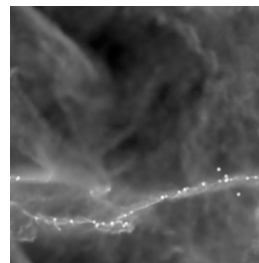# Modelling ISM Dynamics and Star Formation
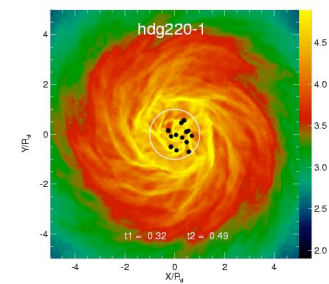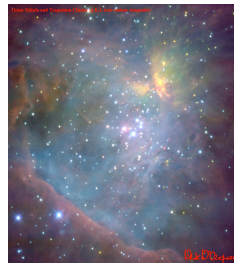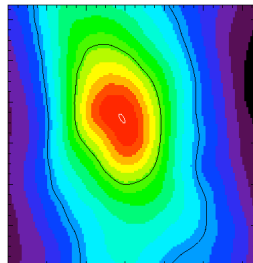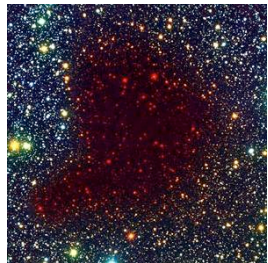
Ralf Klessen

Zentrum für Astronomie der Universität Heidelberg
Institut für Theoretische Astrophysik

thanks to ...

- many thanks to the members of the *star formation group* at the *Institute for Theoretical Astrophysics* at the *Center for Astronomy* of *Heidelberg University*

- Robi Banerjee
- Ingo Berentzen
- Paul Clark
- Christoph Federrath
- Philipp Girichidis
- Simon Glover
- Thomas Greif
- Milica Milosavljevic
- Thomas Peters

- Dominik Schleicher
- Stefan Schmeja
- Bernd Völkl
- and many guests

agenda

- phenomenology

  - stars

  - gas

- theoretical approach

  - what is needed to model the interstellar medium?

  - astrophysical hydrodynamics

  - modelling turbulence

stars

young stars in the Milky Way

© 2000 Axel Mellinger
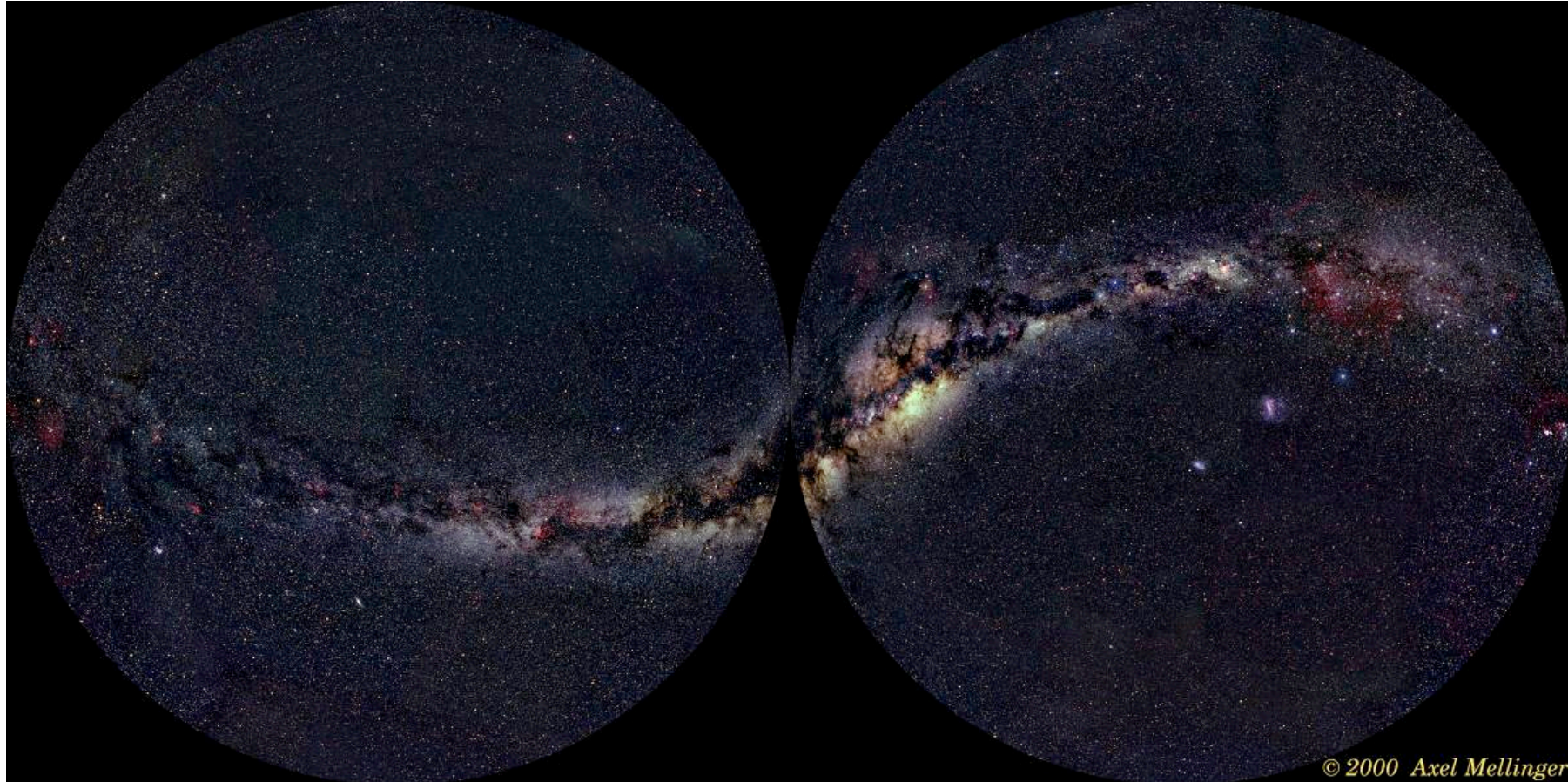
On the night sky, you see *stars* and *dark clouds*:
The brightest stars are massive and therefore young.
→ Star formation is important for understanding the structure of our Galaxy
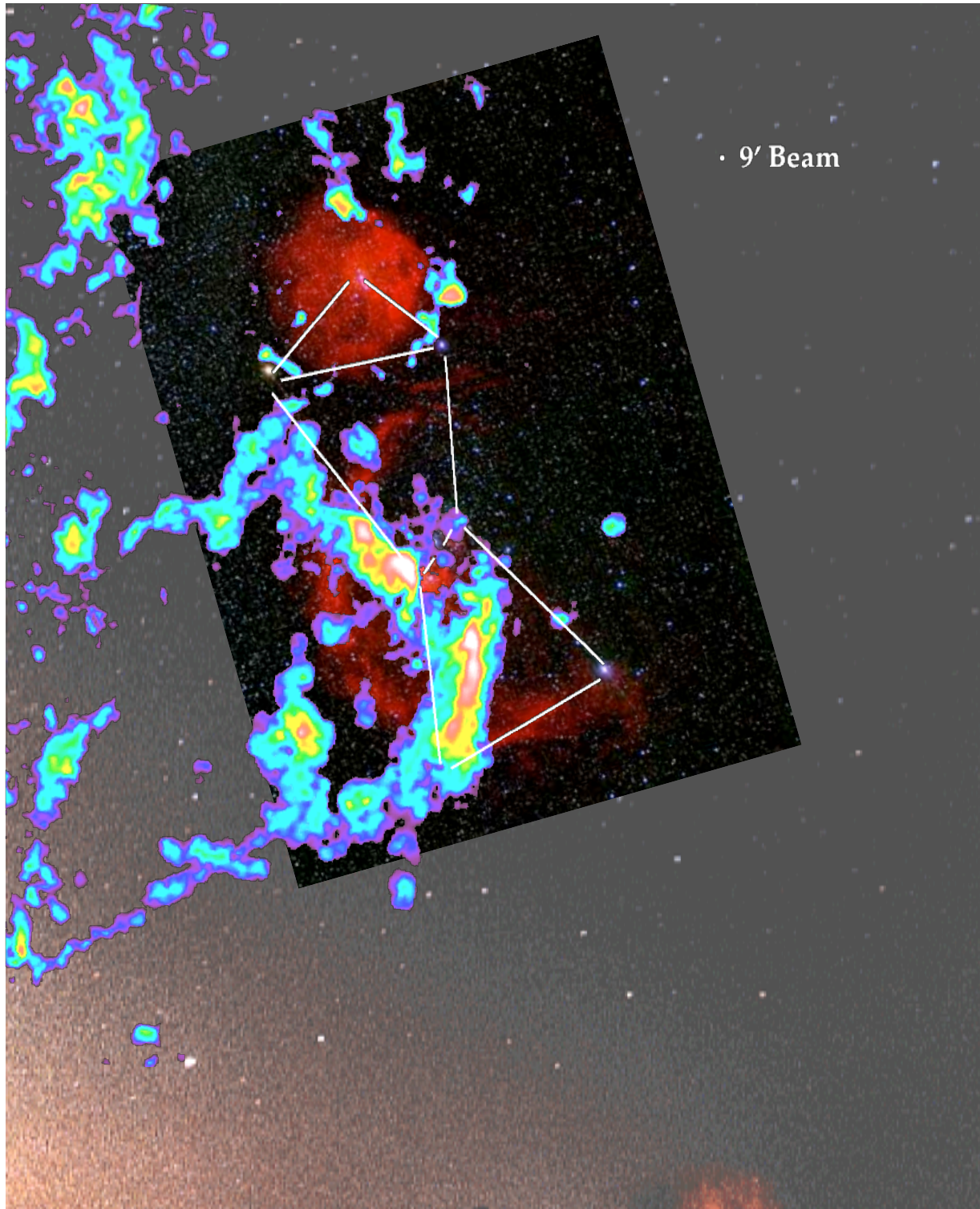
Star formation in Orion

We see

- *Stars* (in visible light)

- Atomic hydrogen
  (in Hα -- red)

- Molecular hydrogen H_2
  (radio emission -- color coded)

9' Beam

Local star forming region: The Trapezium Cluster in Orion
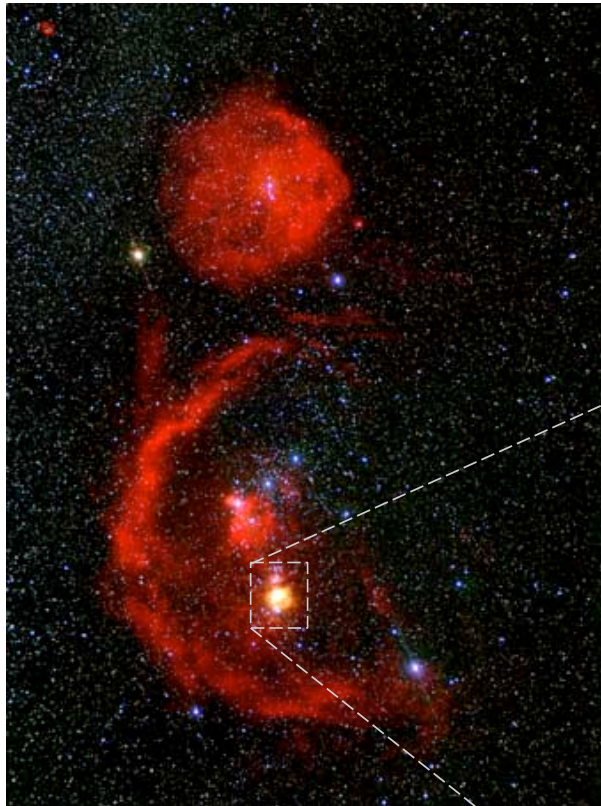
Orion molecular cloud

The Orion molecular cloud is the birth-place of several young embedded star clusters.

The Trapezium cluster is only visible in the IR and contains about 2000 newly born stars.

Orion Nebula and Trapezium Cluster (J,K,L true colour composite)
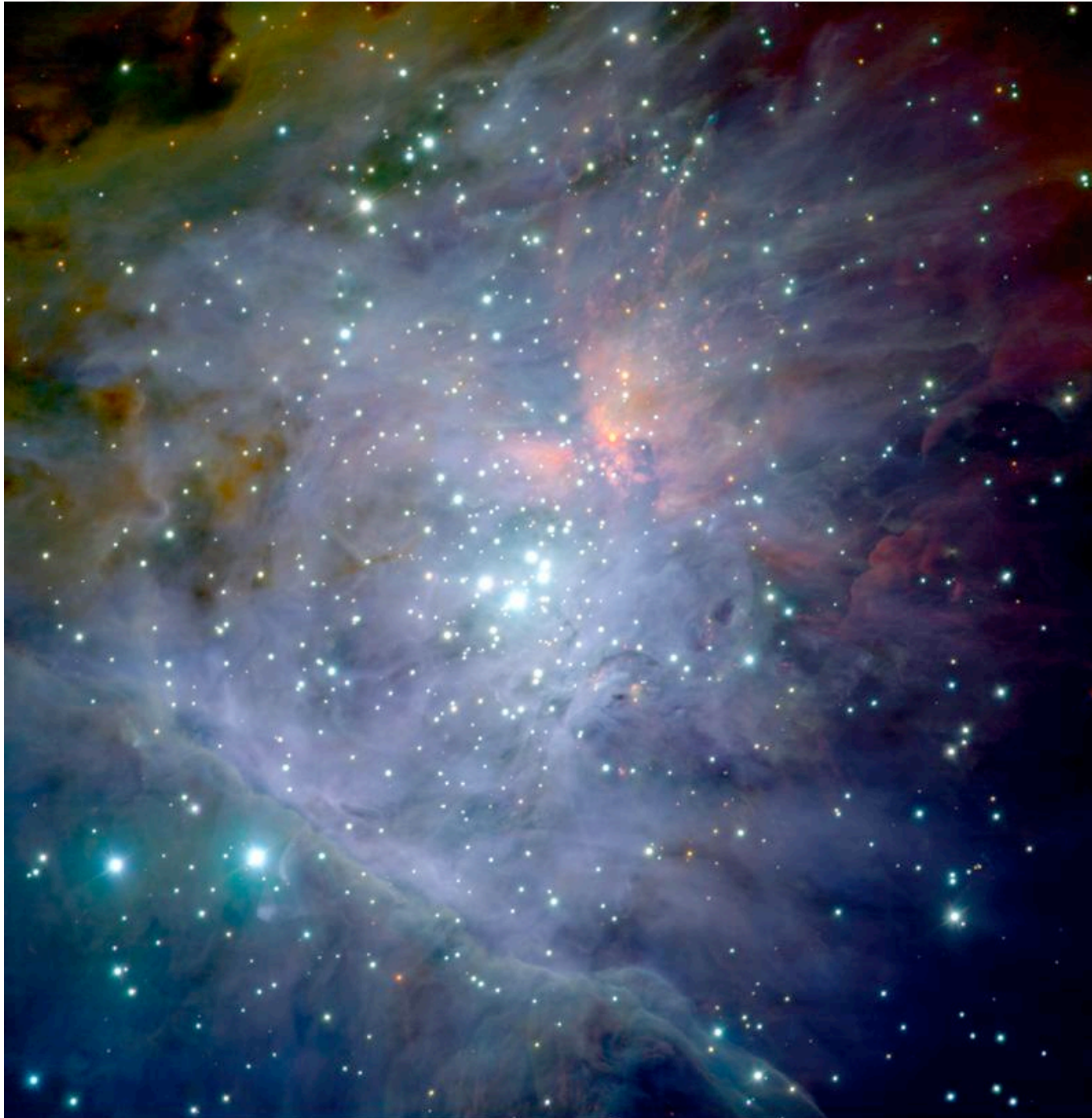
Trapezium cluster

Trapezium Cluster

(detail)

- stars form in clusters
- stars form in molecular clouds
- (proto)stellar feedback is important

(color composite J,H,K by M. McCaughrean, VLT, Paranal, Chile)

Trapezium Cluster: Central Region

Ionizing radiation from central star
Θ**1C Orionis**
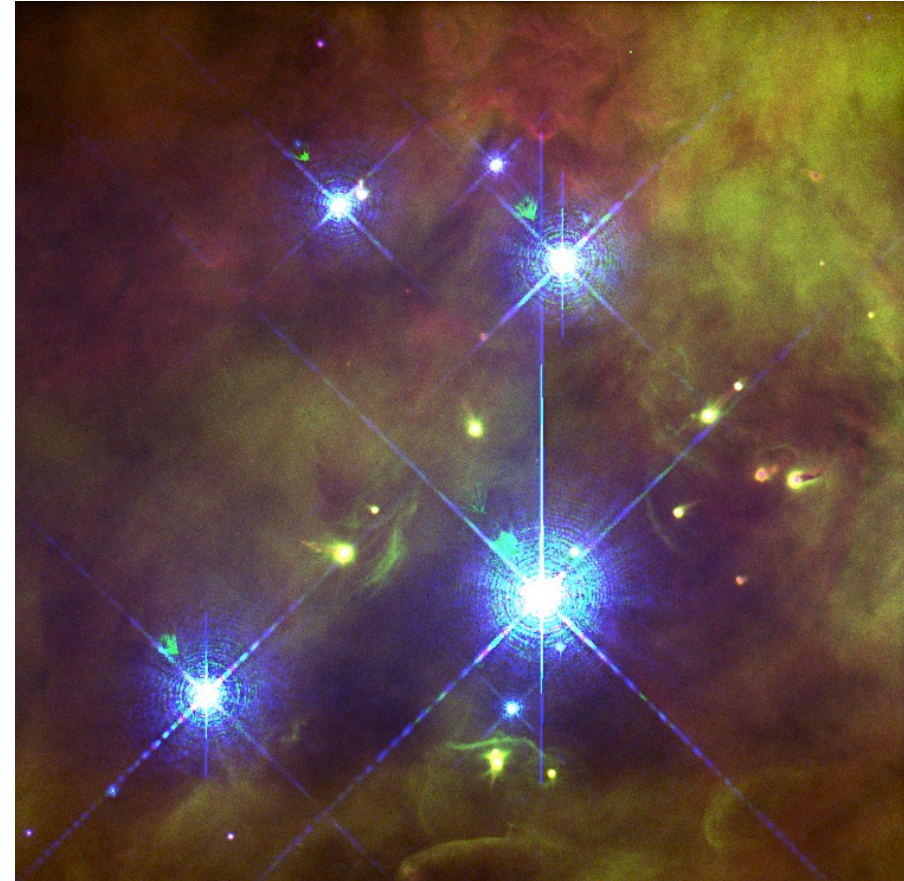
Proplyds: Evaporating ``protoplanetary´´ disks around young low-mass protostars

(images: Doug Johnstone et al.)

