_i$)

the fluid equations in SPH

- *velocity* — Navier Stokes equation (conservation of momentum)

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = \sum_i \vec{F}_i = \vec{F}_{\text{pressure}} + \vec{F}_{\text{viscosity}} + \vec{F}_{\text{gravity}}$$

rate of change of momentum of fluid element depends on sum of all forces acting on it.

the fluid equations in SPH

- *velocity* — Navier Stokes equation (conservation of momentum)

  → consider for now *only* pressure contributions: Euler's equation

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} p = -\vec{\nabla}\left(\frac{p}{\rho}\right) - \frac{p}{\rho^2}\vec{\nabla}\rho \qquad (*)$$

  here, the identity $\vec{\nabla}(p\rho^{-1}) = \rho^{-1}\vec{\nabla} p - p\rho^{-2}\vec{\nabla}\rho$ is used

  → in the SPH formalism this reads as

$$\frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right) \vec{\nabla}_i W(r_{ij}, h_{ij})$$

  where the first term in (*) is neglected because it leads to surface terms in the averaging procedure; it is assumed that either the pressure or the kernel becomes zero at the integration border; if this is not the case *correction terms* need to be added above.

the fluid equations in SPH

- *velocity* — Navier Stokes equation (conservation of momentum)
  - → the SPH implementation of the standard artificial viscosity is

$$\vec{F}_i^{\rm visc} = -\sum_{j=1}^{N_i} m_j \Pi_{ij} \vec{\nabla}_i W(r_{ij}, h_{ij}) \,,$$

where the viscosity tensor Π_{ij} is defined by

$$\Pi_{ij} = \begin{cases} (-\alpha c_{ij}\mu_{ij} + \beta\mu_{ij}^2)/\rho_{ij} & \text{for} \quad \vec{v}_{ij}\cdot\vec{r}_{ij} \le 0\,, \\ 0 & \text{for} \quad \vec{v}_{ij}\cdot\vec{r}_{ij} > 0\,, \end{cases}$$

where

$$\mu_{ij} = \frac{h\vec{v}_{ij}\cdot\vec{r}_{ij}}{\vec{r}_{ij}^{\,2} + 0.01h^2} \,.$$

with $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, mean density $\rho_{ij} = (\rho_i + \rho_j)/2$, and mean sound speed $c_{ij} = (c_i + c_j)/2$.

the fluid equations in SPH

- *velocity* — Navier Stokes equation (conservation of momentum)
  - → if self-gravity is taken into account, the gravitational force needs to be added on the RHS

$$\vec{F}_{\mathrm{G}} = -\vec{\nabla}\phi_i = -G\sum_{j=1}^{N}\frac{m_j}{r_{ij}^2}\frac{r_{ij}}{r_{ij}}$$

    note that the sum needs to be taken over *ALL* particles ⟵ computationally expensive

  - → set together, the momentum equation is

$$\frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij}\right)\vec{\nabla}_i W(r_{ij}, h_{ij}) - \nabla\phi_i$$

the fluid equations in SPH

- *energy equation* (conservation of momentum)
  - → recall the hydrodynamic energy equation:

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \vec{\nabla} \epsilon = \frac{ds}{dt} - \frac{p}{\rho} \vec{\nabla} \cdot \vec{v}$$

  - → for *adiabatic* systems ($c = \mathrm{const}$) the SPH form follows as

$$\frac{d\epsilon_i}{dt} = \frac{p_i}{\rho_i^2} \sum_{j=1}^{N_i} m_j \, \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}, h_{ij}) \,,$$

(note that the alternative form

$$\frac{d\epsilon_i}{dt} = \frac{1}{2} \sum_{j=1}^{N_i} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}, h_{ij})$$

can lead to unphysical solutions, like negative internal energy)

the fluid equations in SPH

- *energy equation* (conservation of momentum)

  → *dissipation* due to (artificial) viscosity leads to a term

$$\frac{d\epsilon_i}{dt} = \frac{1}{2}\sum_{j=1}^{N_i} m_j \Pi_{ij}\, \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}.h_{ij})$$

  → the presence of *heating* sources or *cooling* processes can be incorporated into a function Γ_i.

  → altogether:

$$\boxed{\frac{d\epsilon_i}{dt} = \frac{p_i}{\rho_i^2}\sum_{j=1}^{N_i} m_j\, \vec{v}_{ij} \cdot \vec{\nabla}_i W_{ij} \; + \; \frac{1}{2}\sum_{j=1}^{N_i} m_j \Pi_{ij}\, \vec{v}_{ij} \cdot \vec{\nabla}_i W_{ij} \;\; + \;\; \Gamma_i}$$

  can lead to unphysical solutions, like negative internal energy)

fully conservative formulation using Lagrange multipliers

- the Lagrangian for compressible flows which are generated by the thermal energy $\epsilon(\rho, s)$ acts as effective potential is

$$\mathcal{L} = \int \rho \left\{ \frac{1}{2} v^2 - u(\rho, s) \right\} d^3r.$$

equations of motion follow with $s = \mathrm{const}$ from

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \vec{v}} - \frac{\partial \mathcal{L}}{\partial \vec{r}} = 0$$

- after some SPH arithmetics, one can derive the following acceleration equation for particle i

$$\frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j \left\{ \frac{1}{f_i}\frac{p_i}{\rho_i^2}\vec{\nabla}_i W(r_{ij}, h_i) + \frac{1}{f_j}\frac{p_j}{\rho_j^2}\vec{\nabla}_i W(r_{ij}, h_j) \right\}$$

where

$$f_i = \left[1 + \frac{h_i}{3\rho_i}\frac{\partial \rho_i}{\partial h_i} \right]$$

Large-eddy simulations

- We use ***LES*** to models the large-scale dynamics
- Principal problem: only large scale flow properties
  - Reynolds number: Re = LV/ν ***(Re_{nature} >> Re_{model})***
  - dynamic range much smaller than true physical one
  - need ***subgrid model*** (in our case simple: only dissipation) more complex when processes (chemical reactions, nuclear burning, etc) on subgrid scale determine large-scale dynamics
- Also: stochasticity of the flow ⇒ unpredictable when and where "interesting things" happen
  - occurance of localized collapse
  - location and strength of shock fronts
  - etc.

LES with SPH

- For self-gravitating gases SPH is probably okay …
  - fully Lagrangian (particles are free to move where needed)
  - good resolution in high-density regions (in collapse)
  - particle based --> good for transition from hydrodynamics to stellar dynamics
- BUT:
  - low resolution in low-density region
  - difficult to reach very high levels of refinement (however, particle splitting may be promising path)
  - dissipative and need for artificial viscosity
  - how to handle subgrid scales?

Gravoturbulent SF with SPH

- Comparison between particle-based and grid-based methods: SPH vs. ZEUS

    Klessen, Heitsch, Mac Low (2000)
    Heitsch, Mac Low, Klessen (2001)
    Ossenkopf, Klessen, Heitsch (2001)

- Both methods are complementary…

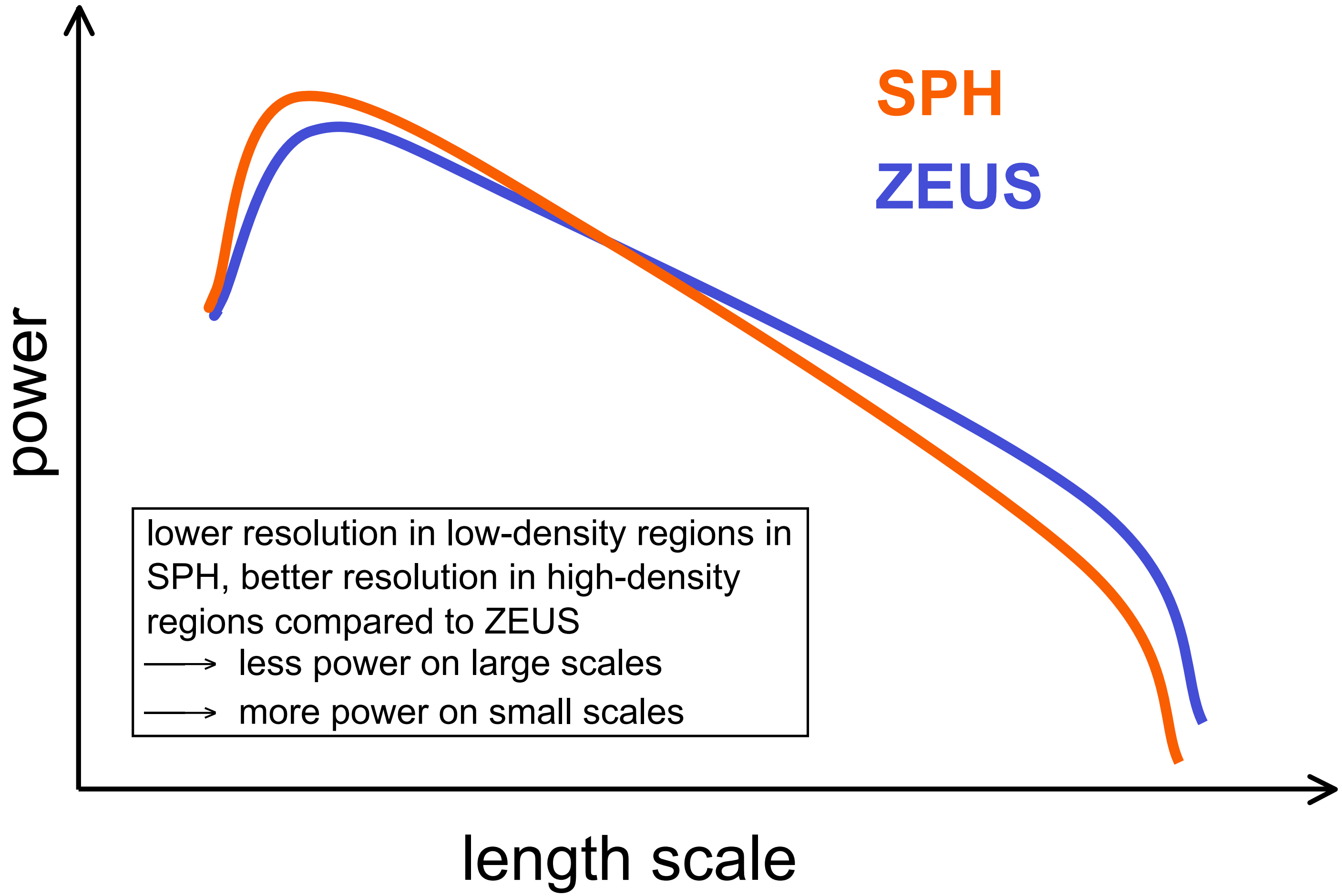
  ⟶ Bracketing reality!