Futher Details: Siluette Disks in Orion
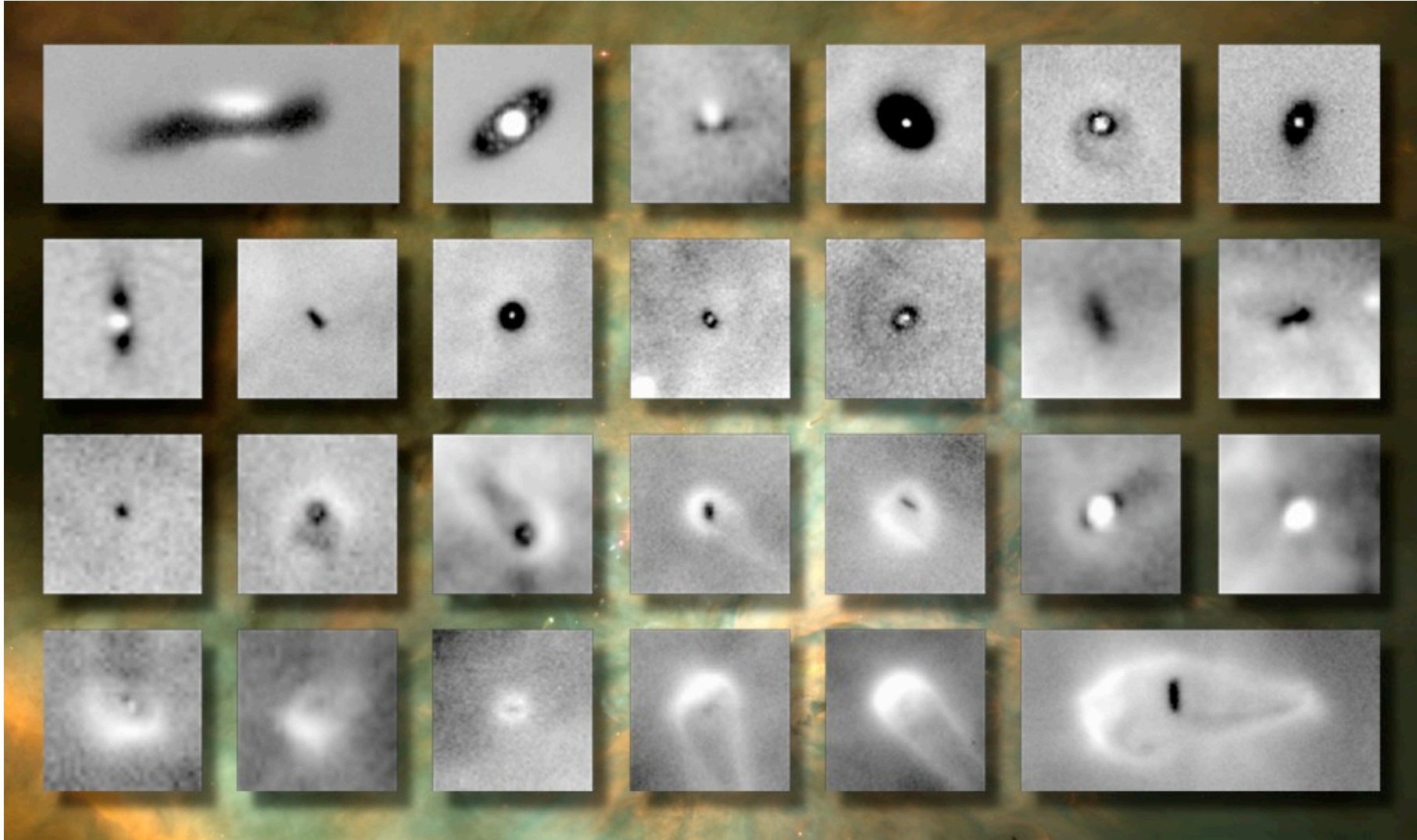
protostellar disks: dark shades in front of the photodissociation region in the background. Each image is 750 AU x 750 AU.

(data: Mark McCaughrean)
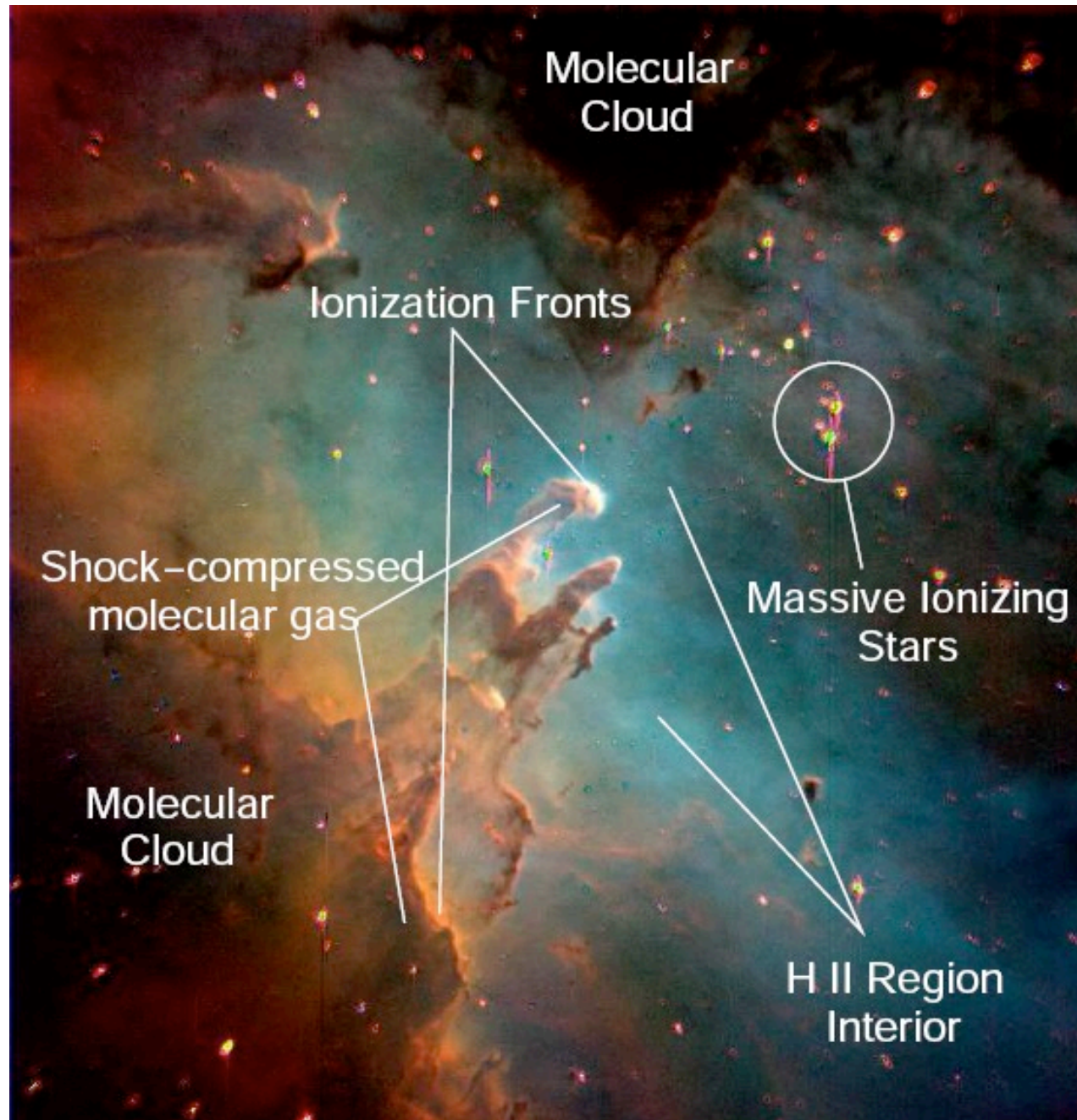
alles in einem Bild
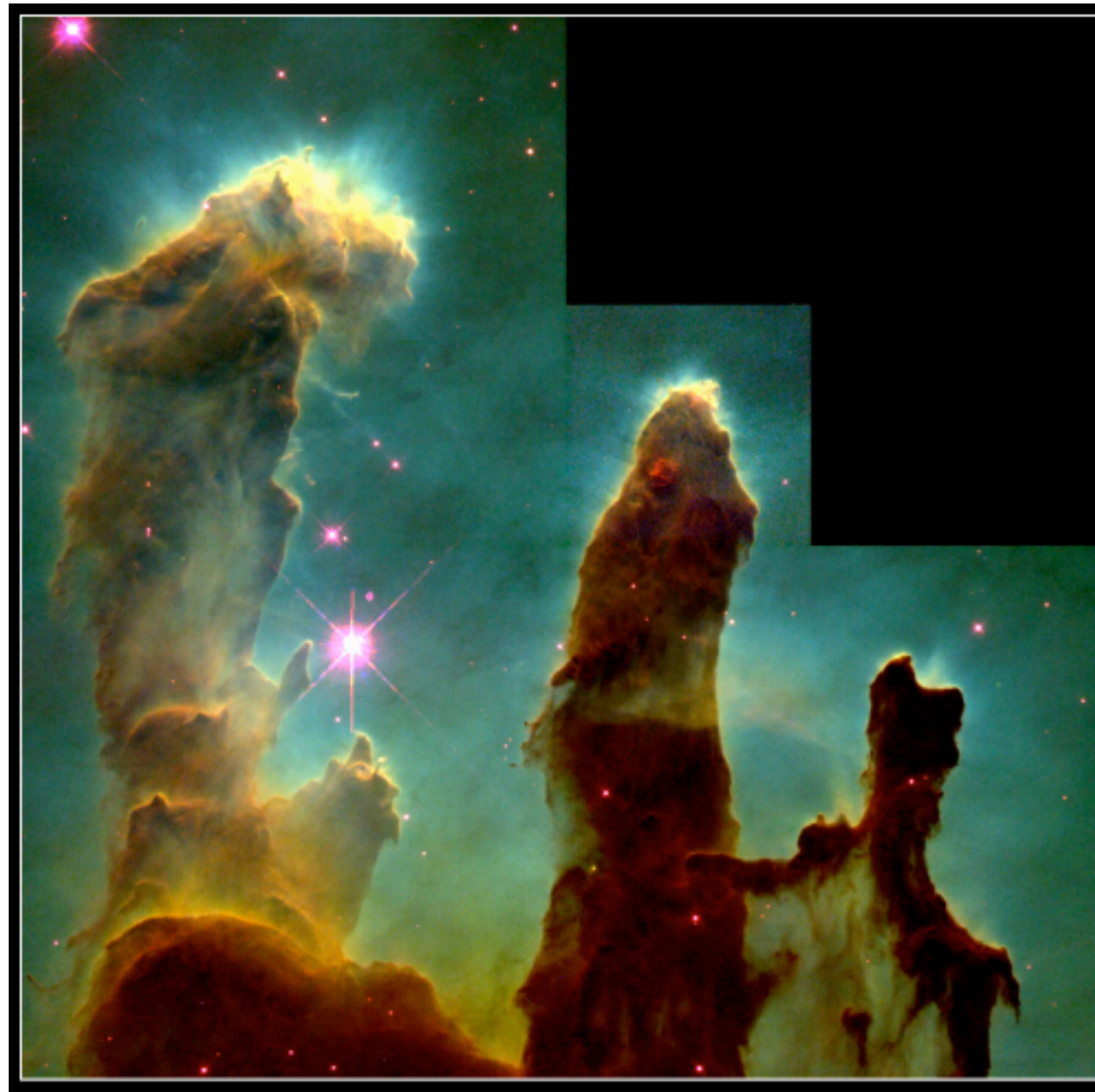
Pillars of God (in Eagle Nebula): Formation of small groups of young stars in the tips of the columns of gas and dust ….

HST Aufnahme

Infrared
observation

IR observation with ESO-VLT

Pillars of God (in Eagle Nebula): Formation of small groups of young stars in the tips of the columns of gas and dust ....
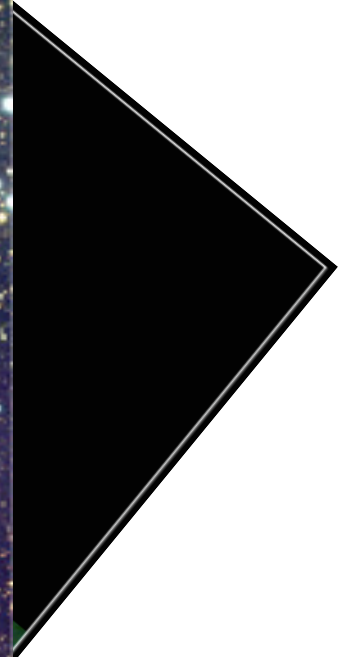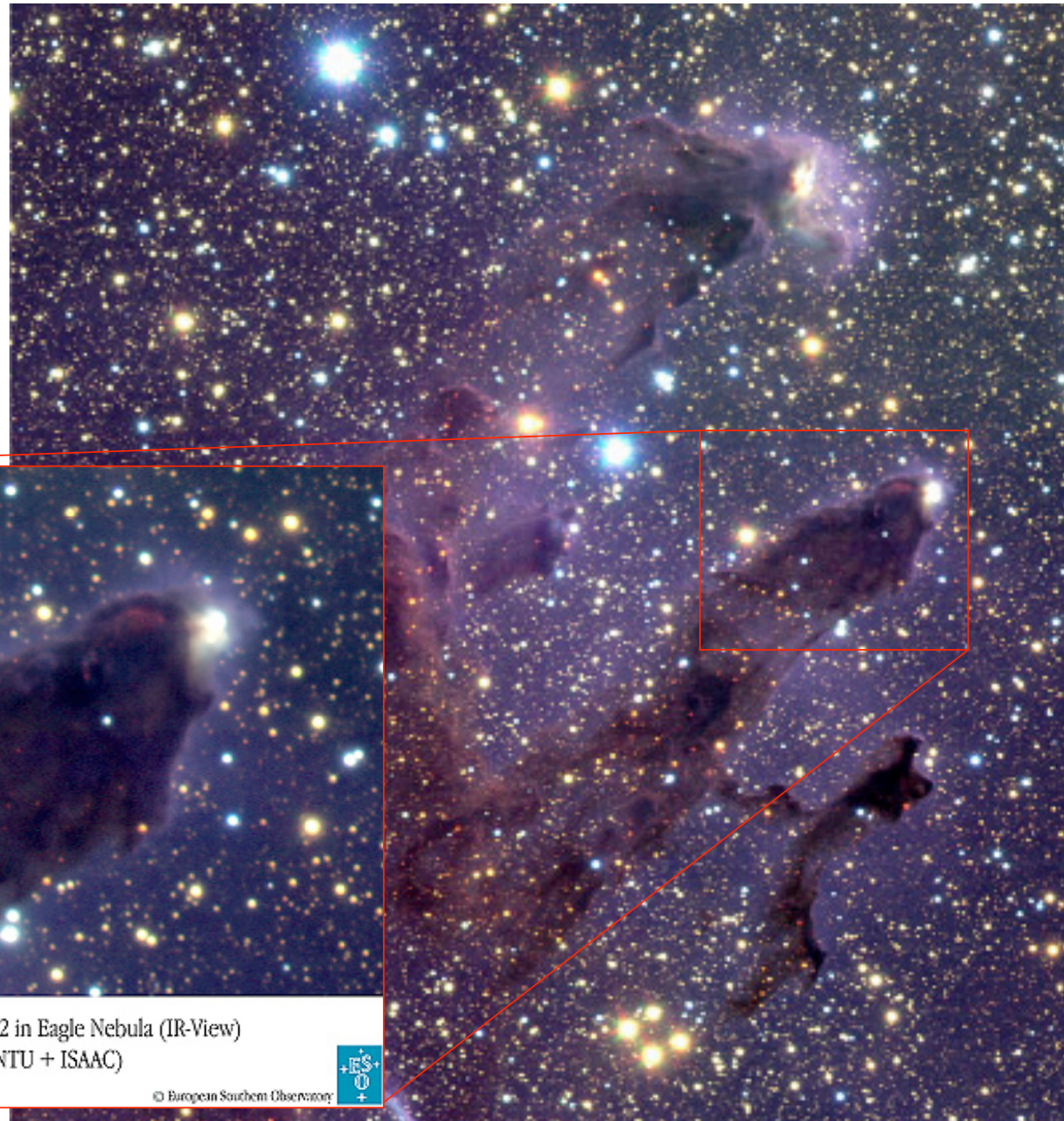
Head of Column No.2 in Eagle Nebula (IR-View)
(VLT ANTU + ISAAC)