- As a crude estimage:

  SPH is better in high-density regions

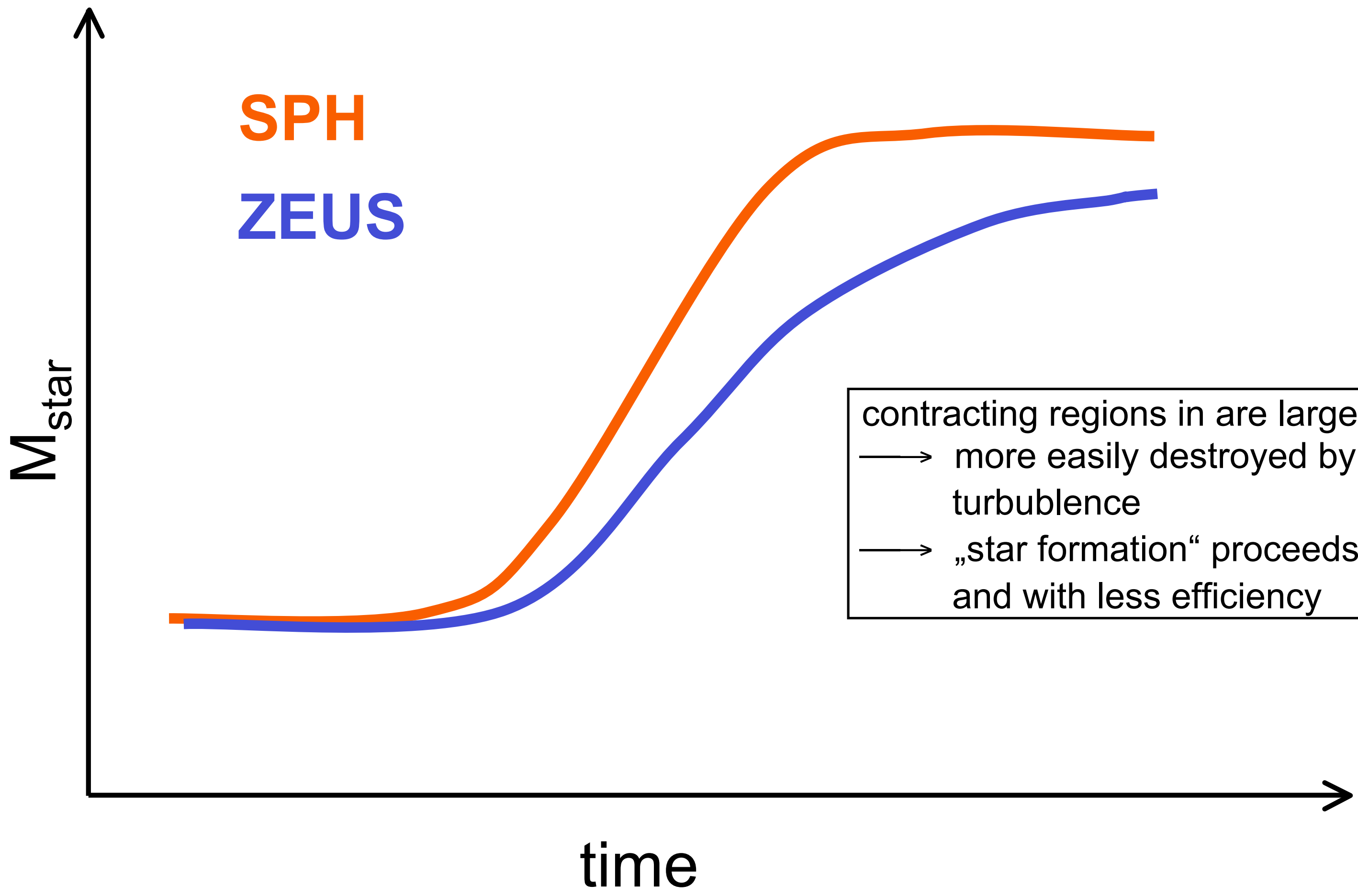
  ZEUS is better in low-density regions

SPH vs. ZEUS



(Ossenkopf, Klessen, Heitsch 2001)

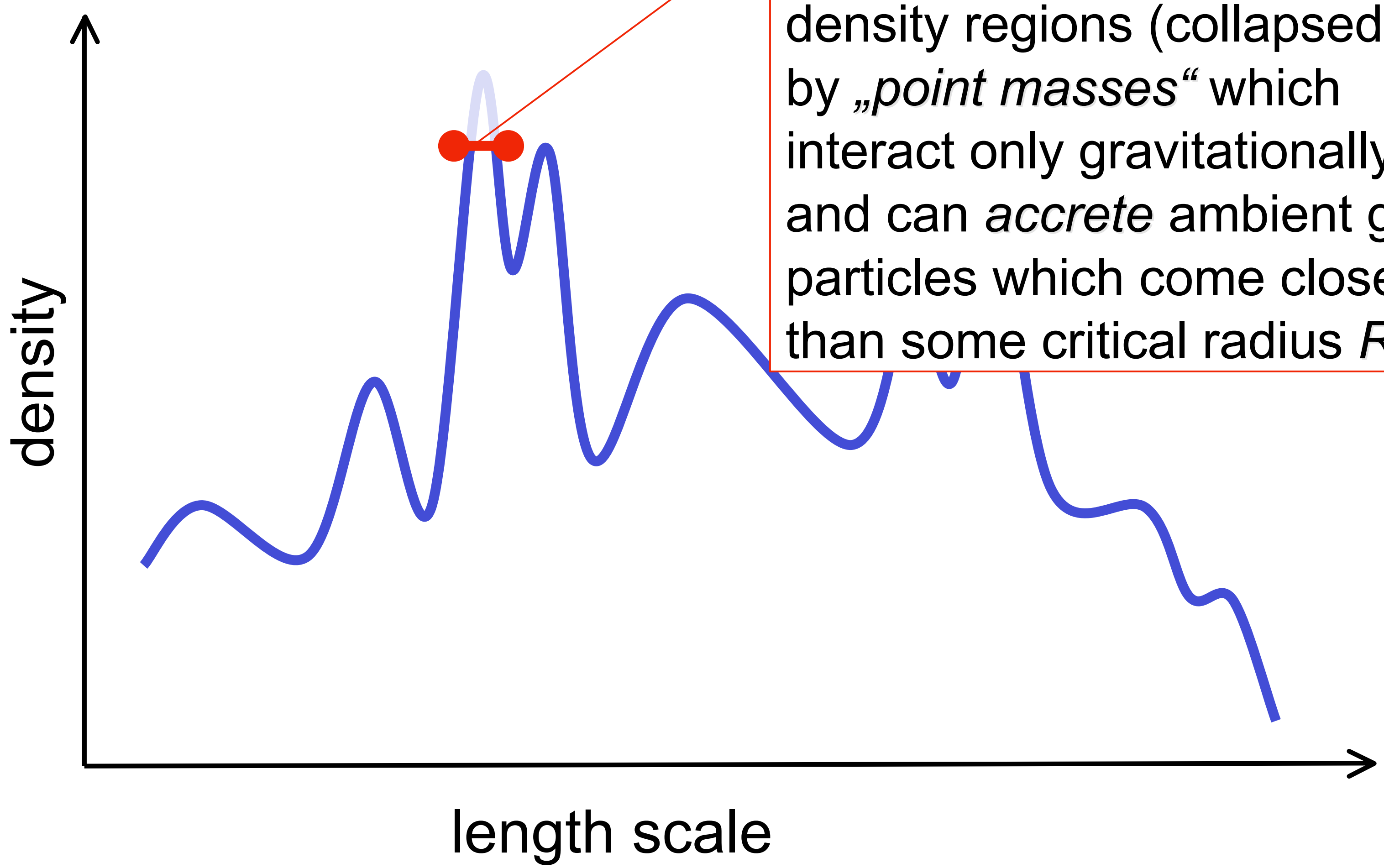
SPH vs. ZEUS



contracting regions in are larger in ZEUS
→ more easily destroyed by ambient turbublence
→ „star formation" proceeds slower and with less efficiency

(Ossenkopf, Klessen, Heitsch 2001)

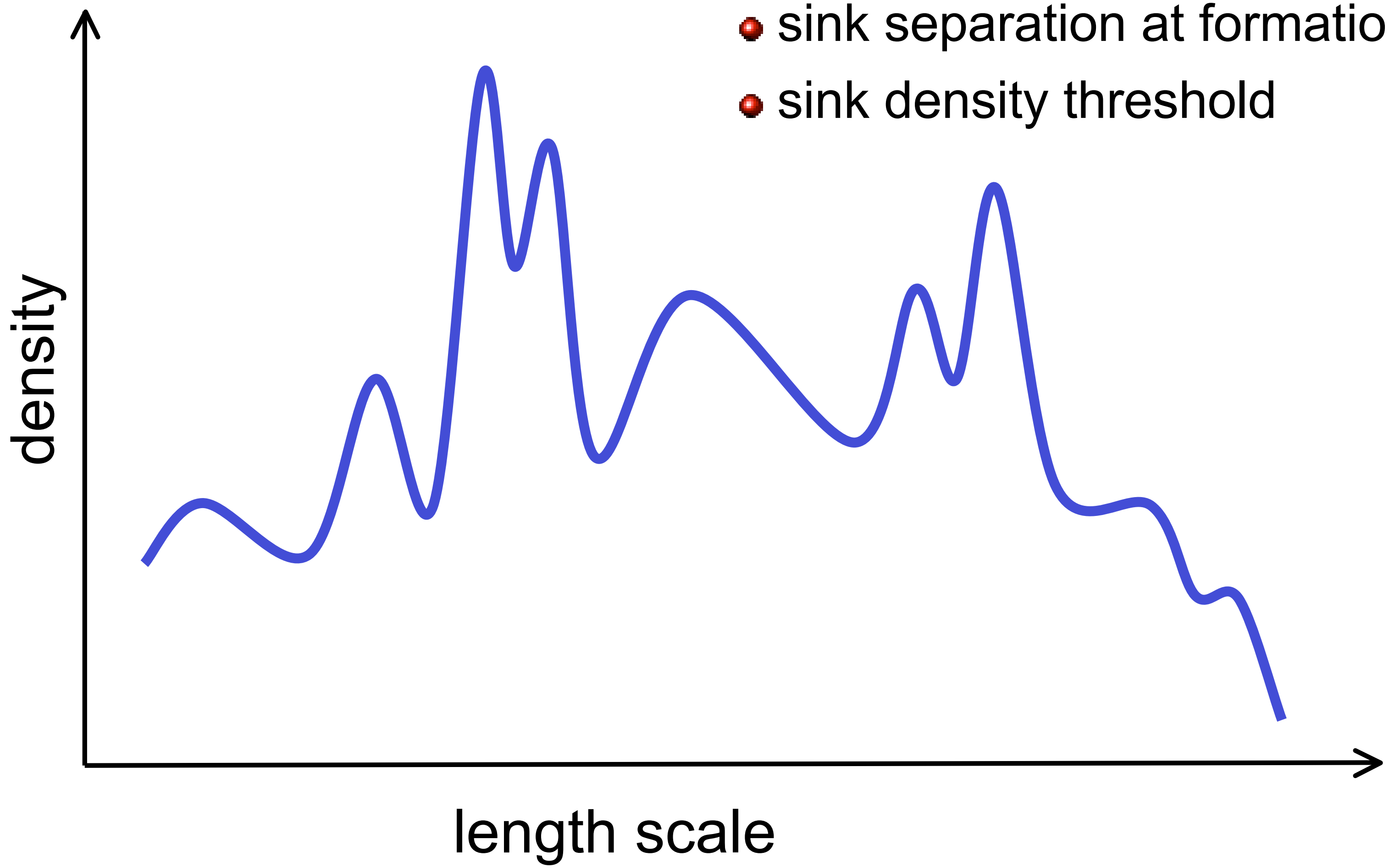
SPH with sink particles I



sink particles *replace* high-density regions (collapsed) by *„point masses“* which interact only gravitationally and can *accrete* ambient gas particles which come closer than some critical radius *R*.

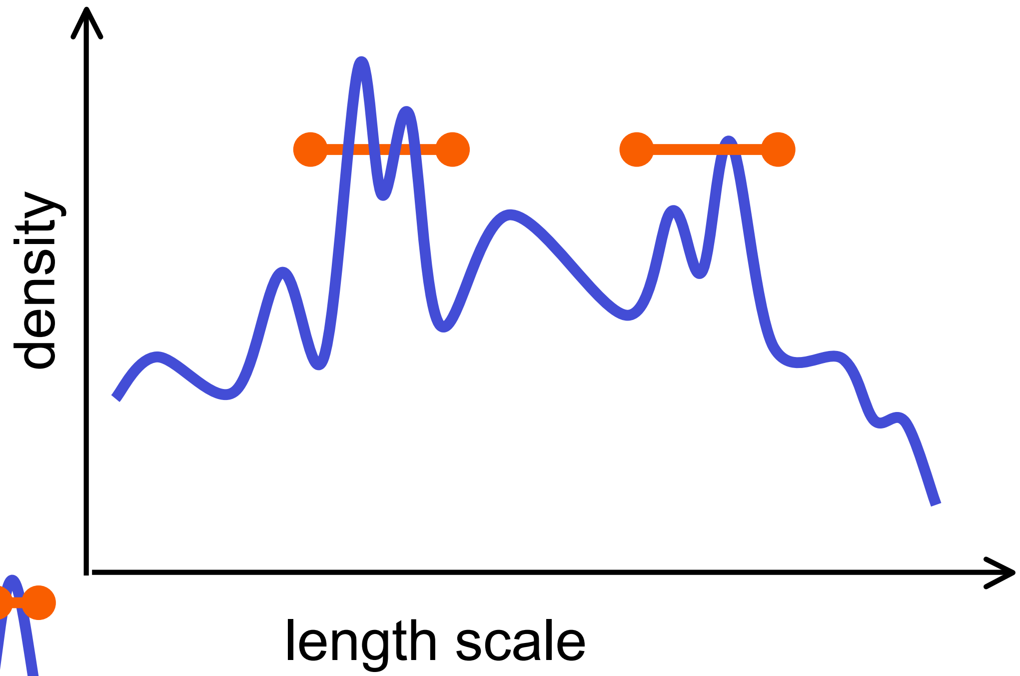
SPH with sink particles I

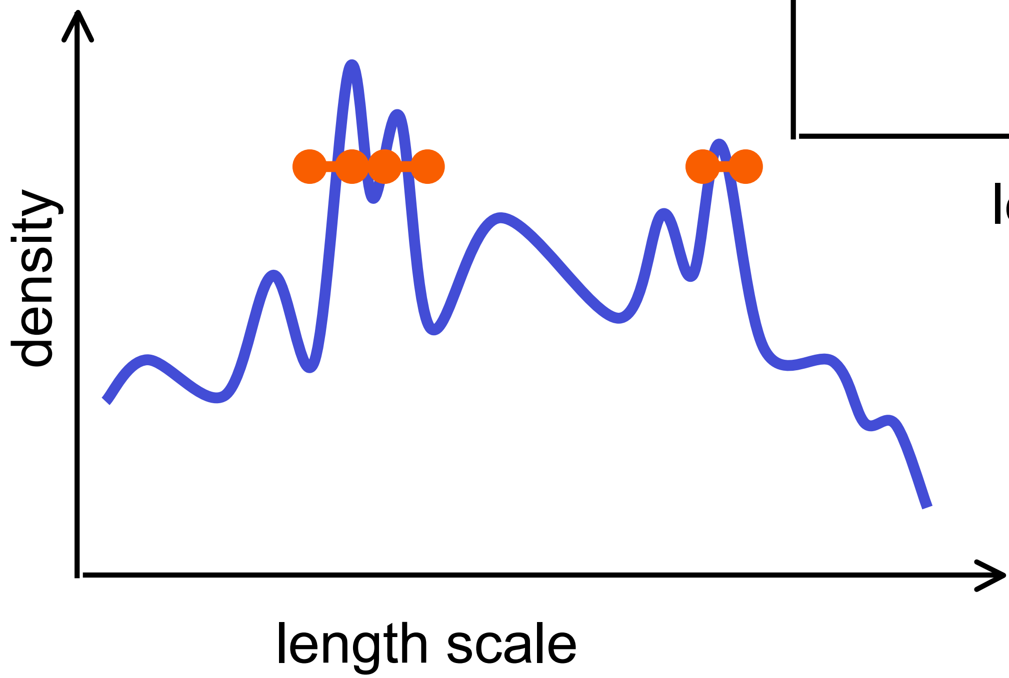
- sink particle diameter
- sink separation at formation
- sink density threshold