ESO PR Photo 37d/01 (20 December 2001)   © European Southern Observatory

Pillars of God (in Eagle Nebula): Formation of small groups of young stars in the tips of the columns of gas and dust ….

gas

Interstellar Matter: ISM

Abundances, scaled to 1.000.000 H atoms

element		atomic number	abundance
Wasserstoff	H	1	1.000.000
Deuterium	$_1H^2$	1	16
Helium	He	2	68.000
Kohlenstoff	C	6	420
Stickstoff	N	7	90
Sauerstoff	O	8	700
Neon	Ne	10	100
Natrium	Na	11	2
Magnesium	Mg	12	40
Aluminium	Al	13	3
Silicium	Si	14	38
Schwefel	S	16	20
Calcium	Ca	20	2
Eisen	Fe	26	34
Nickel	Ni	28	2

Hydrogen is by far the most abundant element (more than 90% in number).

Phases of the ISM

A_V

HII   HI   H$_2$

Dichte- / Säulendichte nimmt zu
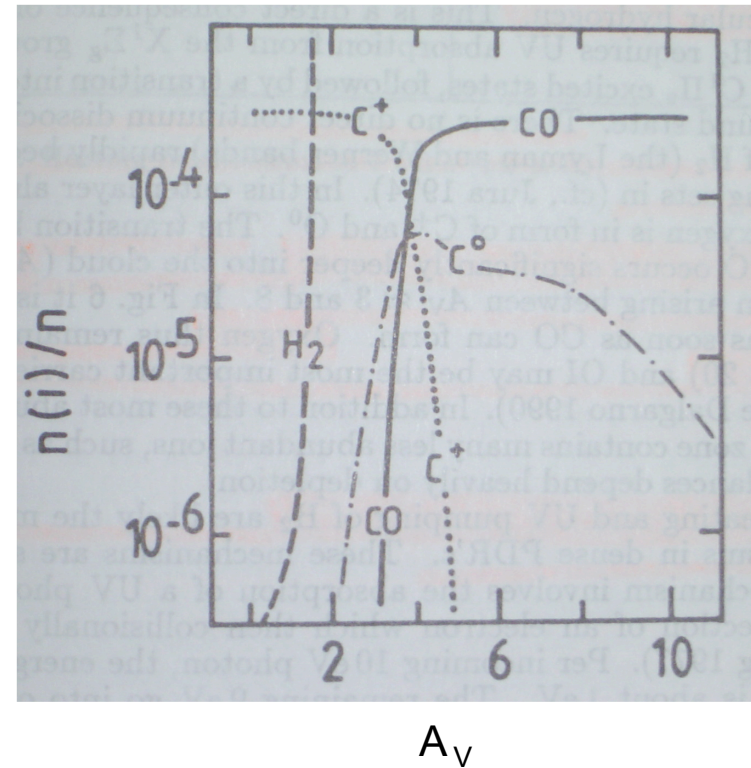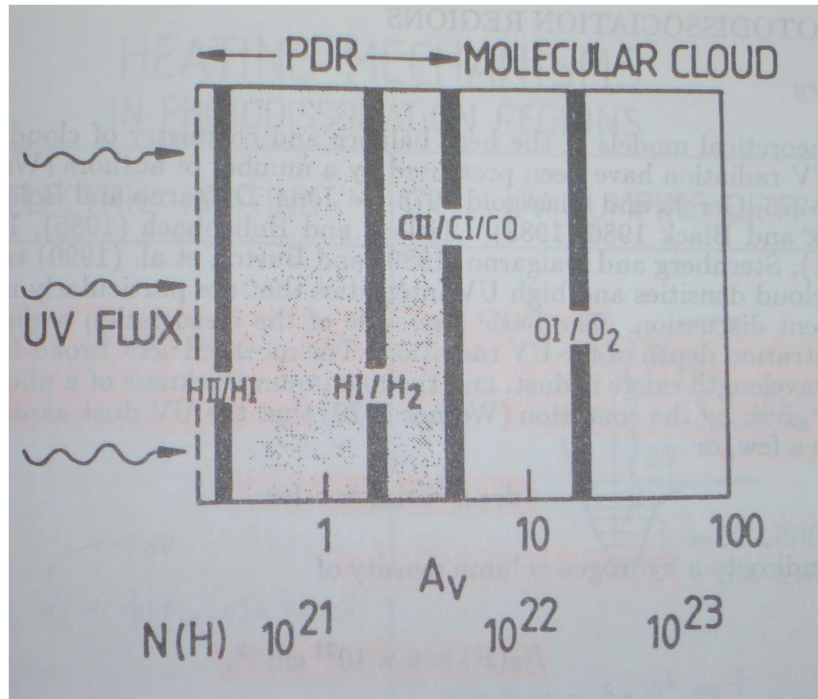
A_V bezeichnet die Extinktion, dh. die Abschwächung der einfallenden Strahlung.

nearby clouds

Pipe · Ophiuchus · Taurus · Perseus · Orion
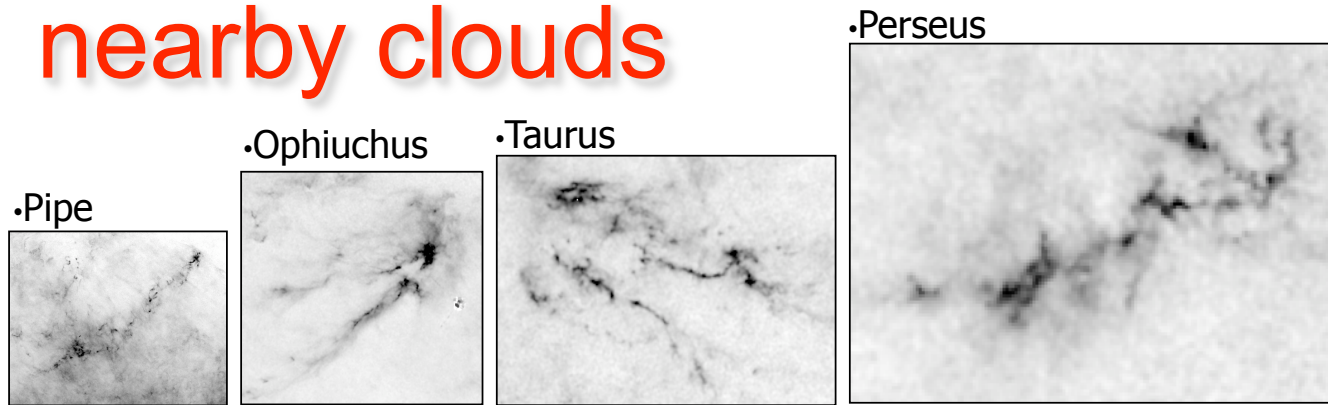
scales to same scale

10 pc
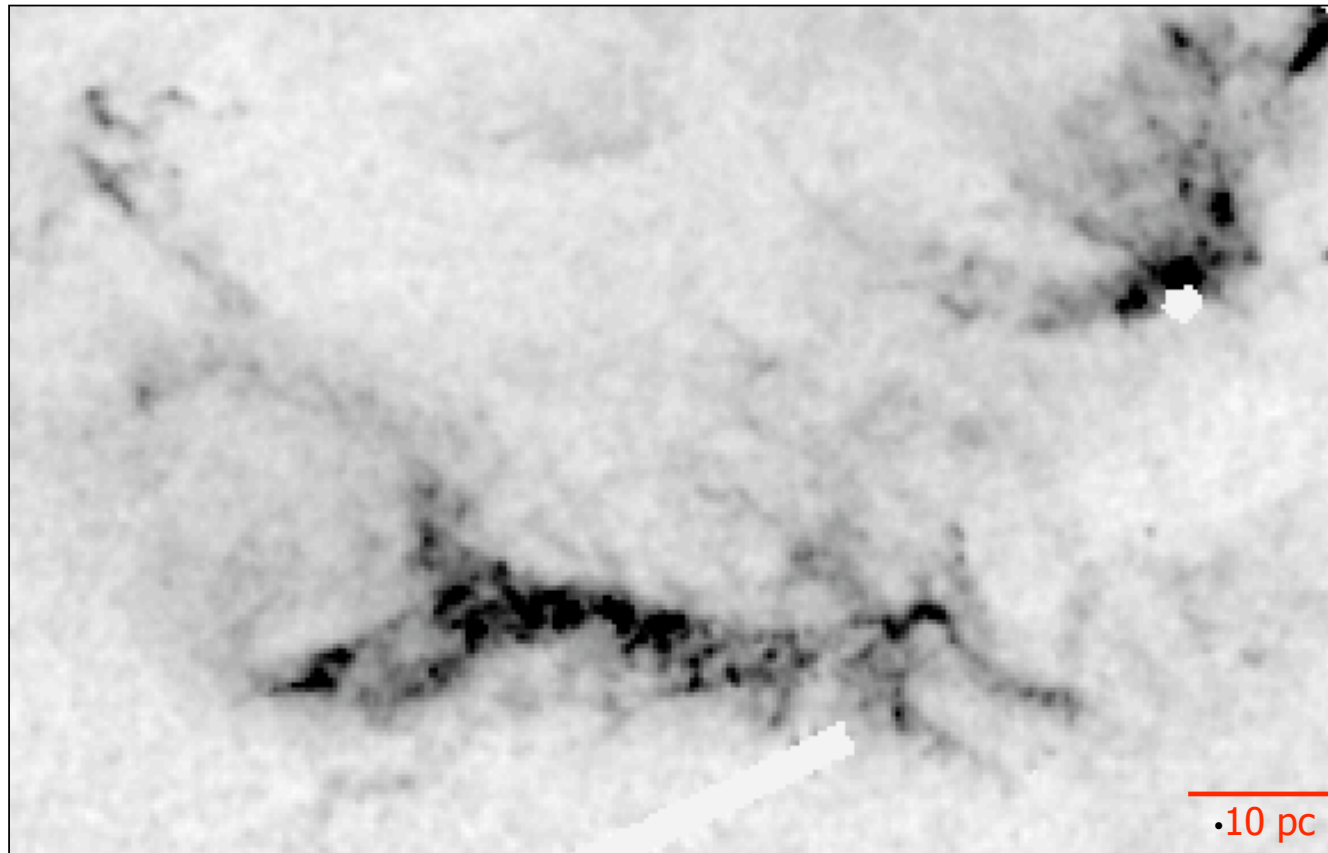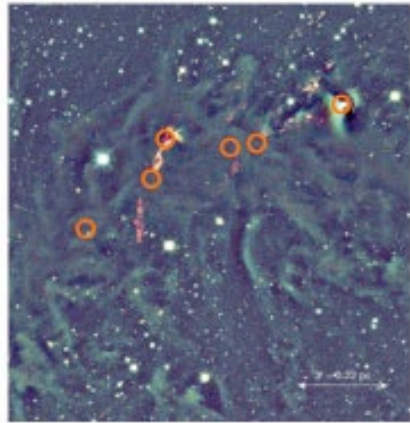
(from A. Goodman)

J,H,K Near-IR image
of Cloudshine
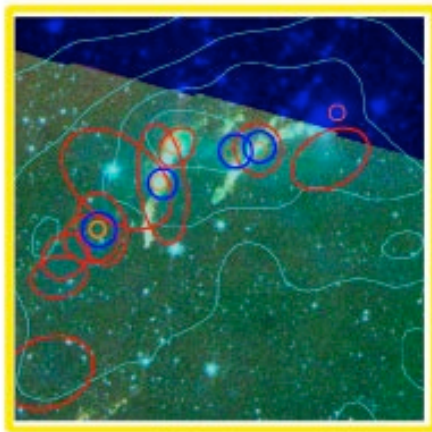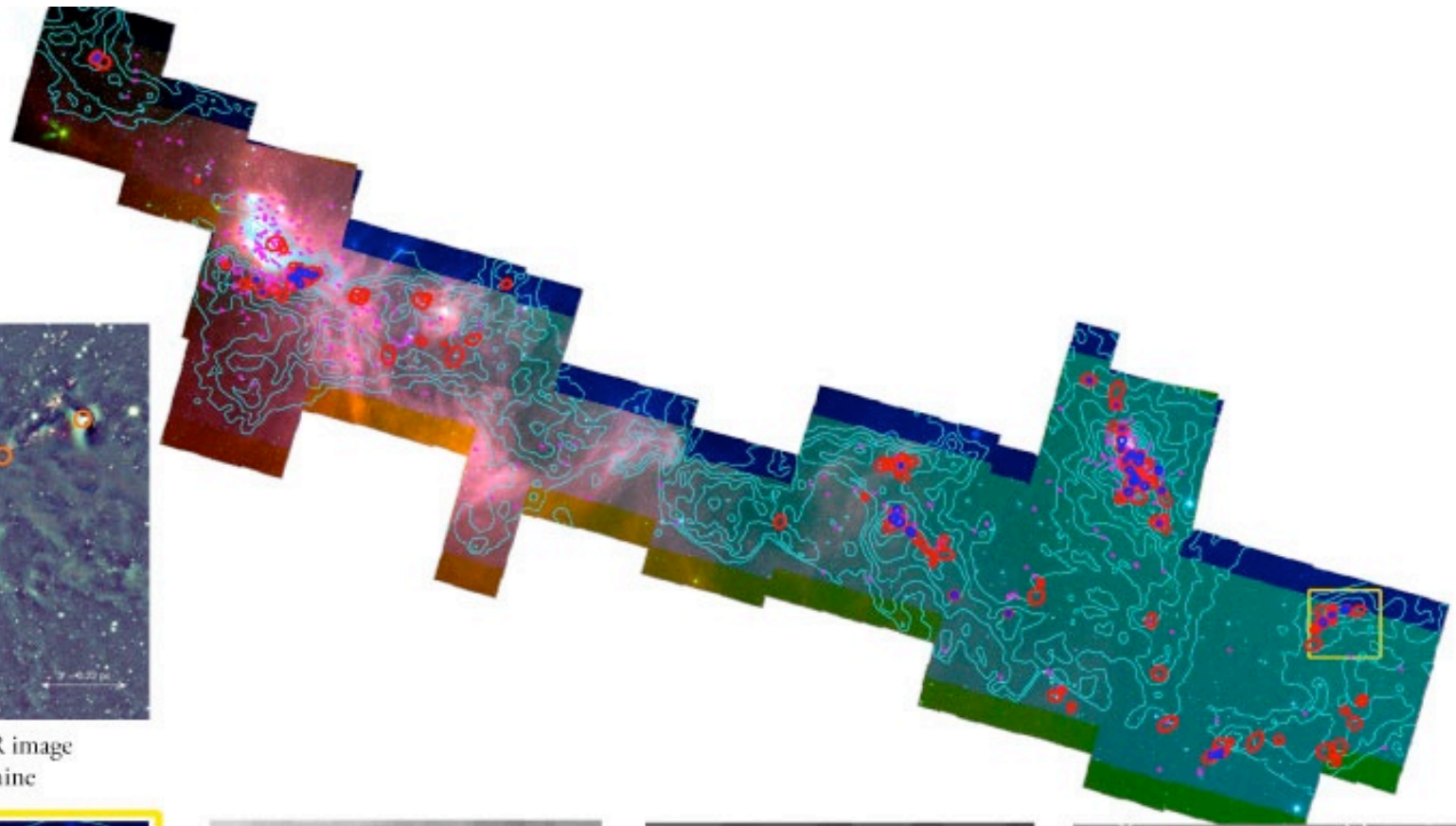
C⦿ 850 micron and 1.1 mm
clumps on a c2d IRAC
3-color image
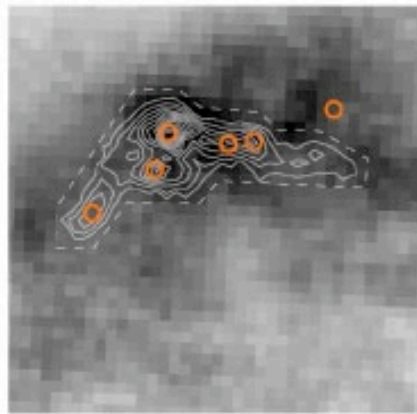
ⱮPL N₂H⁺ on ¹³CO
integrated intensity
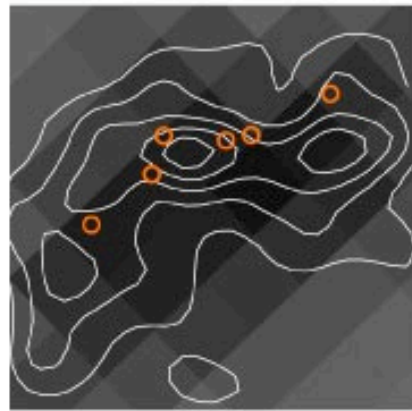
E Deep NIR Extinction on
2MASS Extinction
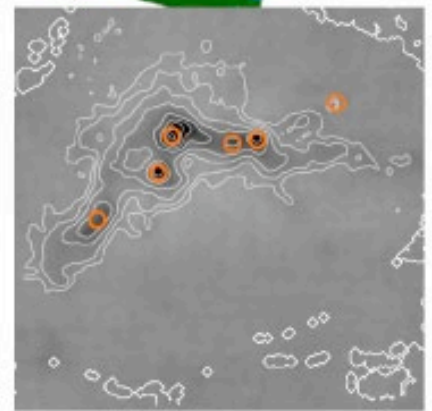
TE 1.2 mm (IRAM) on 850
micron (SCUBA)
continuum