SPH with sink particles II

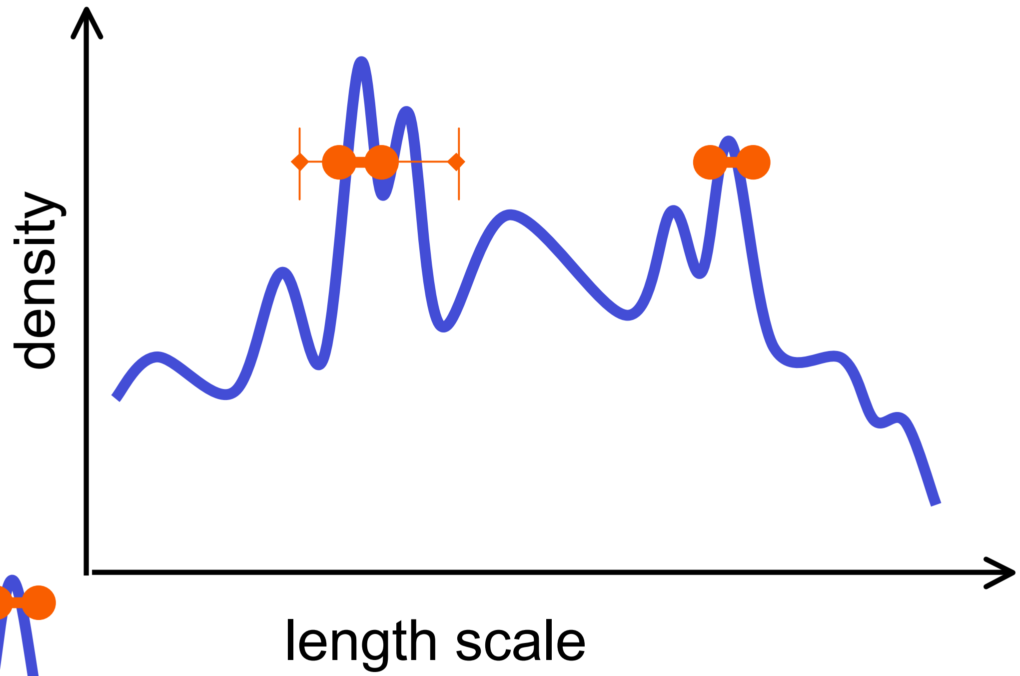
- sink particle diameter
- sink separation at formation
- sink density threshold

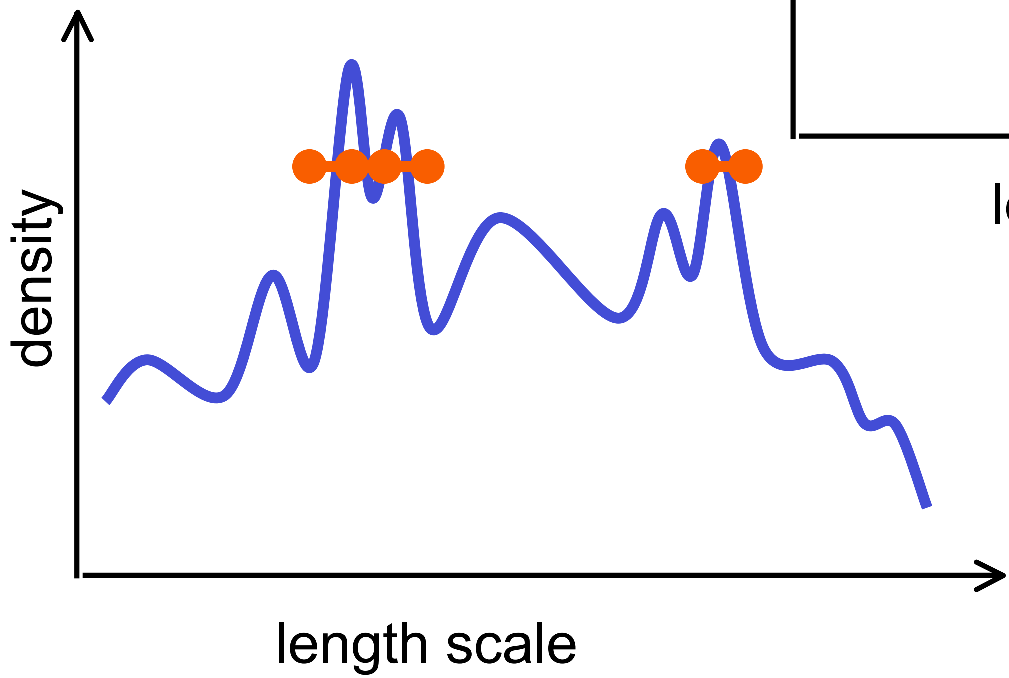




SPH with sink particles III

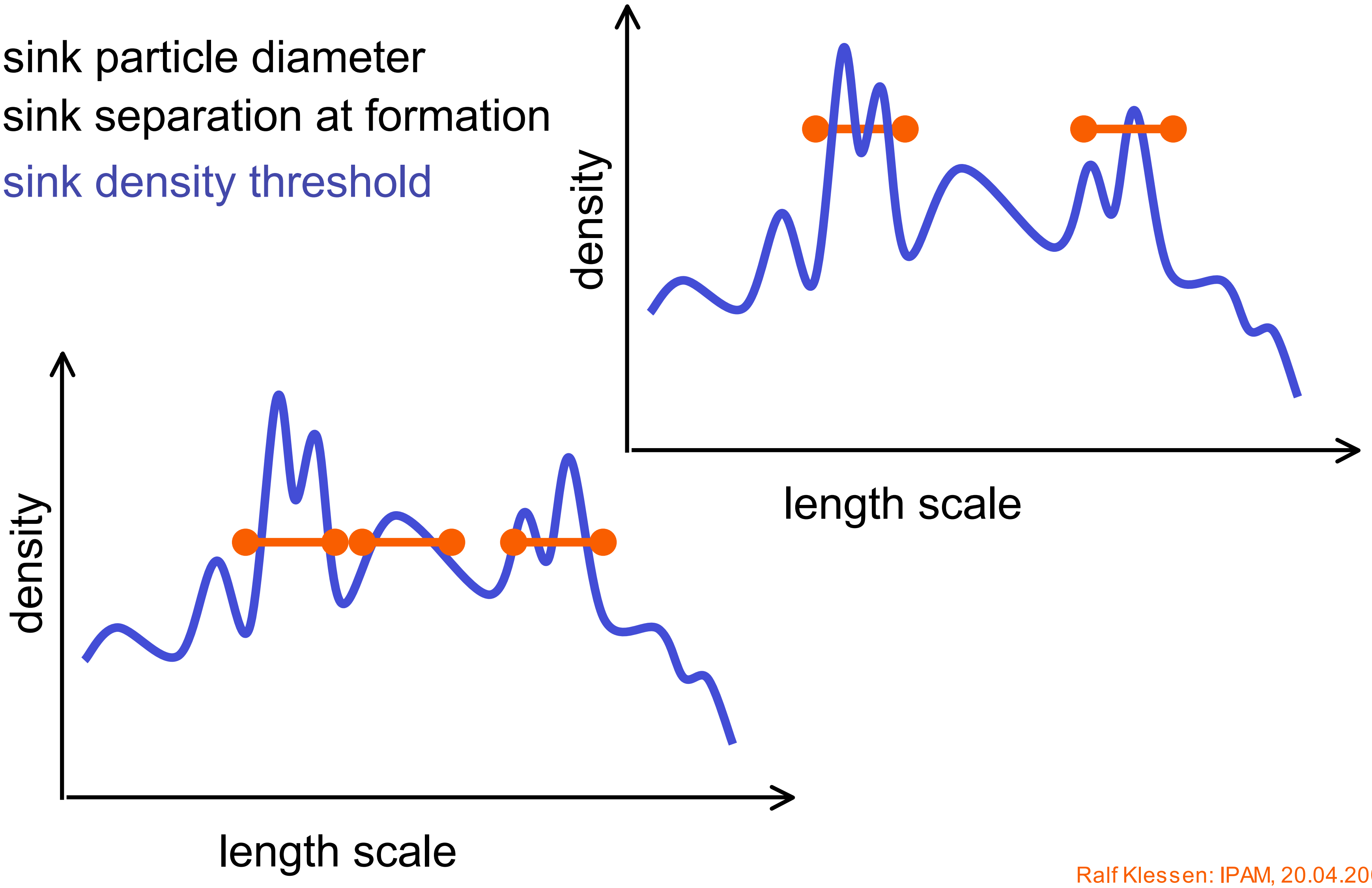
- sink particle diameter
- sink separation at formation
- sink density threshold





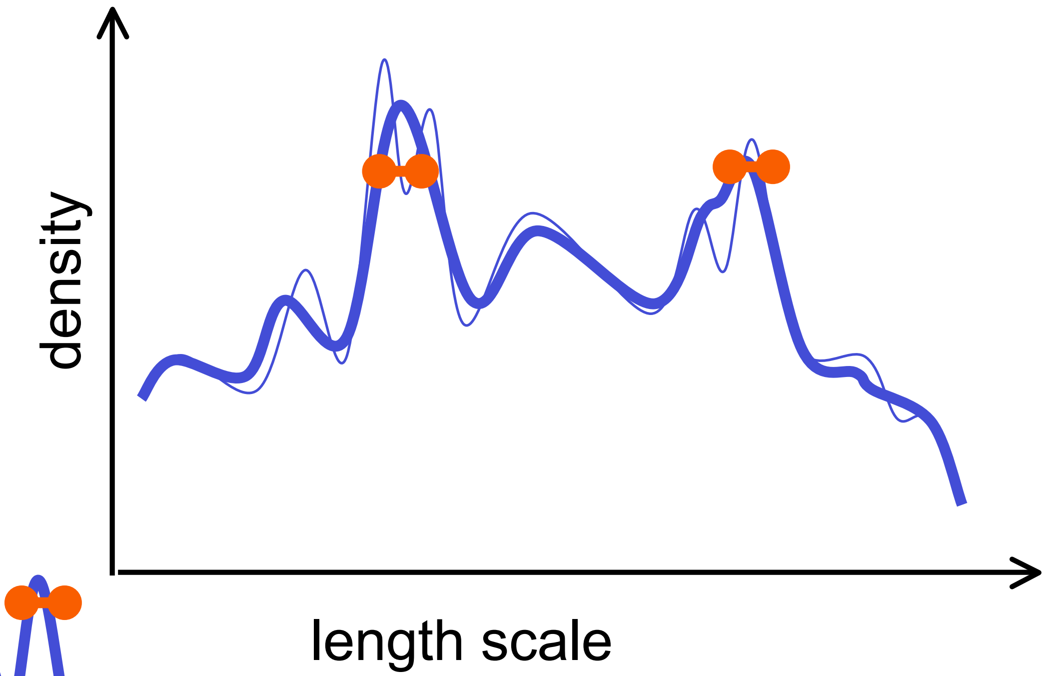
SPH with sink particles IV

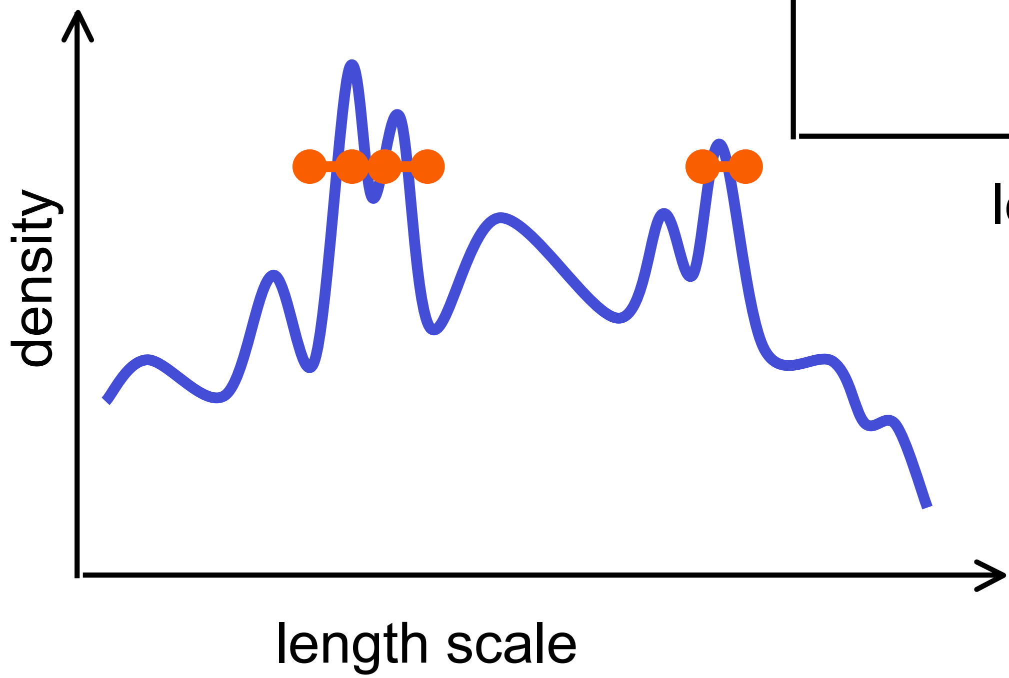
- sink particle diameter
- sink separation at formation
- sink density threshold



SPH with sink particles V

high-resolution runs vs.
low-resolution calculations





Some final remarks…

- *GRAVOTURBULENT STAR FORMATION:*
  This dynamic theory can explain and reproduce many features of star-forming regions on small as well as on large galactic scales.

- Some open questions:
  - role of magnetic fields?
  - role of thermodynamic state of the gas?
  - what drives turbulence?
  - how are small scales (local molecular clouds) connected to large-scale dynamics?
  - what terminates star formation locally?

Some final remarks…

- *NUMERICS:*
  SPH appears able to describe gravoturbulent fragmentation and star formation in molecular clouds.
- Pro:
  - *Lagrangian* character of method.
  - can resolve *large density contrasts*.
  - good for transition from hydro- to stellar dynamics
    --> accreting sink particles describe protostars
- Con:
  - low resolution in low-density regions.
  - difficulties with shock-capturing and treating B-fields.
- Next steps:
  - particle-splitting to locally increase resolution,
  - GPM, XSPH with "physical" viscosity

Outlook & First Examples

- *WHAT WE <u>REALLY</u> NEED: more physics!!*
  We need good *subgrid-scale models* for unresolved scales in our calculations.

- 2 Examples:

  - LOCAL SCALES: so far, we use dumb sink particles to protostellar collapse --> we do not know how the star “inside” forms and how it backreacts onto the ambient environment
    --> combine 3D hydro with 1D/2D PMS models

  - GALACTIC SCALES: can gravoturbulent models give us some handle on star-formation efficiency?
    --> some thoughts...

Example 1

Towards a complete picture...

COMBINE:

- *3D* hydrodynamic simulations of the *turbulent fragmentation* of *entire molecular cloud regions*.
  (using SPH with GRAPE: Klessen & Burkert 2000, 2001, Klessen, Heitsch & Mac Low 2000, Klessen 2001b)

WITH:

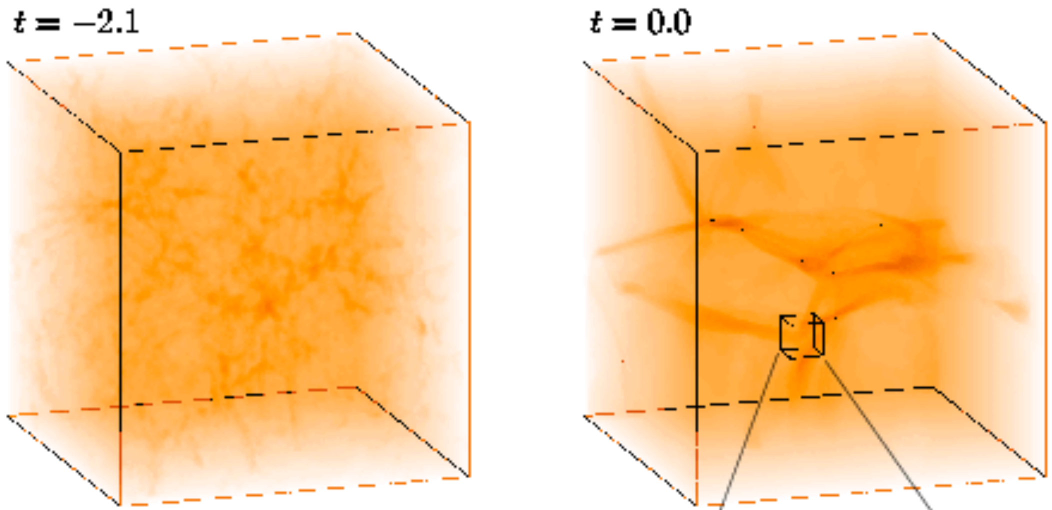
- detailed *2D* hydrodynamic modelling of *protostellar accretion disks.* (e.g. Yorke & Bodenheimer 1999)
  → with rad. transfer → *SED, T_{bol}, L_{bol}, etc.*

AND/OR:

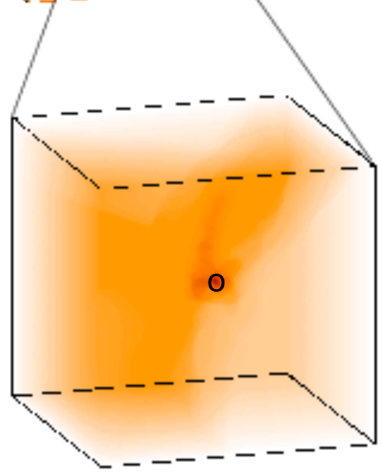
- implicit *1D* radiation-hydrodynamic scheme with time-dependend convection and D network following the *collapse of individual cores towards the MS* (Wuchterl & Tscharnuter 2002)
  → *PMS tracks, absolute stellar ages for cluster stars*

Formation of a $1M_{\odot}$ star

Dynamical evolution of a molecular cloud region of size $(0.32\mathrm{pc})^3$ containing 200 $M_{\odot}$ of gas
(from Klessen & Burkert 2000)