images from Alyssa Goodman

Ralf Klessen: PI -- 04.05.2009

modelling

LEBEN

PLANETENSYSTEME

Planetenentstehung ②

Sternentstehung ①

③

PROTOSTERNE

④

Staub
Moleküle
Gas
Physikalische Chemie
Thermodynamik
Vielteilchenphysik
Atomphysik
Hydrodynamik
Komplexität
Inter Stellare Materie

ROTE RIESEN

Staub + Gas ⑤

PLANETARISCHE NEBEL

kleine Sternmassen ⑥

⑨ große Sternmassen

Gas + Staub ⑦

⑧

⑪

SUPERNOVAE

⑩

WEISSE ZWERGE
NEUTRONENSTERNE
SCHWARZE LÖCHER

(from AG meeting in Berlin 2002)

What do we need to study ISM?
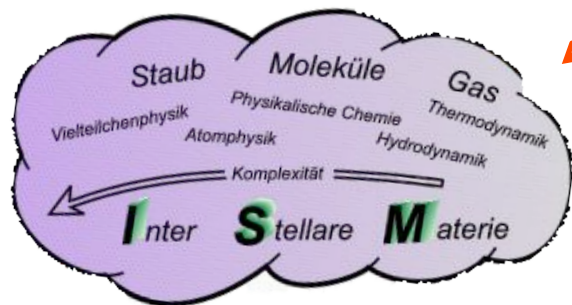
magneto-hydrodynamics
  (multi-phase, non-ideal MHD,
  turbulence)

chemistry (gas + dust, heating + cooling)

radiation (continuum + lines)

stellar dynamics
  (collisional: star clusters,
  collisionless: galaxies, DM)

stellar evolution
  (feedback: radiation, winds, SN)

+ **laboratory work**
  (reaction rates, cross sections,
  dust coagulation properties, etc.)

What do we need to study ISM?

- massive parallel codes
- particle-based: SPH with improved algorithms (XSPH with turb. subgrid model, GPM, particle splitting, MHD-SPH?)
- grid-based: AMR (FLASH, ENZO, RAMSES, Nirvana3, etc), subgrid-scale models (FEARLESS)
- BGK methods

magneto-hydrodynamics
(multi-phase, non-ideal MHD, turbulence)

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(collisional: star clusters, collisionless: galaxies, DM)

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+ **laboratory work**
(reaction rates, cross sections, dust coagulation properties, etc.)

What do we need to study ISM?

magneto-hydrodynamics
  (multi-phase, non-ideal MHD,
  turbulence)

- ever increasing chemical networks

chemistry (gas + dust, heating + cooling)

- working reduced networks for time-dependent chemistry in combination with hydro-dynamics

radiation (continuum + lines)

stellar dynamics
  (collisional: star clusters,
  collisionless: galaxies, DM)

- improved data on reaction rates (laboratory + quantum mechanical calculations)

stellar evolution
  (feedback: radiation, winds, SN)

+ **laboratory work**
  (reaction rates, cross sections,
  dust coagulation properties, etc.)

What do we need to study ISM?

magneto-hydrodynamics
  (multi-phase, non-ideal MHD,
  turbulence)

chemistry (gas + dust, heating + cooling)

- continuum vs. lines
- Monte Carlo, characteristics
- approximative methods
- combine with hydro

radiation (continuum + lines)

stellar dynamics
  (collisional: star clusters,
  collisionless: galaxies, DM)

stellar evolution
  (feedback: radiation, winds, SN)

+ laboratory work
  (reaction rates, cross sections,
  dust coagulation properties, etc.)

What do we need to study ISM?

magneto-hydrodynamics
(multi-phase, non-ideal MHD,
turbulence)

chemistry (gas + dust, heating + cooling)

radiation (continuum + lines)

- statistics: number of
  stars (collisional: 10^6,
  collisionless: 10^{10})
- transition from gas to
  stars
- binary orbits
- long-term integration

stellar dynamics
(collisional: star clusters,
collisionless: galaxies, DM)

stellar evolution
(feedback: radiation, winds, SN)

+ **laboratory work**
(reaction rates, cross sections,
dust coagulation properties, etc.)

What do we need to study ISM?

magneto-hydrodynamics
   (multi-phase, non-ideal MHD,
   turbulence)

chemistry (gas + dust, heating + cooling)

radiation (continuum + lines)

- very early phases (pre main sequence tracks)
- massive stars at late phases
- role of rotation
- primordial star formation
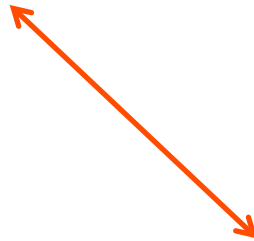
stellar dynamics
   (collisional: star clusters,
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stellar evolution
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What do we need to study ISM?

magneto-hydrodynamics
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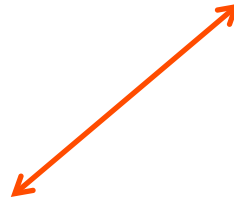**stellar evolution**
  (feedback: radiation, winds, SN)

+ **laboratory work**
  (reaction rates, cross sections,
  dust coagulation properties, etc.)

methods
need to be
combined!
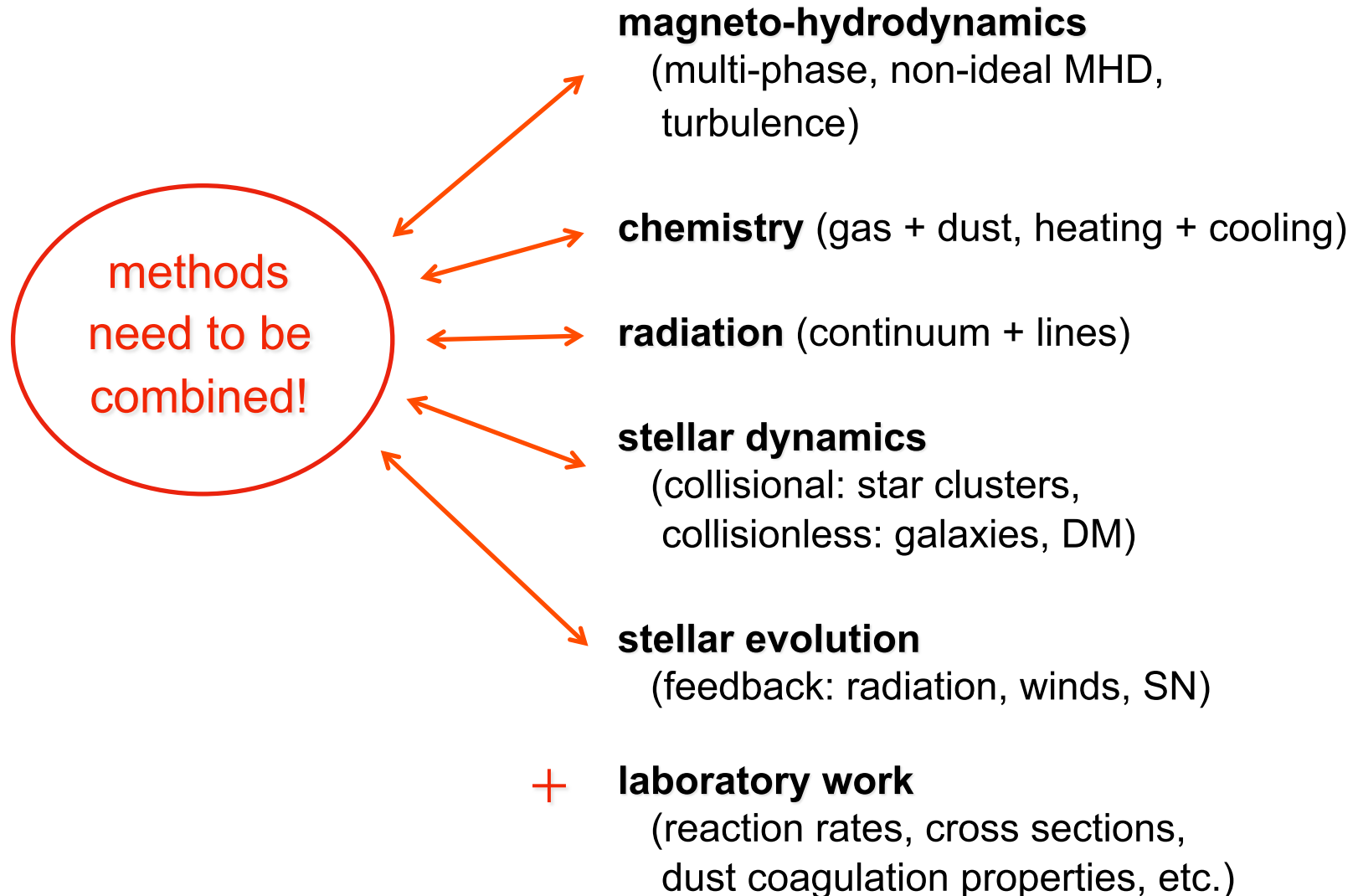
some
technical
details

Goal

- We want to understand the formation of star clusters in turbulent interstellar gas clouds.

--> We want to describe the transition from a hydro-dynamic system (the self-gravitating gas cloud) to one that is dominated by (collisional) stellar dynamics (the final star cluster).

- How can we do that?

Numerical approach I

- Problem of star formation is very complex. It involves many scales (10^7 in length, and 10^{20} in density), and many physical processes → *NO* analytic solution

  → NUMERICAL APPROACH

- BUT, we need to…

  - solve the MHD equations in 3 dimensions

  - solve Poisson's equation (self-gravity)

  - follow the full turbulent cascade (in the ISM + in stellar interior)

  - follow chemical evolution (time-dependent chemical network)

  - include heating / cooling processes (internal degrees of freedom)

  - treat radiation transfer

IMPOSSIBLE

Numerical approach II

- Simplify!
  Divide problem into little bits and pieces….

- **GRAVOTURBULENT CLOUD FRAGMENTATION**

- We try to…

  - solve the HD equations in 3 dimensions

  - solve Poisson's equation (self-gravity)

  - include a (humble) approach to supersonic turbulence

  - include simple chemical network & tabulated cooling functions

  - follow collapse: include "sink particles"
    (this will "handle" our subgrid-scale physics)

the equations of hydrodynamics

○ hydrodynamics ≡ book keeping problem

One must keep track of the 'change' of a fluid element due to various physical processes acting on it. How do its 'properties' evolve under the influence of compression, heat sources, cooling, etc.?

○ Eulerian vs. Lagrangian point of view

Eulerian

Lagrangian
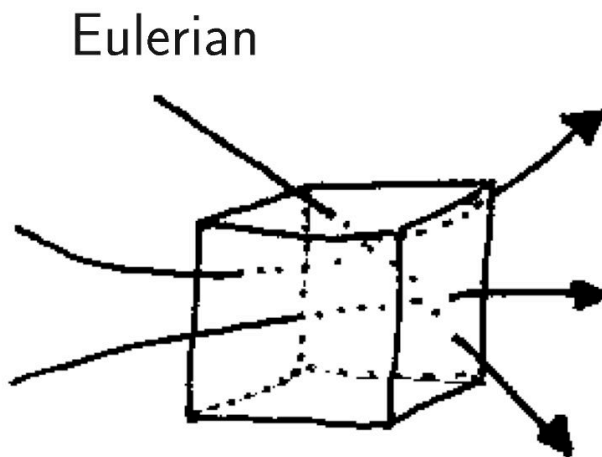
consider spatially fixed volume element
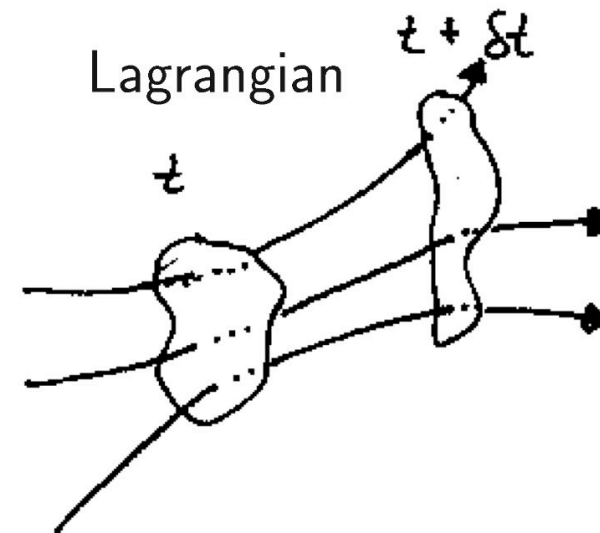
following motion of fluid element

○ hydrodynamic equations $=$ set of equations for the five conserved quantities $(\rho, \rho\vec{v}, \rho\vec{v}^{\,2}/2)$ plus closure equation (plus transport equations for 'external' forces if present, e.g. gravity, magnetic field, heat sources, etc.)