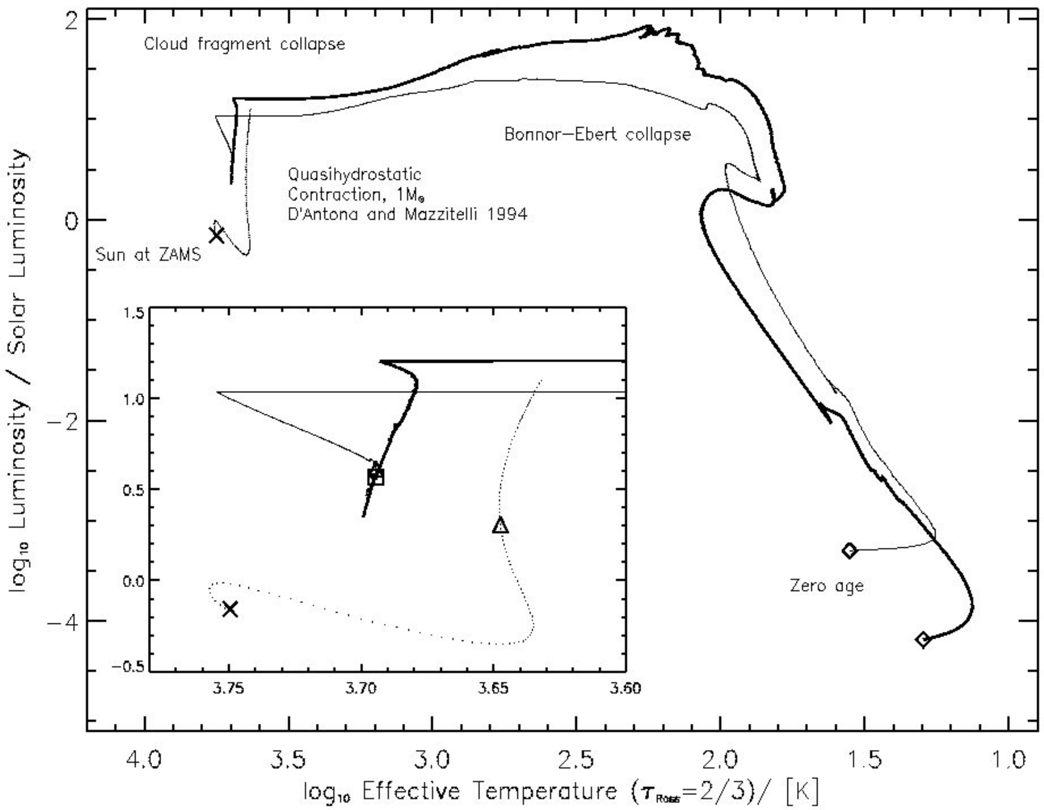
Within two free-fall times the system builds up a *cluster* of deeply embedded accreting *protostellar cores.*
We select the protostellar core with mass closest to $1M_{\odot}$ and use its mass accretion rate as *input* for the detailed 1D-RHD calculation (see Wuchterl & Tscharnuter 2002)



The system is shown initially, and at a stage when the $1M_{\odot}$-fragment reaches zero age (i.e. when it becomes optically thick for the first time).

(from Wuchterl & Klessen 2001)

PMS tracks



core in cluster environment

isolated Bonnor-Ebert core

D'Antona & Mazzitelli track

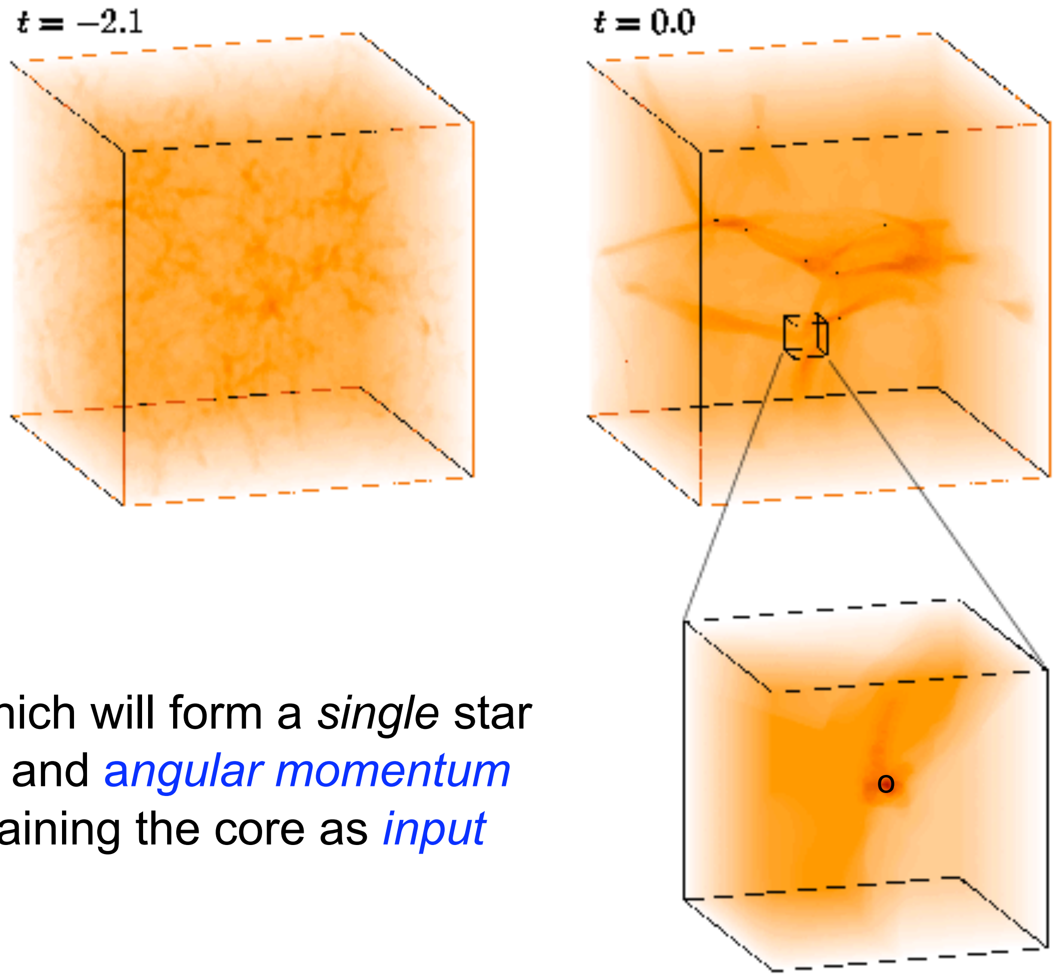
For ages less than 10^6 years, different collapse conditions lead to different evolutionary tracks. Later the dynamical tracks converge.

There are large differences to the hydrostatic track, due to different stellar structure.

(from Wuchterl & Klessen 2001)

SED's of a 1 $M_\odot$ star

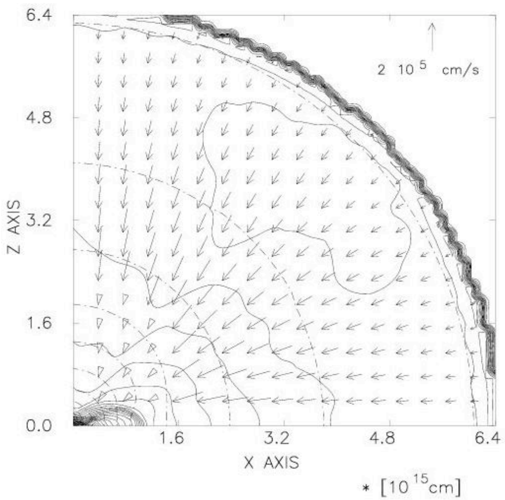
Dynamical evolution of a molecular cloud region of size $(0.32pc)^3$ containing 200 $M_\odot$ of gas
(from Klessen & Burkert 2000)



Within one to two free-fall times the system builds up a *cluster* of deeply embedded accreting *protostellar cores.*

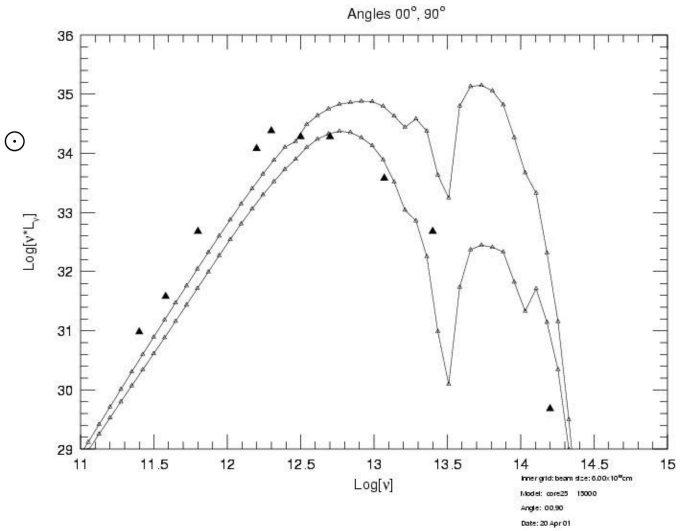
We select a protostellar core which will form a *single* star and use its *mass accretion rate* and *angular momentum gain* into a controll volume containing the core as *input* for a detailed 2D calculation
(Bodenheimer & Klessen, in preparation)

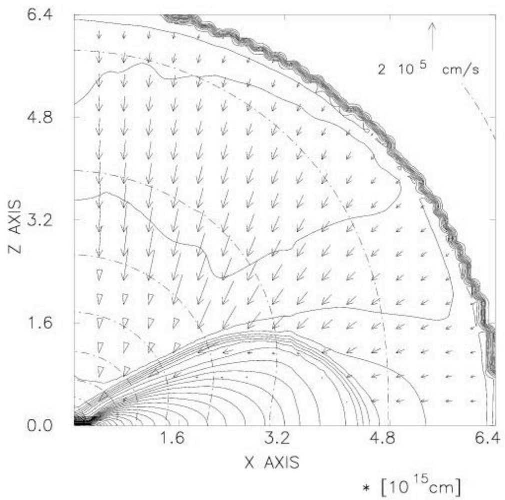
SED's of a 1 $M_\odot$ star



$t \approx 15000$yr

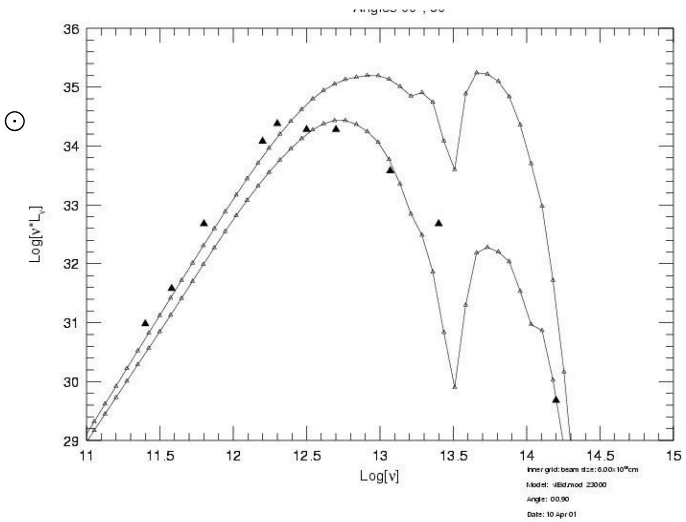
$M_{tot} \approx 0.75 M_\odot$





$t \approx 30000$yr

$M_{tot} \approx 0.89 M_\odot$



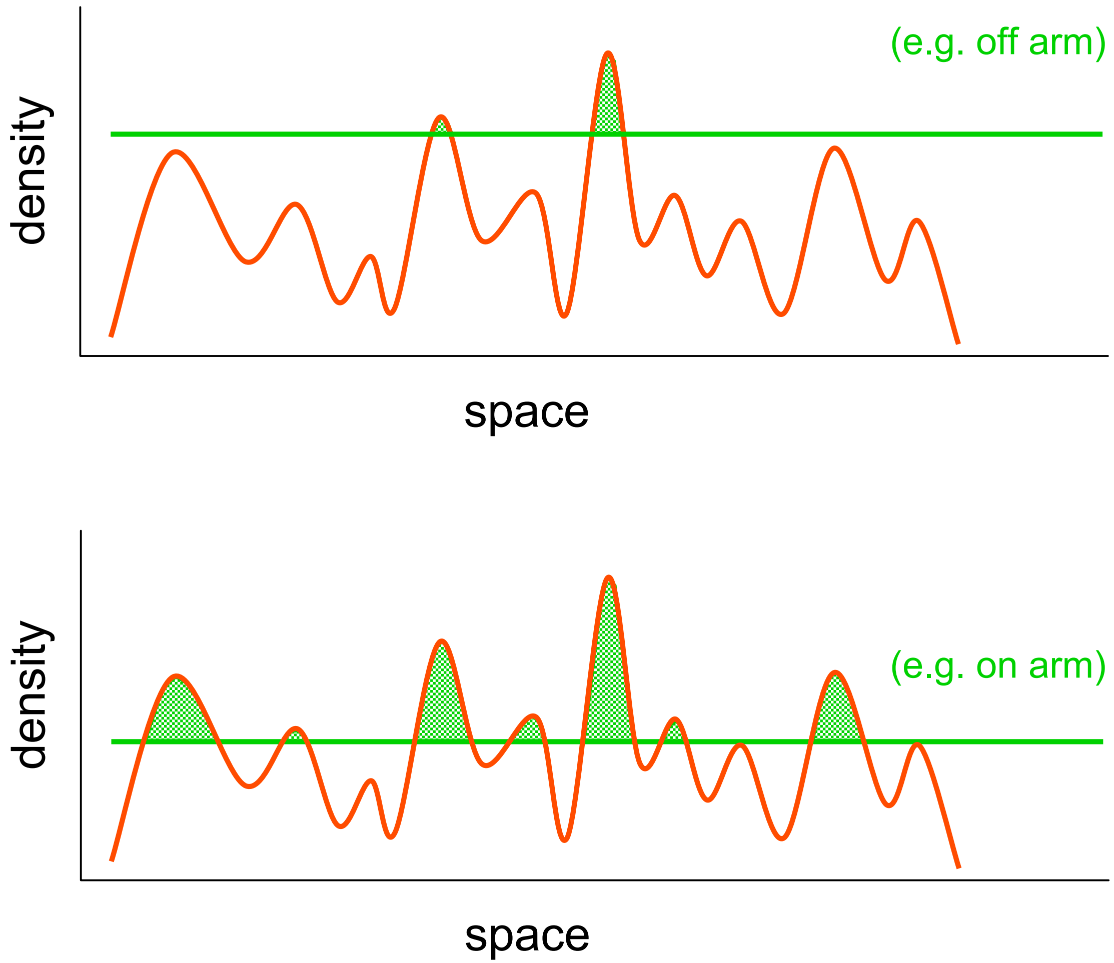
(Bodenheimer & Klessen, in preparation)

Example 2

Star formation on *global* scales

- *SF on global scales* $\stackrel{!}{=}$ *formation of molecular clouds*
- MC's form at *stagnation points* of *convergent large-scale flows* (need ~0.5kpc^3 of gas) → high density → enhanced cooling → fast H_2 formation & gravitational instability → local collapse and star formation
- External perturbations *increase* the local likelihood of MC formation (e.g. in spiral density waves, galaxy interactions, etc.)