○ equations of hydrodynamics

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho = -\rho \vec{\nabla} \cdot \vec{v} \qquad \text{(continuity equation)}$$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \eta \vec{\nabla}^2 \vec{v} + \left(\zeta + \frac{\eta}{3}\right) \vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$

(Navier-Stokes equation)

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \vec{\nabla} \epsilon = T \frac{ds}{dt} - \frac{p}{\rho} \vec{\nabla} \cdot \vec{v} \qquad \text{(energy equation)}$$

$$\vec{\nabla}^2 \phi = 4\pi G \rho \qquad \text{(Poisson's equation)}$$

$$p = \mathcal{R} \rho T \qquad \text{(equation of state)}$$

○ equations of hydrodynamics

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho = -\rho \vec{\nabla} \cdot \vec{v}$$ (continuity equation)

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi$$ *often replaced by artificial / numerical viscosity*

(Navier-Stokes equation)

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \vec{\nabla} \epsilon = T \frac{ds}{dt} - \frac{p}{\rho} \vec{\nabla} \cdot \vec{v}$$ (energy equation)

$$\vec{\nabla}^2 \phi = 4\pi G \rho$$ (Poisson's equation)

$$p = \mathcal{R} \rho T$$ (equation of state)

$$\vec{F}_B = -\vec{\nabla}\frac{\vec{B}^2}{8\pi} + \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B} \quad \text{(magnetic force)}$$

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) \qquad \text{(Lorentz equation)}$$

ρ = density, $\vec{v}$ = velocity, p = pressure, ϕ = gravitational poten-tial, ζ and η viscosity coefficients, $\epsilon = \rho\vec{v}^2/2$ = kinetic energy density, T = temperature, s = entropy, $\mathcal{R}$ = gas constant, $\vec{B}$ = magnetic field (cgs units)

○ mass transport – continuity equation

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}\rho = -\rho\vec{\nabla} \cdot \vec{v}$$

(conservation of mass)

○ transport equation for momentum – Navier Stokes equation

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\phi + \eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$

momentum change due to

→ pressure gradients: $\quad (-\rho^{-1}\vec{\nabla}p)$

→ (self) gravity: $\quad -\vec{\nabla}\phi$

→ viscous forces (internal friction, contains $\mathrm{div}(\partial v_i/\partial x_j)$ terms):
$\eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla} \cdot \vec{v})$

(conservation of momentum, general form of momentum transport: $\partial_t(\rho v_i) = -\partial_j \Pi_{ij}$)

○ transport equation for internal energy

$$\frac{d\epsilon}{dt} = \frac{\partial\epsilon}{\partial t} + \vec{v}\cdot\vec{\nabla}\epsilon = T\frac{ds}{dt} - \frac{p}{\rho}\vec{\nabla}\cdot\vec{v}$$

$\rightarrow$ follows from the thermodynamic relation $d\epsilon = T\,ds - p\,dV = T\,ds + p/\rho^2 d\rho$ which described changes in ϵ due to entropy changed and to volume changes (compression, expansion)

$\rightarrow$ for adiabatic gas the first term vanishes ($s =$constant)

$\rightarrow$ heating sources, cooling processes can be incorporated in ds

(conservation of energy)

○ closure equation – equation of state

→ general form of equation of state $p = p(T, \rho, ...)$

→ ideal gas: $p = \mathcal{R}\rho T$

→ special case – isothermal gas: $p = c_s^2 T$ (as $\mathcal{R}T = c_s^2$)

Note:
- in reality, computing the EOS is VERY complex!
- depends on detailed *balance* between *heating* and *cooling*
- these depend on *chemical composition* (which atomic and molecular species, dust)
- and on the ability to radiate away „cooling lines" and black body radiation
  --> problem of *radiation transfer* (see, e.g., IPAM III)

two approaches to hydrodynamics

- Eulerian schemes

  - classical grid-based approach

  - cartesian grid with adaptive mesh refinement (AMR)

  - many codes: FLASH, RAMSES, ENZO, Pluto

- Lagrangian schemes

  - particle-based approach:
    smoothed particle hydrodynamics (SPH)

  - some codes: GADGET, Gasoline, Exeter code

astrophysical jets with FLASH ...
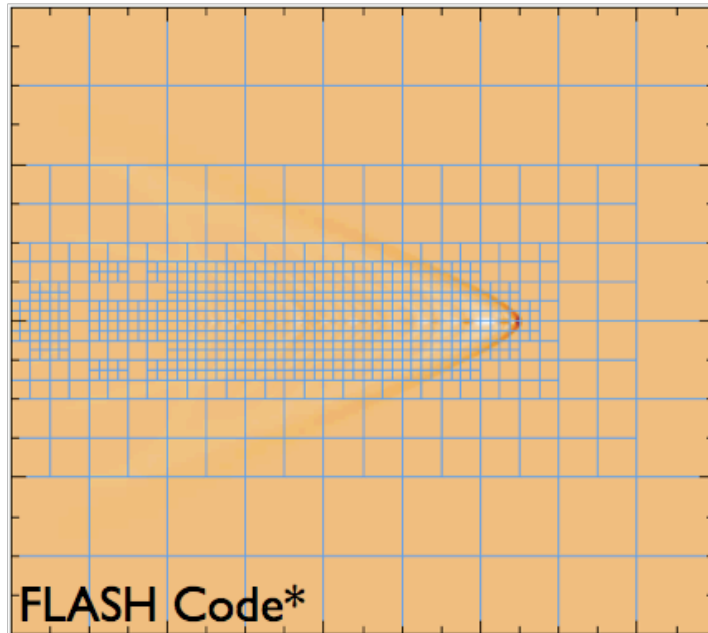
FLASH Code*

*Alliance Center for Astrophysical Thermonuclear Flashes (ASC), University of Chicago

- momentum injection from the side of the simulation box
- variation of: jet speed, injection duration density, w/wo magnetic fields, 2D/3D, EOS

SUMMARY OF THE PARAMETERS OF THE SIMULATIONS

Run	Dim.	Mach	Duration	δ	Clump?	MHD?
M5c...	2D	5	∞	1	No	No
M5t (g1.4).................................	2D	5	1.3	1	No	No
M10c...	2D	10	∞	1	No	No
M20tCl......................................	2D	20	0.6	1	Yes, $\delta_{cl} = 10$	No
M5t3D.......................................	3D	5	1.3	1	No	No
M10tOd3D.................................	3D	10	1.3	10	No	No
M10tMpll3D..............................	3D	10	1.3	1	No	Yes, parallel field
M10tMpe3D...............................	3D	10	1.3	1	No	Yes, perpendicular field

Banerjee, Klessen, Fendt (2008)

... as model for Orion cloud

(Spitzer: Megeath et al.)

Orion A

Ophiuchus

Parsecs

Parsecs

Bonnell & Clark 2008

Ralf Klessen: PI -- 04.05.2009
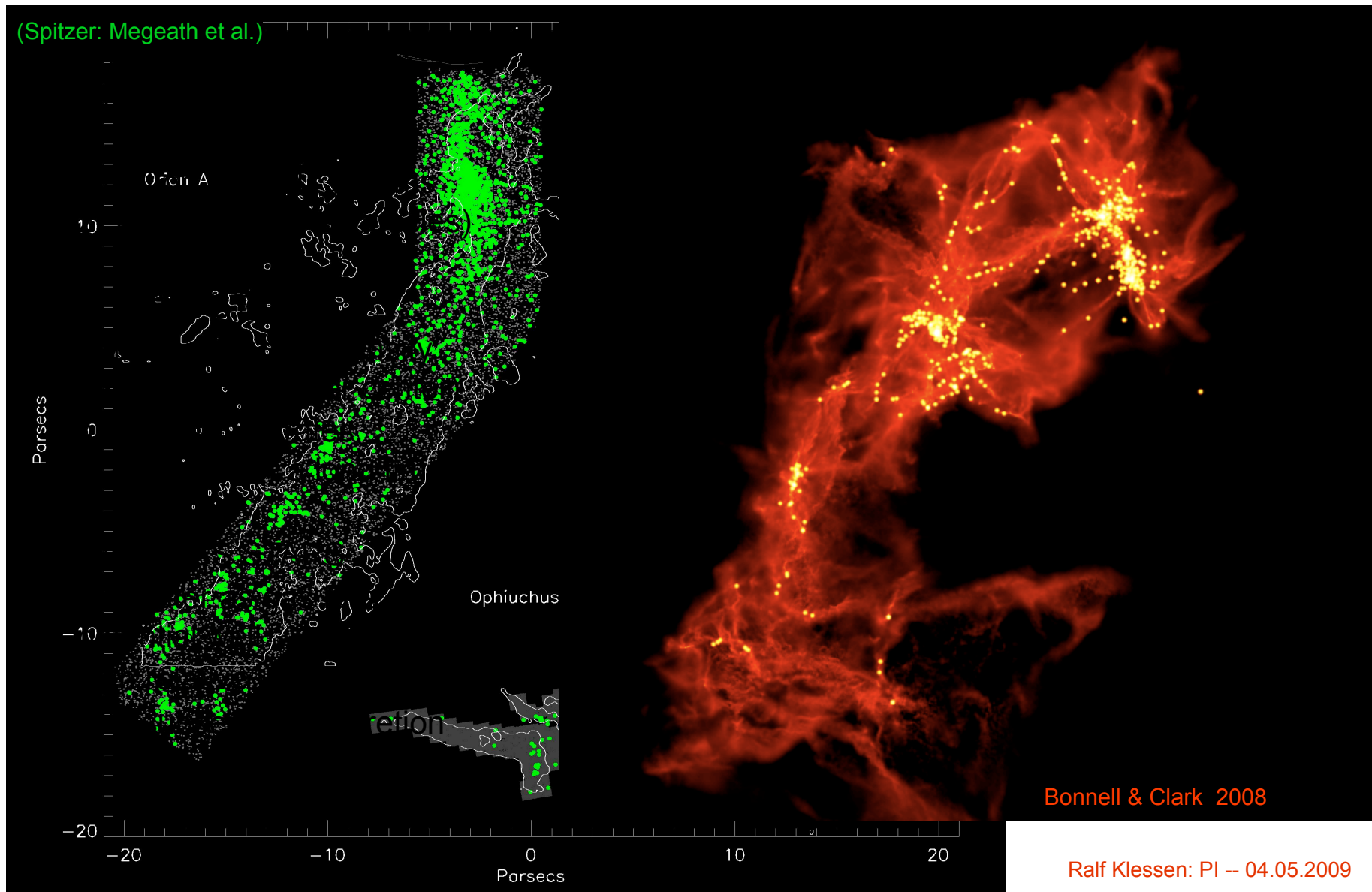
turbulence

Properties of turbulence

- laminar flows turn *turbulent* at *high Reynolds* numbers

$$\text{Re} = \frac{\text{advection}}{\text{dissipation}} = \frac{VL}{\nu}$$

V= typical velocity on scale L, ν = viscosity,   Re > 1000

- *vortex streching* --> turbulence is *intrinsically anisotropic*
  (only on large scales you *may* get homogeneity & isotropy in a statistical sense; see Landau & Lifschitz, Chandrasekhar, Taylor, etc.)

  (ISM turbulence: shocks & B-field cause additional inhomogeneity)

Turbulent cascade

$\log E$

$k^{-5/3}$

transfer

L^{-1}

η_K^{-1}

$\log k$

energy input scale

energy dissipation scale

inertial range:
scale-free behavior of turbulence

„size" of inertial range:

$$\frac{L}{\eta_K} \approx \mathrm{Re}^{3/4}$$

Turbulent cascade

Shock-dominated turbulence

$\log E$

k^{-2}

transfer

L^{-1}

η_K^{-1}

$\log k$

energy
input
scale

energy
dissipation
scale

inertial range:
scale-free behavior
of turbulence

„size" of inertial range:

$$\frac{L}{\eta_K} \approx \mathrm{Re}^{3/4}$$

Turbulent cascade in ISM

energy source & scale
NOT known
(supernovae, winds,
spiral density waves?)

$\sigma_{rms} \ll 1$ km/s
$M_{rms} \leq 1$
$L \approx 0.1$ pc

dissipation scale not known
(ambipolar diffusion,
molecular diffusion?)

Large-eddy simulations

- We use **LES** to model the large-scale dynamics

- Principal problem: only large scale flow properties

  - Reynolds number: Re = LV/ν **(Re_nature >> Re_model)**

  - dynamic range much smaller than true physical one

  - need *subgrid model* (in our case simple: only dissipation)

  - but what to do for more complex when processes on subgrid scale determine large-scale dynamics
    (chemical reactions, nuclear burning, etc)

  - Turbulence is "space filling" --> difficulty for AMR (don't know what criterion to use for refinement)

- How *large* a Reynolds number do we need to catch basic dynamics right?

dynamic range
of model

true dynamic range

$\log E$

L^{-1} η_K^{-1}
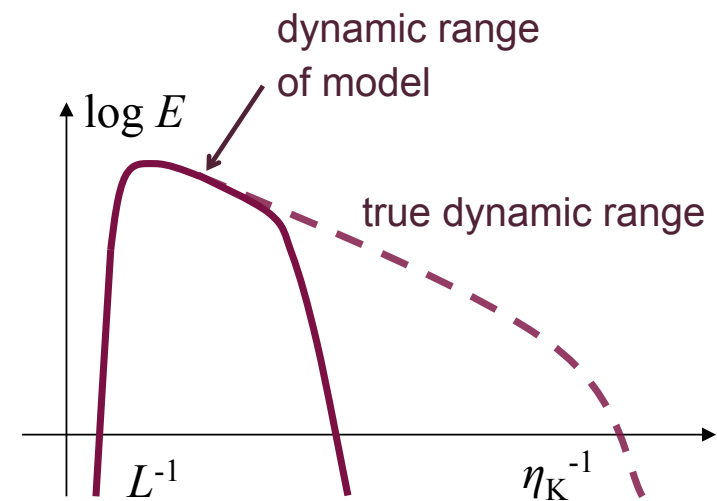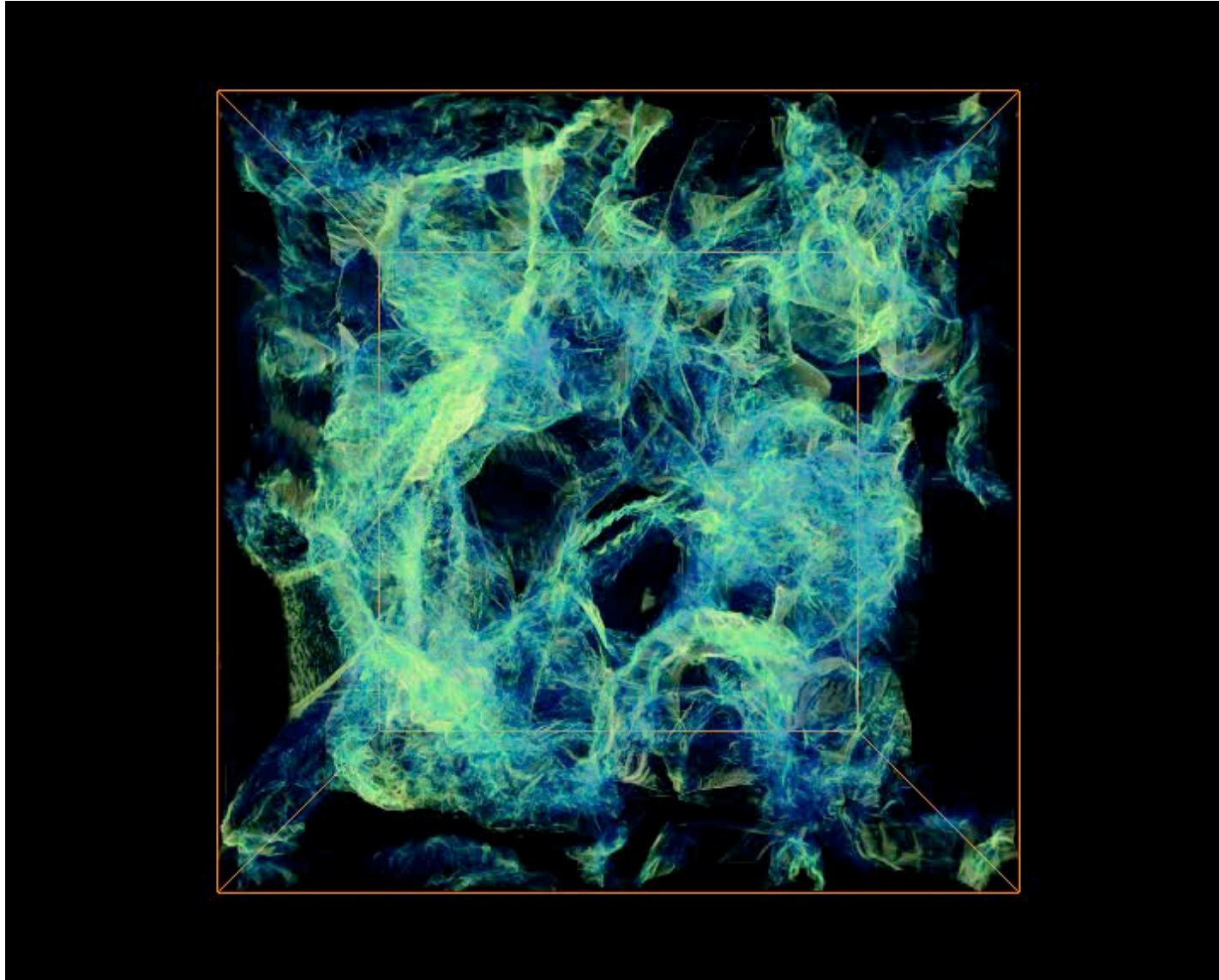
(movie from Christoph Federrath)

compressive vs. rotational driving

- statistical characteristics of turbulence depend strongly on „type" of driving

- example: dilatational vs. solenoidal driving

- question: what drives ISM turbulence on different scales?
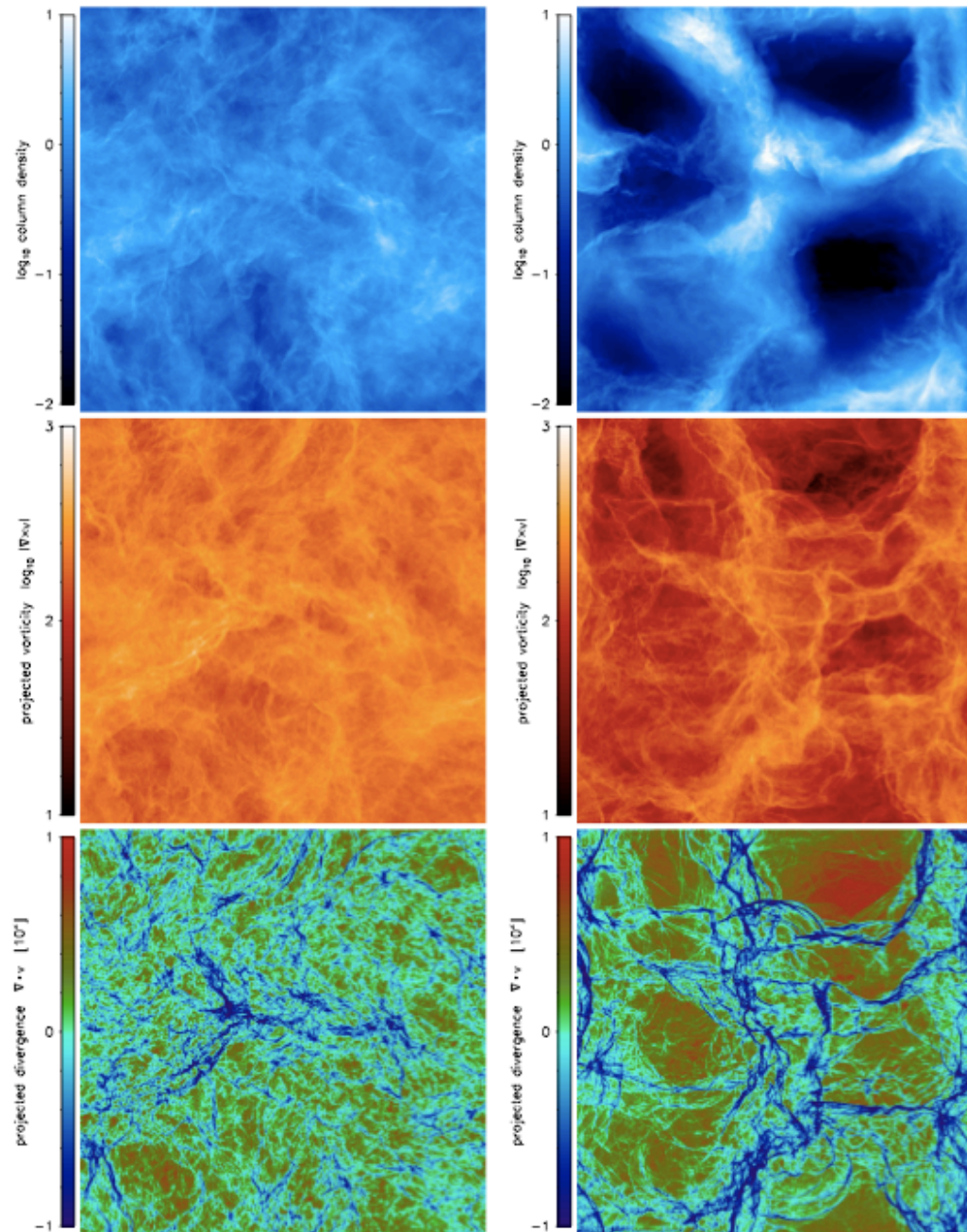
column density

projected vorticity

projected divergence

Fig. 1. Maps showing density, vorticity and divergence in projection along the z-axis at time $t = 2\,T$ as an example for the regime of statistically fully developed compressible turbulence for solenoidal forcing (*left*) and compressive forcing (*right*). *Top panels:* Column density fields in units of the mean column density. Both maps show three orders of magnitude in column density with the same scaling and magnitudes for direct comparison. *Middle panels:* Projections of the modulus of the vorticity $|\nabla \times v|$. Regions of intense vorticity appear to be elongated filamentary structures often coinciding with positions of intersecting shocks. *Bottom panels:* Projections of the divergence of the velocity field $\nabla \cdot v$ showing the positions of shocks. Negative divergence corresponds to compression, while positive divergence corresponds to rarefaction.

dilatational vs. solenoidal
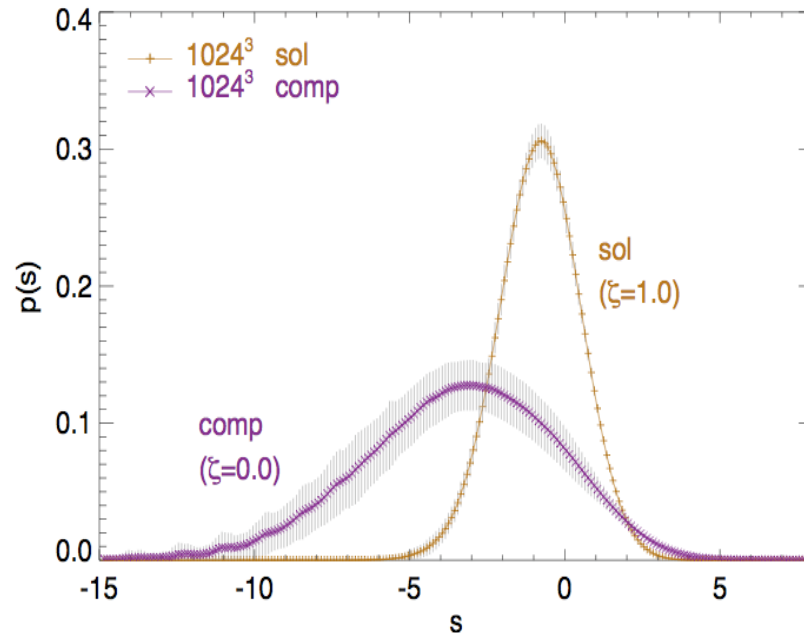
FIG. 2.— Volume-weighted density PDFs $p_s(s)$ in linear scaling where $s = \ln(\rho/\rho_0)$. The PDF obtained by compressive forcing (comp, $\zeta = 0.0$) is much broader compared to the solenoidal one (sol, $\zeta = 1.0$) at the same rms Mach number. The peak is shifted due to mass conservation (Vázquez-Semadeni 1994). Gray error bars indicate 1-sigma temporal fluctuations of the PDF. A sample of $\sim 10^{11}$ datapoints contribute to each PDF.

Federrath, Klessen, Schmidt (2008a)

- density pdf depends on "dimensionality" of driving

  - relation between width of pdf and Mach number

$$\sigma_\rho / \rho_0 = b\mathcal{M}$$

  - with b depending on ζ via

$$b = 1 + \left[\frac{1}{D} - 1\right]\zeta = \begin{cases} 1 - \frac{2}{3}\zeta & , \text{ for } D = 3 \\ 1 - \frac{1}{2}\zeta & , \text{ for } D = 2 \\ 1 & , \text{ for } D = 1 \end{cases}$$

  - with ζ being the ratio of dilatational vs. solenoidal modes:

$$\mathcal{P}_{ij}^\zeta = \zeta \mathcal{P}_{ij}^\perp + (1-\zeta)\mathcal{P}_{ij}^\parallel = \zeta\delta_{ij} + (1-2\zeta)\frac{k_i k_j}{|k|^2}$$

dilatational vs. solenoidal
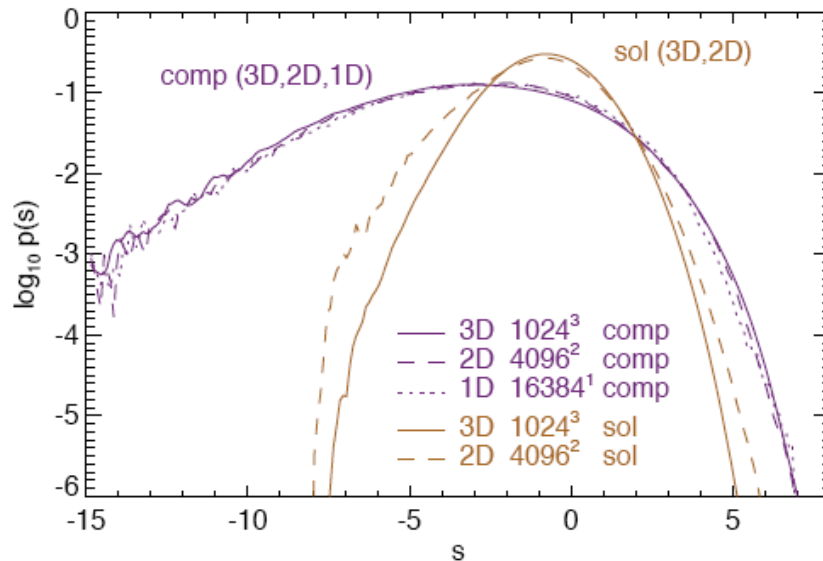
FIG. 3.— Volume-weighted density PDFs $p(s)$ obtained from 3D, 2D and 1D simulations with compressive forcing and from 3D and 2D simulations using solenoidal forcing. Note that in 1D, only compressive forcing is possible as in the study by Passot & Vázquez-Semadeni (1998). As suggested by eq. (5), compressive forcing yields almost identical density PDFs in 1D, 2D and 3D with $b \sim 1$, whereas solenoidal forcing leads to a density PDF with $b \sim 1/2$ in 2D and with $b \sim 1/3$ in 3D.

Federrath, Klessen, Schmidt (2008a)

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dilatational vs. solenoidal
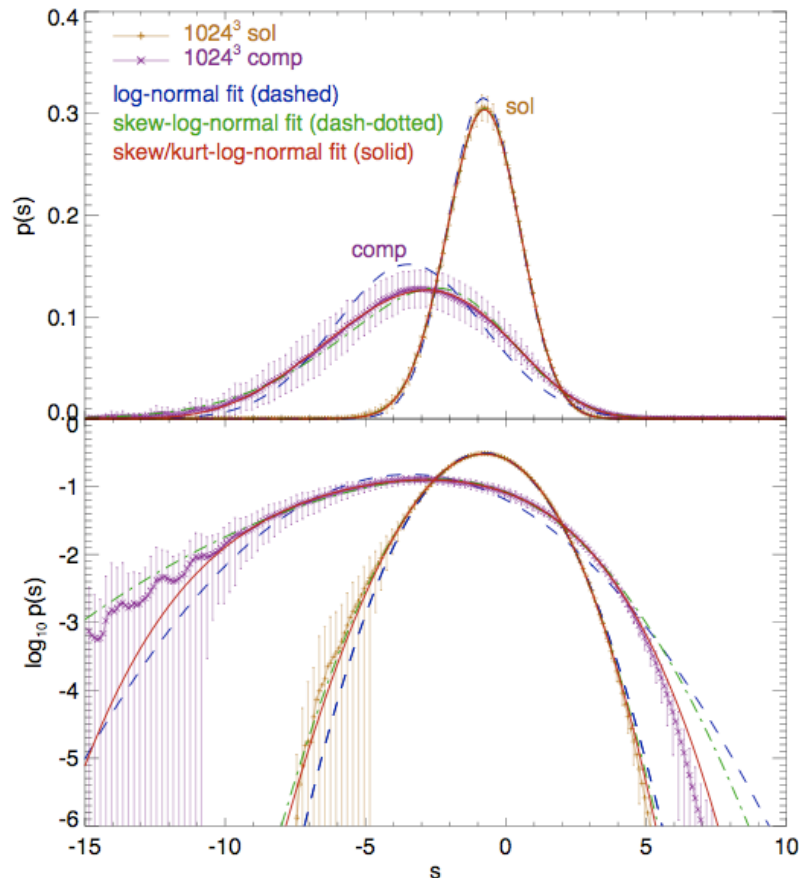
good fit needs 3rd and 4th moment of distribution!

Federrath, Klessen, Schmidt (2008b)

- density pdf depends on "dimensionality" of driving
  → is that a problem for the Krumholz & McKee model of the SF efficiency?

- density pdf of compressive driving is *NOT log-normal*
  → is that a problem for the Padoan & Nordlund IMF model?

- most "physical" sources should be *compressive* (convergent flows from spiral shocks or SN)

dilatational vs. solenoidal

- density power spectrum differs between dilatational and solenoidal driving!

  → dilatational driving leads to break at sonic scale!

- can we use that to determine driving sources from observations ?
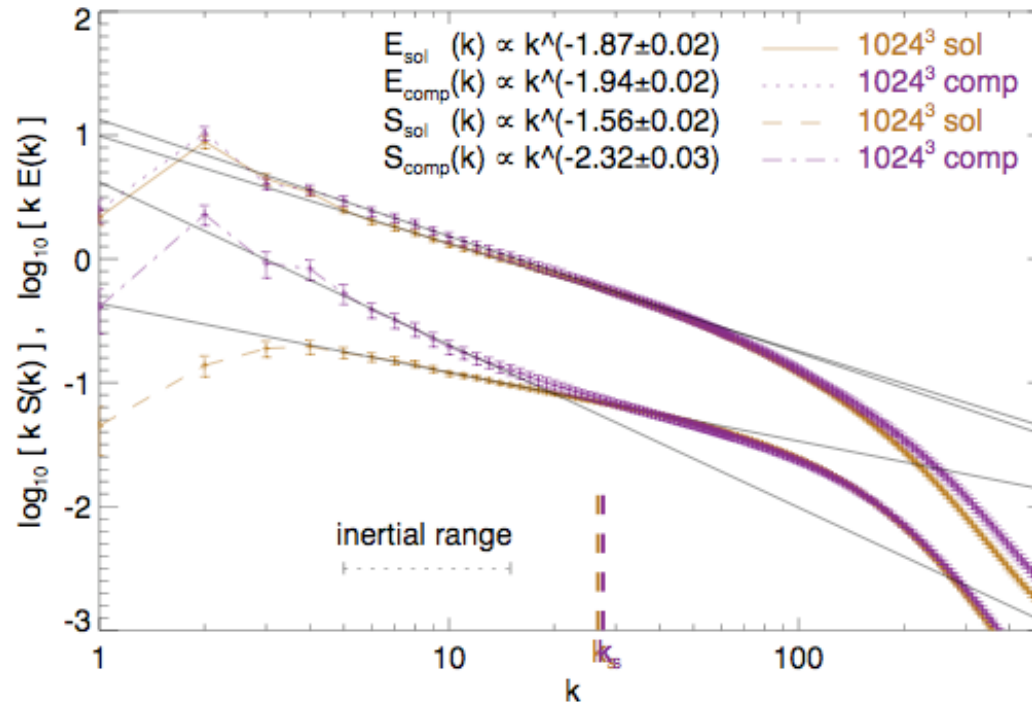
compensated density spectrum kS(k) shows clear break at sonic scale. below that shock compression no longer is important in shaping the power spectrum ...

Federrath, Klessen, Schmidt (2008b)

dilatational vs. solenoidal
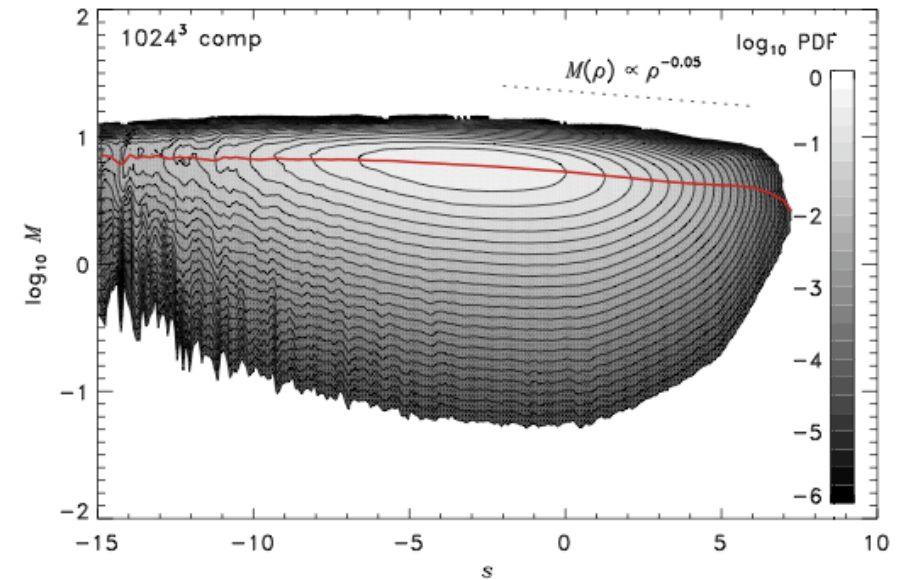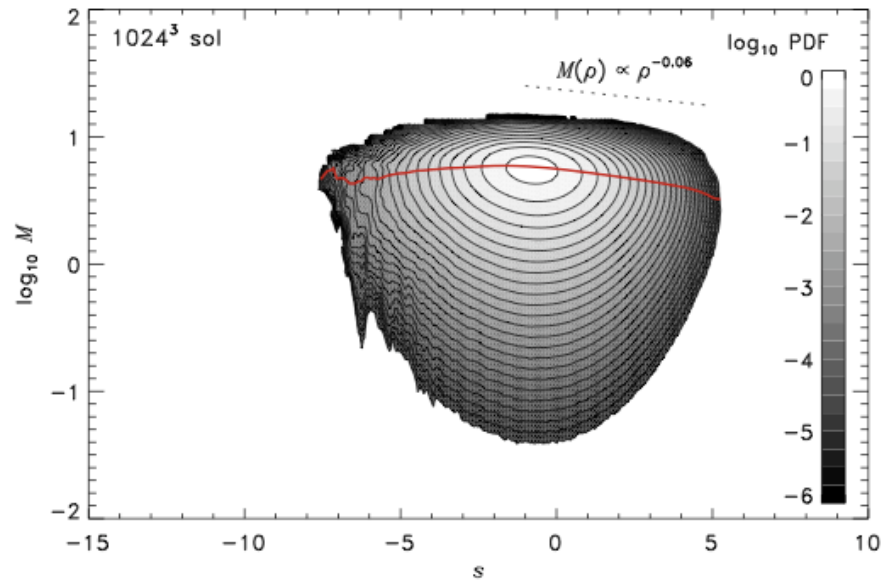
there is a weak *log density – log Mach number* relation ...