Star formation on *global* scales

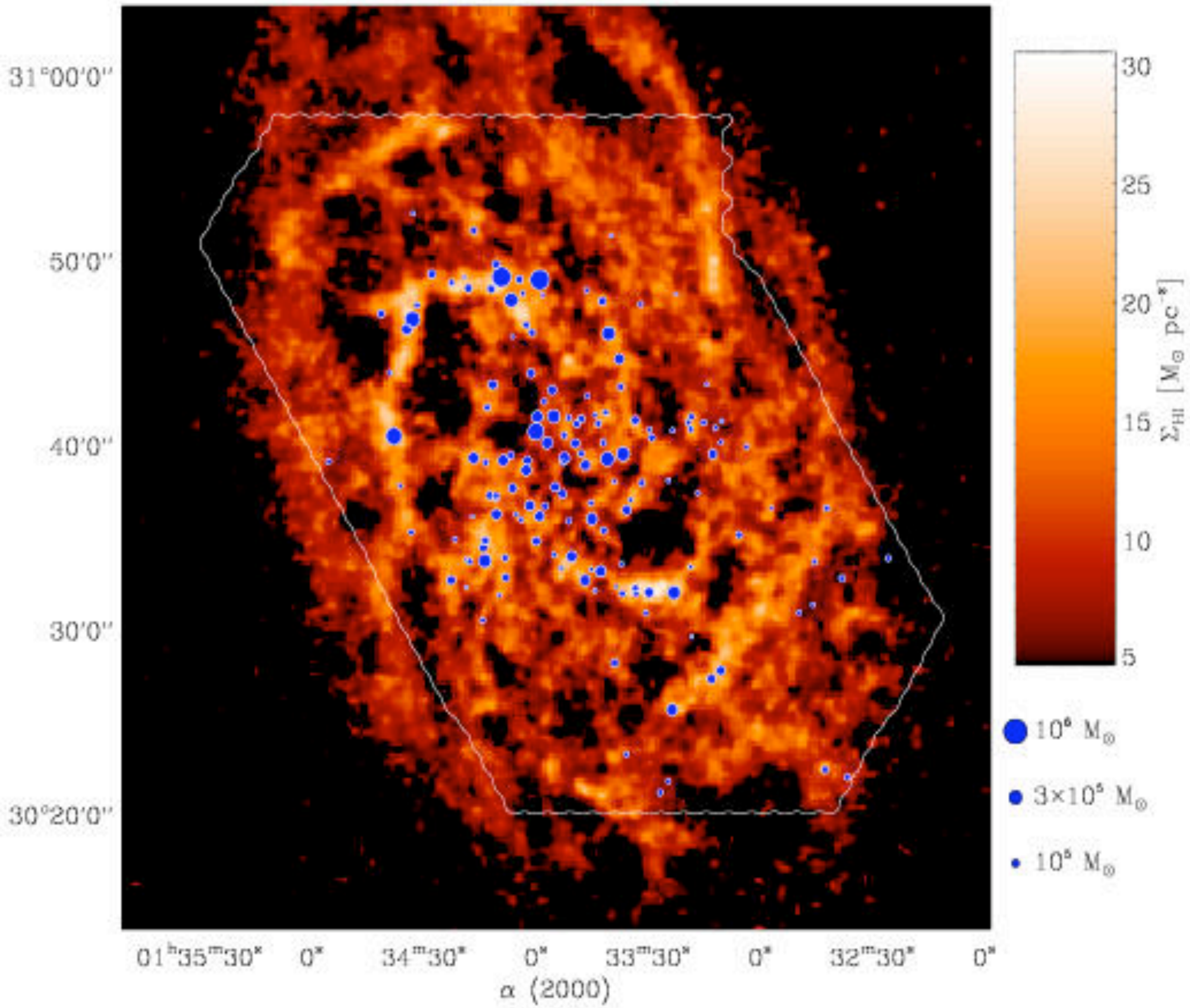


density fluctuations in warm atomar ISM caused by supersonic turbulence

some are dense enough to form H2 within “reasonable timescale” →molecular clouds

external perturbuations (i.e. potential changes) increase likelihood

Correlation between H_2 and HI



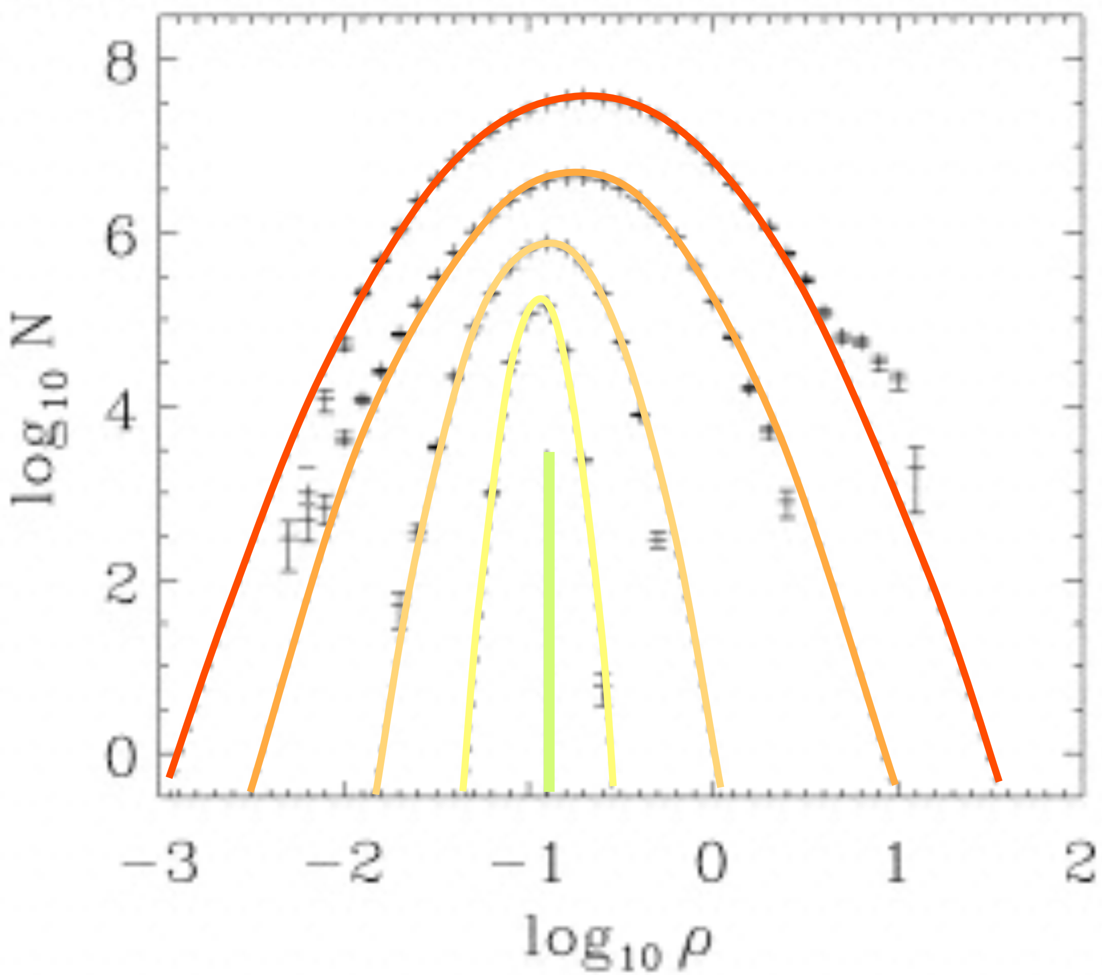
Compare H_2 - HI in M33:

- H_2: BIMA-SONG Survey, see Blitz et al.
- HI: Observations with Westerbork Radio T.

H_2 clouds are seen in regions of high HI density (in spiral arms and filaments)

(Deul & van der Hulst 1987, Blitz et al. 2004)

Star formation on *global* scales



mass weighted ρ-pdf, each shifted by $\Delta \log N = 1$

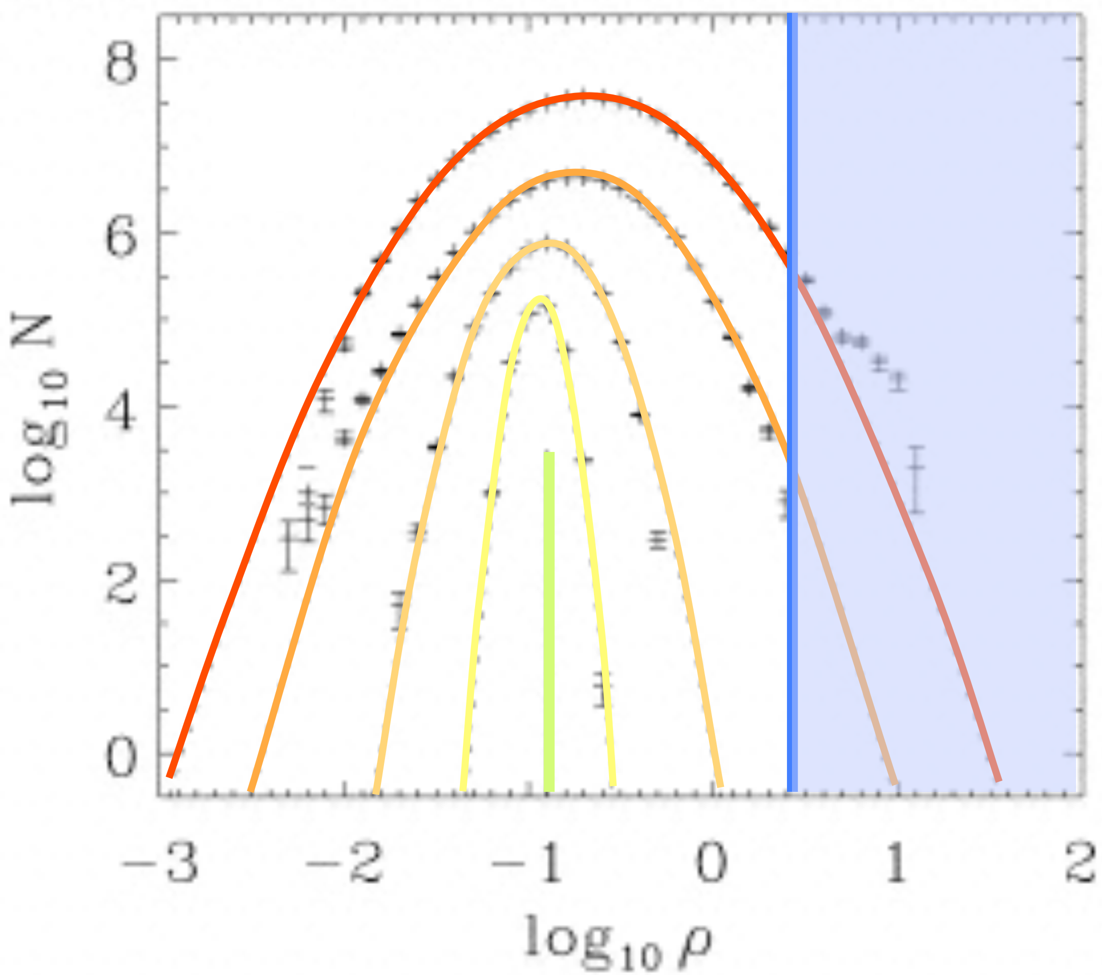
(from Klessen, 2001)

probability distribution function of density *(ρ-pdf)* for decaying supersonic turbulence

varying rms Mach numbers:

M1 > M2 > M3 > M4 > 0

Star formation on *global* scales



mass weighted ρ-pdf, each shifted by $\Delta \log N = 1$

(from Klessen, 2001; rate from Hollenback, Werner, & Salpeter 1971)

H_2 formation rate:

$$\tau_{H_2} \approx \frac{1.5\,\mathrm{Gyr}}{n_H / 1\mathrm{cm}^{-3}}$$

For $n_H \geq 100\ \mathrm{cm}^{-3}$, H_2 forms within 10Myr, this is about the lifetime of typical MC's.

What fraction of the galactic ISM reaches such densities?

THANKS