Federrath, et al. (2009)
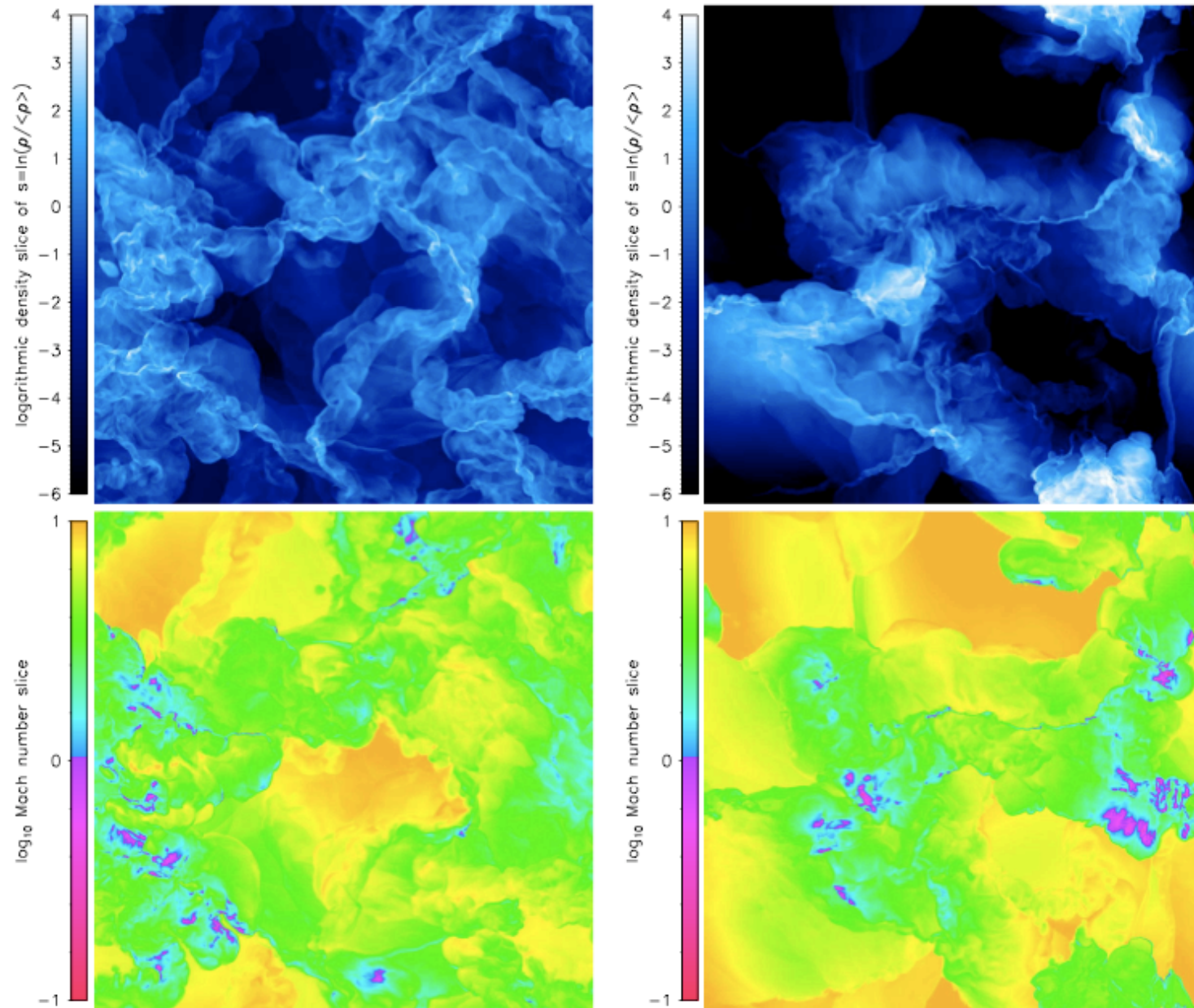
Fig. 14. z-slices through the local density (*top panels*) and Mach number fields (*bottom panels*) at $z = 0$ and $t = 2\,T$ for solenoidal forcing (*left*), and compressive forcing (*right*). Regions with subsonic velocity dispersions (Mach < 1) are distinguished from regions with supersonic velocity dispersions (Mach > 1) in the colour scheme used. The correlation between density and Mach number is quite weak. However, as shown in Fig. 4, high-density regions exhibit smaller Mach numbers on average.

dilatational vs. solenoidal
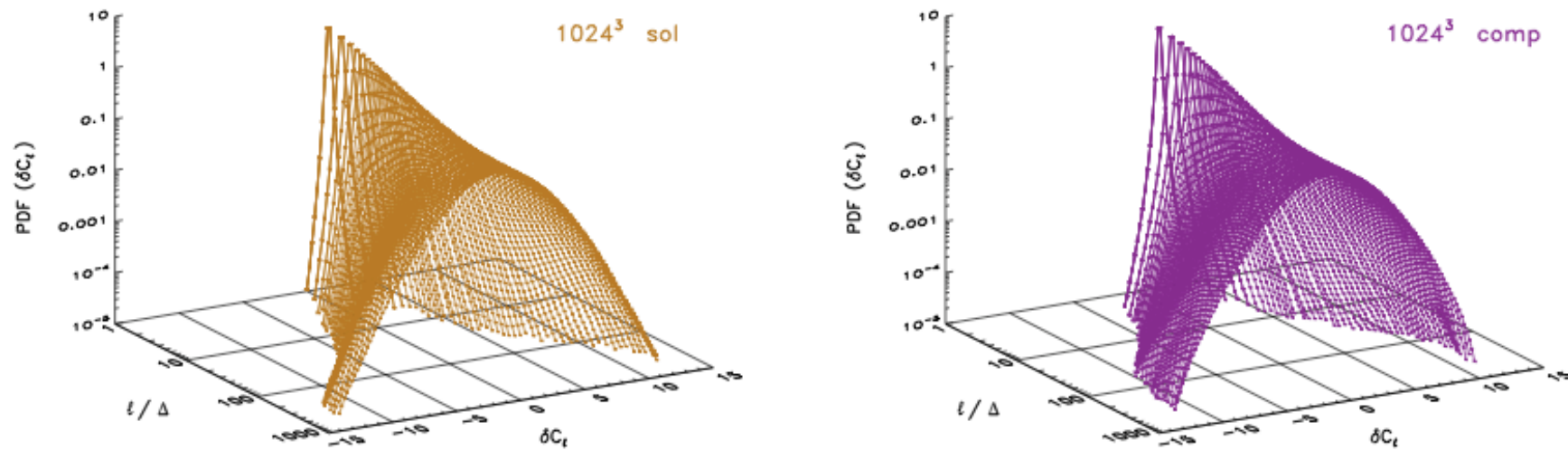
Fig. 7. PDFs of centroid velocity increments computed using equations (18) and (19) are shown as a function of the lag ℓ in units of grid cells $\Delta = L/1024$ for solenoidal forcing (*left*) and compressive forcing (*right*). The PDFs are very close to Gaussian distributions for large lags ℓ, whereas for small lags, they develop exponential tails, which is a manifestation of intermittency (e.g., Hily-Blant et al. 2008).

Federrath, et al. (2009)

summary

Summary I

- interstellar gas is highly *inhomogeneous*

  - *thermal instability*
  - *gravitational instability*
  - *turbulent compression* (in shocks $\delta\rho/\rho \propto M^2$; in atomic gas: $M \approx 1...3$)

- cold *molecular clouds* can form rapidly in high-density regions at *stagnation points* of *convergent large-scale flows*

  - chemical *phase transition*: atomic → molecular
  - process is *modulated* by large-scale *dynamics* in the galaxy

- inside *cold clouds:* turbulence is highly supersonic ($M \approx 1...20$)
  → *turbulence* creates density contrast, *gravity* selects for collapse

  ─────────→ *GRAVOTUBULENT FRAGMENTATION*

- *turbulent cascade:* local compression *within* a cloud provokes collapse → formation of individual *stars* and *star clusters*

- *star cluster:* gravity dominates in large region (--> competitive accretion)

(e.g. Mac Low & Klessen, 2004, Ballesteros-Paredes et al. 2006, McKee & Ostriker 2007)
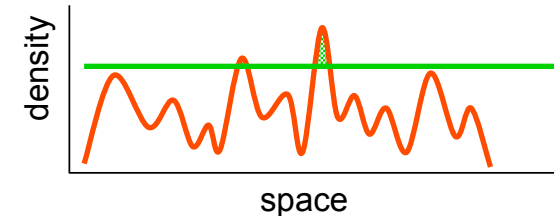
Summary II

- *thermodynamic response* (EOS) determines fragmentation behavior

  - characteristic stellar mass from fundamental atomic and molecular parameters
    --> explanation for quasi-universal IMF?

- *stellar feedback* is important

  - accretion heating may reduce degree of fragmentation

  - ionizing radiation will set efficiency of star formation

- *CAVEATS*:

  - star formation is *multi-scale, multi-physics* problem --> VERY difficult to model

  - in simulations: very small turbulent inertial range (Re < 1000)

  - can we use EOS to describe thermodynamics of gas, or do we need time-dependent chemical network and radiative transport?

  - stellar feedback requires (at least approximative) radiative transport, most numerical calculations so far have neglected that aspect

additional stuff!

Ralf Klessen: PI -- 04.05.2009

FLASH
(AMR MHD)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{9.30}$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B}) + \nabla p_* = \rho \mathbf{g} + \nabla \cdot \tau \tag{9.31}$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\mathbf{v}(\rho E + p_*) - \mathbf{B}(\mathbf{v} \cdot \mathbf{B})) = \rho \mathbf{g} \cdot \mathbf{v} + \nabla \cdot (\mathbf{v} \cdot \tau + \sigma \nabla T) + \nabla \cdot (\mathbf{B} \times (\eta \nabla \times \mathbf{B})) \tag{9.32}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = -\nabla \times (\eta \nabla \times \mathbf{B}) \tag{9.33}$$

where

$$p_* = p + \frac{B^2}{2}, \tag{9.34}$$

$$E = \frac{1}{2}v^2 + \varepsilon + \frac{1}{2}\frac{B^2}{\rho}, \tag{9.35}$$

$$\tau = \mu \left((\nabla \mathbf{v}) + (\nabla \mathbf{v})^{\mathsf{T}} - \frac{2}{3}(\nabla \cdot \mathbf{v}) \right) \tag{9.36}$$

are total pressure, specific total energy and viscous stress respectively. Also, ρ is the density of a magnetized fluid, $\mathbf{v}$ is the fluid velocity, p is the fluid thermal pressure, T is the temperature, ε is the specific internal energy, $\mathbf{B}$ is the magnetic field, $\mathbf{g}$ is the body force per unit mass, for example due to gravity, μ is the viscosity, σ is the heat conductivity, and η is the resistivity. The thermal pressure is a scalar quantity, so that the code is suitable for simulations of ideal plasmas

With viscosity, the momentum is the quantity that is diffused

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) + \nabla P = \rho \mathbf{g} + \nabla \cdot (\nu \nabla \mathbf{v}) .$$

(9.15)

The fluxes are calculated as in the thermal diffusion, although there is one flux for each velocity component. There are two viscosity modules in `source/materials/viscosity`. In `viscosity/constant`, the viscosity ν is assumed constant and set by the runtime parameter `diff_visc_nu`. `viscosity/spitzer` uses a viscosity computed according to the classical Spitzer (1962) prescription. Total energy fluxes are not updated by viscosity, since it is assumed the effect is small.
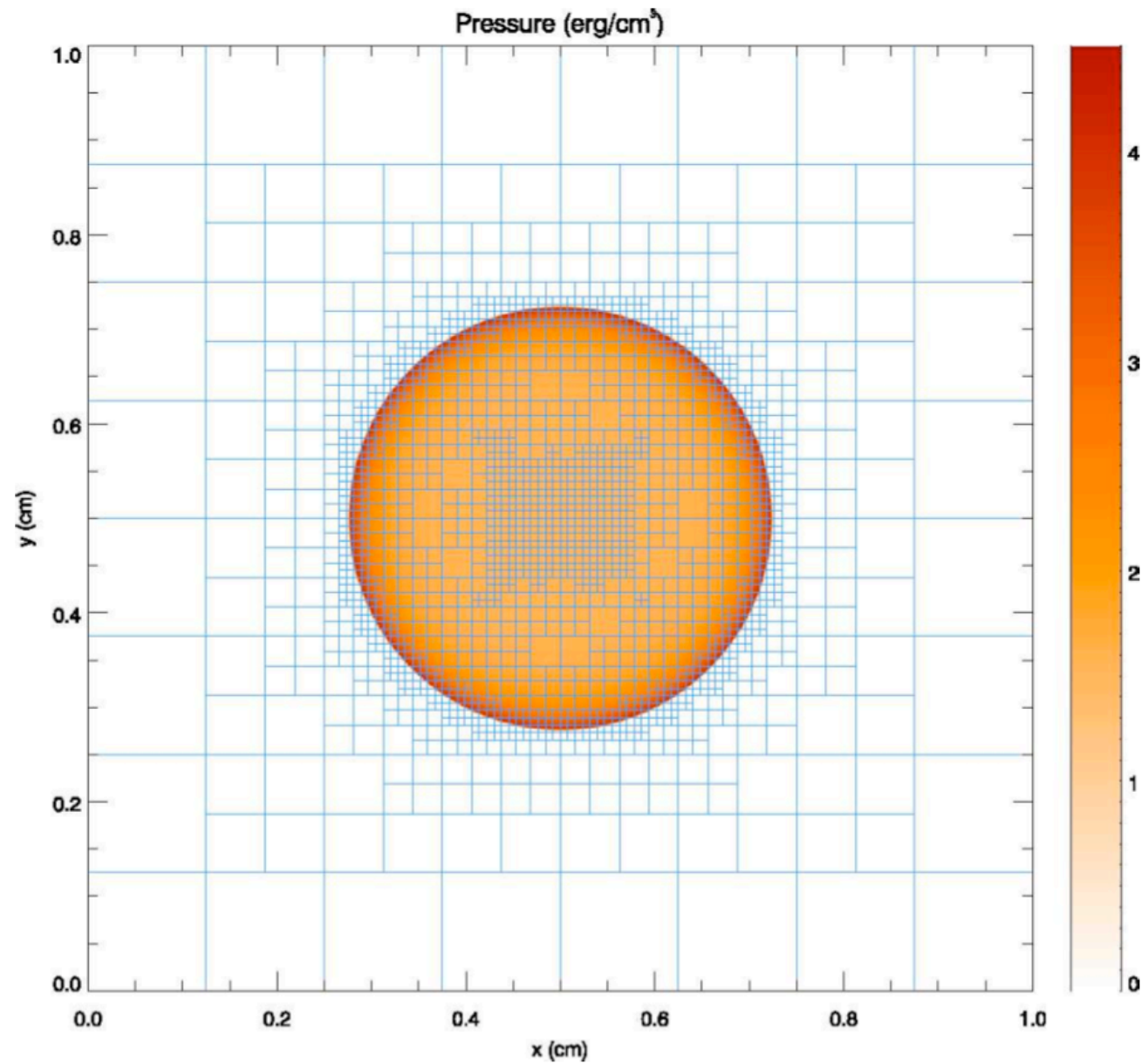
Sedov solution of explosion
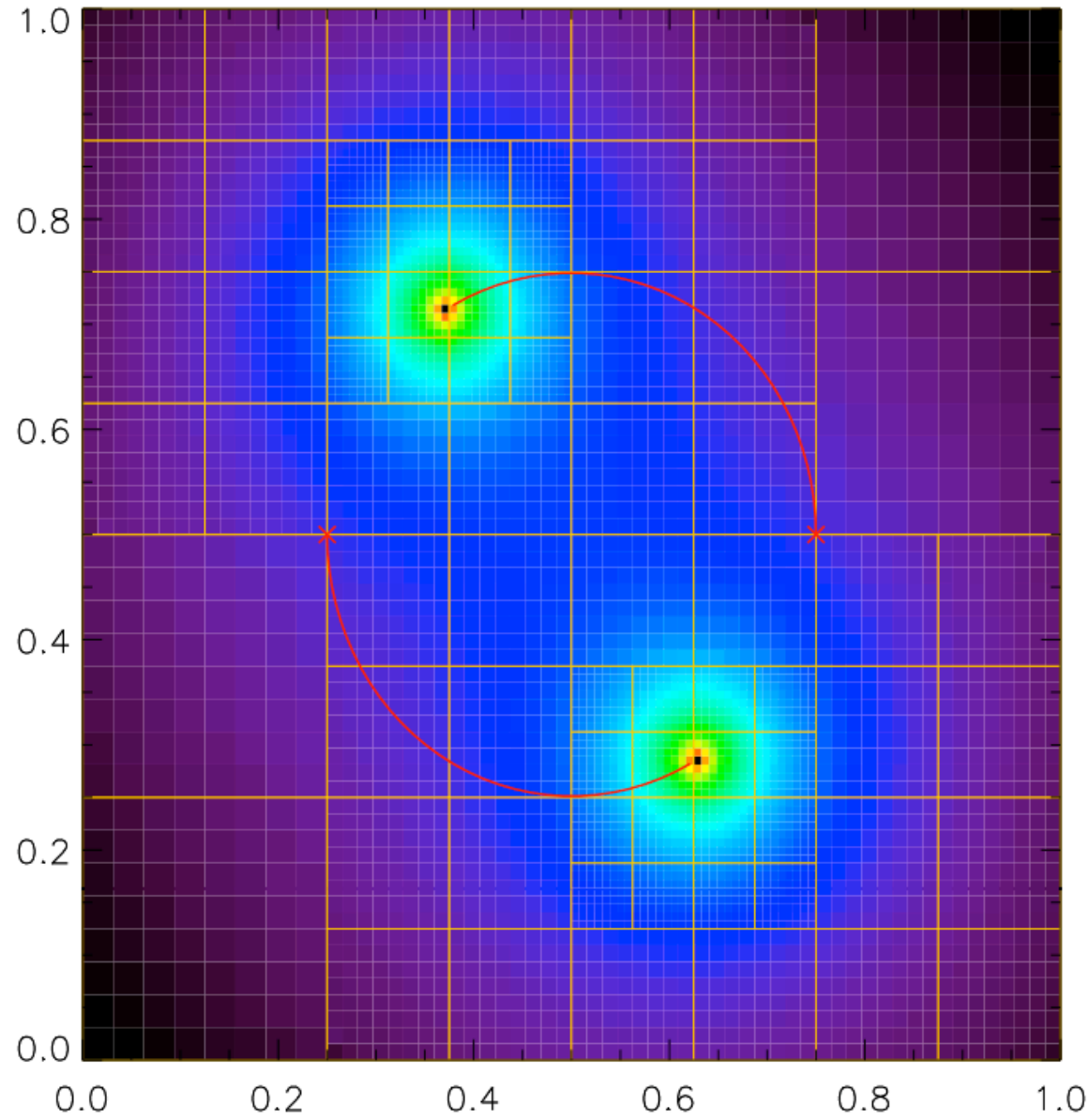
Two orbiting gas blobs (gravity test)

SPH

concept of SPH

- ○ 'invented' independently by Lucy (1977) and Gingold & Monaghan (1977)

- ○ originally proposed as Monte Carlo approach to calculate the time evolution of gaseous systems

- ○ more intuitively understood as interpolation scheme:

> The fluid is represented by an ensemble of particles i, each carrying mass m_i, momentum $m_i \vec{v}_i$, and hydrodynamic properties (like pressure p_i, temperature T_i, internal energy ϵ_i, entropy s_i, etc.). The time evolution is governed by the equation of motion plus additional equations to modify the hydrodynamic properties of the particles. Hydrodynamic observables are obtained by a local averaging process.

properties of local averaging processes

○ local averages $\langle f(\vec{r}) \rangle$ for any quantity $f(\vec{r})$ can be obtained by convolution with an appropriate smoothing function $W(\vec{r}, \vec{h})$:

$$\langle f(\vec{r}) \rangle \equiv \int f(\vec{r}') W(\vec{r} - \vec{r}', \vec{h}) \, d^3 r' \ .$$

the function $W(\vec{r}, \vec{h})$ is called smoothing kernel

○ the kernel must satisfy the following two conditions:

$$\int W(\vec{r}, \vec{h}) \, d^3 r = 1 \quad \text{and} \quad \langle f(\vec{r}) \rangle \longrightarrow f(\vec{r}) \ \text{for} \ \vec{h} \to 0$$

the kernel W therefore follows the same definitions as Dirac's delta function $\delta(\vec{r})$: $\quad \lim_{h \to 0} W(\vec{r}, h) = \delta(\vec{r})$.

○ most SPH implementations use spherical kernel functions

$$W(\vec{r}, \vec{h}) \equiv W(r, h) \quad \text{with} \quad r = |\vec{r}| \ \text{and} \ h = |\vec{h}|.$$

(one could also use triaxial kernels, e.g. Martel et al. 1995)

properties of local averaging processes

○ as the kernel function W can be seen as approximation to the δ-function for small but finite h we can expand the averaged function $\langle f(\vec{r})\rangle$ into a Taylor series for h to obtain an estimate for $f(\vec{r})$; if W is an even function, the first order term vanishes and the errors are second order in h

$$\langle f(\vec{r})\rangle = f(\vec{r}) + \mathcal{O}(h^2)$$

this holds for functions f that are smooth and do not exhibit steep gradients over the size of W ($\rightarrow$ problems in shocks).

(more specifically the expansion is $\langle f(\vec{r})\rangle = f(\vec{r}) + \kappa h^2 \vec{\nabla}^2 f(\vec{r}) + \mathcal{O}(h^3)$)

properties of local averaging processes

○ within its intrinsic accuracy, the smoothing process therefore is a linear function with respect to summation and multiplication:

$$\langle f(\vec{r}) + g(\vec{r}) \rangle = \langle f(\vec{r}) \rangle + \langle g(\vec{r}) \rangle$$
$$\langle f(\vec{r}) \cdot g(\vec{r}) \rangle = \langle f(\vec{r}) \rangle \cdot \langle g(\vec{r}) \rangle$$

(one follows from the linearity of integration with respect to summation, and two is true to $\mathcal{O}(h^2)$)

○ derivatives can be 'drawn into' the averaging process:

$$\frac{d}{dt}\langle f(\vec{r}) \rangle = \left\langle \frac{d}{dt} f(\vec{r}) \right\rangle$$
$$\vec{\nabla}\langle f(\vec{r}) \rangle = \langle \vec{\nabla} f(\vec{r}) \rangle$$

Furthermore, the spatial derivative of f can be transformed into a spatial derivative of W (no need for finite differences or grid):

$$\vec{\nabla} \langle f(\vec{r}) \rangle = \langle \vec{\nabla} f(\vec{r}) \rangle = \int f(\vec{r}') \, \vec{\nabla} W(|\vec{r} - \vec{r}'|, h) \, d^3 r' \, .$$

(shown by integrating by parts and assuming that the surface term vanishes; if the solution space is

extended far enough, either the function f itself or the kernel approach zero)

properties of local averaging processes

○ basic concept of SPH is a particle representation of the fluid
$\longrightarrow$ *integration* transforms into *summation* over discrete set of particles; example density ρ:

$$\langle \rho(\vec{r}_i) \rangle = \sum_j m_j W(|\vec{r}_i - \vec{r}_j|, h) \ .$$

in this picture, the mass of each particle is smeared out over its kernel region; the density at each location is obtained by summing over the contributions of the various particles $\longrightarrow$ *smoothed particle hydrodynamics!*

the kernel function

- different functions meet the requirement $\int W(|\vec{r}|, h)\, d^3r = 1$ and $\lim_{h \to 0} \int W(|\vec{r} - \vec{r}'|, h) f(\vec{r}')\, d^3r' = f(\vec{r})$:

  $\rightarrow$ Gaussian kernel:

  $$W(r, h) = \frac{1}{\pi^{3/2}h^3} \exp\left(-\frac{r^2}{h^2}\right)$$

  - *pro:* mathematically sound
  - *pro:* derivatives exist to all orders and are smooth
  - *contra:* all particles contribute to a location

  $\rightarrow$ spline functions with compact support

the kernel function

- different functions meet the requirement $\int W(|\vec{r}|, h)\, d^3r = 1$ and $\lim_{h \to 0} \int W(|\vec{r} - \vec{r}'|, h) f(\vec{r}')\, d^3 r' = f(\vec{r})$:

  $\rightarrow$ the standard kernel: cubic spline

  with $\xi = r/h$ it is defined as

  $$W(r, h) \equiv \frac{1}{\pi h^3} \begin{cases} 1 - \frac{3}{2}\xi^2 + \frac{3}{4}\xi^3, & \text{for } 0 \leq \xi \leq 1; \\ \frac{1}{4}(2 - \xi)^3, & \text{for } 1 \leq \xi \leq 2; \\ 0, & \text{otherwise.} \end{cases}$$

  - *pro:* compact support $\longrightarrow$ all interactions are zero for $r > 2h$ $\longrightarrow$ number of particles involved in the average remains small (typically between 30 and 80)
  - *pro:* second derivative is continuous
  - *pro:* dominant error term is second order in h

the fluid equations in SPH

- ○ there is an infinite number of possible SPH implementations of the hydrodynamic equations!

- ○ some notation: $h_{ij} = (h_i + h_j)/2$, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, and $\vec{\nabla}_i$ is the gradient with respect to the coordinates of particle i; all measurements are taken at particle positions (e.g. $\rho_i = \rho(\vec{r}_i)$)

- ○ *general form of SPH equations:*

$$\langle f_i \rangle = \sum_{j=1}^{N_i} \frac{m_j}{\rho_j} f_j W(r_{ij}, h_{ij})$$

the fluid equations in SPH

○ *density* — continuity equation (conservation of mass)

$$\rho_i = \sum_{j=1}^{N_i} m_j W(r_{ij}, h_{ij})$$

or

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \sum_{j=1}^{N_i} m_j \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}, h_{ij})$$

(the second implementation is almost never used, see however Monaghan 1991 for an application to water waves)

important

density is needed for *ALL* particles *BEFORE* computing other averaged quantities $\longrightarrow$ at each timestep, SPH computations consist of *TWO* loops, first the *density* is obtained for each particle, and then in a second round, all *other* particle properties are updated.

the fluid equations in SPH

- *pressure* is defined via the equation of state (for example for isothermal gas $p_i = c_{\mathrm{s}}^2 \rho_i$)

the fluid equations in SPH

○ *velocity* — Navier Stokes equation (conservation of momentum)

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = \sum_i \vec{F}_i = \vec{F}_{\text{pressure}} + \vec{F}_{\text{viscosity}} + \vec{F}_{\text{gravity}}$$

rate of change of momentum of fluid element depends on sum of all forces acting on it.

the fluid equations in SPH

○ *velocity* — Navier Stokes equation (conservation of momentum)

$\rightarrow$ consider for now *only* pressure contributions: Euler's equation

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p = -\vec{\nabla}\left(\frac{p}{\rho}\right) - \frac{p}{\rho^2}\vec{\nabla}\rho \quad (*)$$

here, the identity $\vec{\nabla}(p\rho^{-1}) = \rho^{-1}\vec{\nabla}p - p\rho^{-2}\vec{\nabla}\rho$ is used

$\rightarrow$ in the SPH formalism this reads as

$$\frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j\left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\vec{\nabla}_i W(r_{ij}, h_{ij})$$

where the first term in (*) is neglected because it leads to surface terms in the averaging procedure; it is assumed that either the pressure or the kernel becomes zero at the integration border; if this is not the case *correction terms* need to be added above.

the fluid equations in SPH

○ *velocity* — Navier Stokes equation (conservation of momentum)

→ the SPH implementation of the standard artificial viscosity is

$$\vec{F}_i^{\text{visc}} = - \sum_{j=1}^{N_i} m_j \Pi_{ij} \vec{\nabla}_i W(r_{ij}, h_{ij}),$$

where the viscosity tensor Π_{ij} is defined by

$$\Pi_{ij} = \begin{cases} (-\alpha c_{ij}\mu_{ij} + \beta\mu_{ij}^2)/\rho_{ij} & \text{for} \quad \vec{v}_{ij} \cdot \vec{r}_{ij} \leq 0, \\ 0 & \text{for} \quad \vec{v}_{ij} \cdot \vec{r}_{ij} > 0, \end{cases}$$

where

$$\mu_{ij} = \frac{h\vec{v}_{ij} \cdot \vec{r}_{ij}}{\vec{r}_{ij}^2 + 0.01h^2}.$$

with $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, mean density $\rho_{ij} = (\rho_i + \rho_j)/2$, and mean sound speed $c_{ij} = (c_i + c_j)/2$.

the fluid equations in SPH

- *velocity* — Navier Stokes equation (conservation of momentum)

  → if self-gravity is taken into account, the gravitational force needs to be added on the RHS

  $$\vec{F}_{\mathrm{G}} = -\vec{\nabla}\phi_i = -G \sum_{j=1}^{N} \frac{m_j}{r_{ij}^2} \frac{r_{ij}}{r_{ij}}$$

  note that the sum needs to be taken over *ALL* particles ⟵ computationally expensive

  → set together, the momentum equation is

  $$\frac{d\vec{v}_i}{dt} = - \sum_{j=1}^{N_i} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij} \right) \vec{\nabla}_i W(r_{ij}, h_{ij}) - \nabla\phi_i$$

the fluid equations in SPH

○ *energy equation* (conservation of momentum)

$\rightarrow$ recall the hydrodynamic energy equation:

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \vec{v} \cdot \vec{\nabla}\epsilon = \frac{ds}{dt} - \frac{p}{\rho}\vec{\nabla} \cdot \vec{v}$$

$\rightarrow$ for *adiabatic* systems $(c = \mathrm{const})$ the SPH form follows as

$$\frac{d\epsilon_i}{dt} = \frac{p_i}{\rho_i^2} \sum_{j=1}^{N_i} m_j\, \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}, h_{ij})\,,$$

(note that the alternative form

$$\frac{d\epsilon_i}{dt} = \frac{1}{2} \sum_{j=1}^{N_i} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}, h_{ij})$$

can lead to unphysical solutions, like negative internal energy)

the fluid equations in SPH

○ *energy equation* (conservation of momentum)

→ *dissipation* due to (artificial) viscosity leads to a term

$$\frac{d\epsilon_i}{dt} = \frac{1}{2} \sum_{j=1}^{N_i} m_j \Pi_{ij} \, \vec{v}_{ij} \cdot \vec{\nabla}_i W(r_{ij}.h_{ij})$$

→ the presence of *heating* sources or *cooling* processes can be incorporated into a function Γ_i.

→ altogether:

$$\frac{d\epsilon_i}{dt} = \frac{p_i}{\rho_i^2} \sum_{j=1}^{N_i} m_j \, \vec{v}_{ij} \cdot \vec{\nabla}_i W_{ij} + \frac{1}{2} \sum_{j=1}^{N_i} m_j \Pi_{ij} \, \vec{v}_{ij} \cdot \vec{\nabla}_i W_{ij} + \Gamma_i$$

can lead to unphysical solutions, like negative internal energy)

fully conservative formulation using Lagrange multipliers

○ the Lagrangian for compressible flows which are generated by the thermal energy $\epsilon(\rho, s)$ acts as effective potential is

$$\mathcal{L} = \int \rho \left\{ \frac{1}{2} v^2 - u(\rho, s) \right\} d^3 r.$$

equations of motion follow with $s = \text{const}$ from

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \vec{v}} - \frac{\partial \mathcal{L}}{\partial \vec{r}} = 0$$

○ after some SPH arithmetics, one can derive the following acceleration equation for particle i

$$\frac{d\vec{v}_i}{dt} = -\sum_{j=1}^{N_i} m_j \left\{ \frac{1}{f_i} \frac{p_i}{\rho_i^2} \vec{\nabla}_i W(r_{ij}, h_i) + \frac{1}{f_j} \frac{p_j}{\rho_j^2} \vec{\nabla}_i W(r_{ij}, h_j) \right\}$$

where

$$f_i = \left[1 + \frac{h_i}{3\rho_i} \frac{\partial \rho_i}{\partial h_i} \right]$$