# Chapter 5

# Formation of structure in the Universe I: Growth of perturbations

Some time during the inflation period tiny perturbations in the matter density were implaneted in the otherwise homogeneous Universe. Due to gravity these perturbations grow in amplitude, and eventually lead to the structure we see in the Universe today: galaxies, stars, planets etc.. In this chapter we will study how this takes place.

## 5.1 Motion of gas in an expanding Universe

### 5.1.1 Newtonian equations of motion for non-relativistic matter

Since the motions of cold matter in the Universe (baryons, cold dark matter) are non-relativisitc ($v \ll c$) we will derive the equations of motion for this matter in a Newtonian setting, see Chapter 2. We will follow the space and time dependent density $\rho(\vec{r}, t)$, its velocity $\vec{v}(\vec{r}, t)$ and the gravitational potential $\Phi(\vec{x}, t)$. The equations of self-gravitating gas dynamics are as follows. The continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{5.1}$$

The velocity field $\vec{v}(\vec{r}, t)$ obeys the Euler equation:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla \Phi \tag{5.2}$$

The gravitational potential obeys the following Poisson equation:

$$\nabla^2 \Phi = 4\pi G \rho \tag{5.3}$$

Since we are concerned with the early Universe, we will ignore any $\Lambda$ force.

### 5.1.2 Verification: The Newtonian expanding Universe solution

Now let us first verify that these equations indeed have the simple expanding Universe model of Chapter 2 as a solution. In this model the density is time-varying, but constant in space ($\nabla \rho = 0$) and the velocity field is given by

$$\vec{v}(\vec{r}, t) = \frac{\dot{a}}{a}\vec{r} = H(t)\vec{r} \tag{5.4}$$

We write the continuity equation, Eq. (5.1), as

$$0 = \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} + (\vec{v} \cdot \nabla)\rho = \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} \tag{5.5}$$

We can evaluate $\nabla \cdot \vec{v}$:

$$\nabla \cdot \vec{v} = \nabla \cdot (H(t)\vec{r}) = 3H(t) \tag{5.6}$$

so that we arrive at

$$\frac{\partial \rho}{\partial t} + 3H\rho = 0 \tag{5.7}$$

With $H = \dot{a}/a$ this gives the familiar result for cold matter that $\rho \propto 1/a^3$. Next we look at the Poisson equation for gravity, Eq. (5.3). Taking $\Phi(\vec{r} = 0) = 0$ and assuming spherical symmetry we obtain

$$\nabla^2 \Phi(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi(r)}{\partial r} \right) = 4\pi G \rho \tag{5.8}$$

with $r = |\vec{r}|$. The solution is

$$\Phi(r) = \frac{2\pi G}{3} \rho r^2 \tag{5.9}$$

Now insert Eq. (5.4) into the Euler equation, Eq. (5.2):

$$\vec{r} \frac{\partial H}{\partial t} + H^2 (\vec{r} \cdot \nabla) \vec{r} = -\nabla \Phi \tag{5.10}$$

which, with $(\vec{r} \cdot \nabla)\vec{r} = \vec{r}$, results in

$$(\dot{H} + H^2)\vec{r} = -\frac{4\pi G}{3} \rho \vec{r} \tag{5.11}$$

which leads to

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho \tag{5.12}$$

which is the second Friedmann equation (Eq. 4.10) for cold matter.

### 5.1.3 Intermezzo: The case of relativistic matter (radiation dominated era)

If we want to follow the equations of motion for perturbations in the very early Universe, when the matter was still relativistic, we should, of course, switch to at least a proper special relativistic, if not general relativistic description. However, one can gain some insight by making some small adaptions to the Newtonian equations to approximate the motion of relativistic matter in the early Universe. The key is to add some terms to Eqs. (5.1, 5.2, 5.3):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left[ \left( \rho + \frac{p}{c^2} \right) \vec{v} \right] = 0 \tag{5.13}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho + p/c^2} - \nabla \Phi \tag{5.14}$$

$$\nabla^2 \Phi = 4\pi G \rho \left( 1 + \frac{3p}{\rho c^2} \right) \tag{5.15}$$

We can only apply these equations if the particles form a gas, i.e. if they have regular collisions so that their energy *and* momenta are thermalized. A freely streaming sea of particles such as the CMB or radiation from stars does not act as a gas and we can not apply the above equations to them. However, before the release of the CMB at $z \lesssim 1100$ the electromagnetic radiation is firmly coupled (both energetically and kinetically) to the baryons. For $z \gg 1100$ the baryon-photon sea is therefore very well approximated by a gas, obeying Eqs. (5.13, 5.14, 5.15). Most of the inertial mass of this gas is dominated by the photons, with a small contribution by the baryons. Though for $z \lesssim 3200$ the energy density of the Universe is dominated cold matter, baryonic matter is only 17% of the "cold" matter in the Universe, so the baryonic matter starts dominating over the radiation energy density only around $z \simeq 550$, i.e. after CMB decoupling. During most of the time before $z \simeq 1100$ the baryons therefore play a role as the medium that makes the "photon sea" a "photon gas"; their contribution to the intertial or gravitational mass is small. During this time the photon gas can produce sound waves (which would not be possible for a free-streaming photon sea). We will study these so-called "baryonic acoustic oscillations" lateron.

### 5.1.4 Linearized equations for non-relativistic matter

Now let us assume that there are small perturbations on the density, velocity and gravitational potential fields:

$$\rho(\vec{r},t) = \rho_0(t) + \delta\rho(\vec{r},t) \tag{5.16}$$

$$\vec{v}(\vec{r},t) = \vec{v}_0(\vec{r},t) + \delta\vec{v}(\vec{r},t) \tag{5.17}$$

$$\Phi(\vec{r},t) = \Phi_0(\vec{r},t) + \delta\Phi(\vec{r},t) \tag{5.18}$$

where $\rho_0$, $\vec{v}_0$ and $\Phi_0$ are the homogeneous isotropic expanding universe solution from Section 5.1.2. We obtain, to first order, the following version of the continuity equation:

$$\begin{aligned} 0 &= \frac{\partial\rho_0}{\partial t} + \frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot\vec{v}_0 + \rho_0\nabla\cdot\delta\vec{v} + \delta\rho\nabla\cdot\vec{v}_0 + \vec{v}_0\cdot\nabla\delta\rho \\ &= \frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\vec{v} + 3H\delta\rho + \vec{v}_0\cdot\nabla\delta\rho \end{aligned} \tag{5.19}$$

the following version of the Euler equation:

$$\begin{aligned} 0 &= \frac{\partial\vec{v}_0}{\partial t} + \frac{\partial\delta\vec{v}}{\partial t} + \vec{v}_0\cdot\nabla\vec{v}_0 + \vec{v}_0\cdot\nabla\delta\vec{v} + \delta\vec{v}\cdot\nabla\vec{v}_0 + \frac{\nabla\delta P}{\rho_0} + \nabla\Phi_0 + \nabla\delta\Phi \\ &= \frac{\partial\delta\vec{v}}{\partial t} + \vec{v}_0\cdot\nabla\delta\vec{v} + \delta\vec{v}\cdot\nabla\vec{v}_0 + \frac{\nabla\delta P}{\rho_0} + \nabla\delta\Phi \end{aligned} \tag{5.20}$$

and the following version of the Poisson equation

$$\begin{aligned} 0 &= \nabla^2\Phi_0 + \nabla^2\delta\Phi - 4\pi G\rho_0 - 4\pi G\delta\rho \\ &= \nabla^2\delta\Phi - 4\pi G\delta\rho \end{aligned} \tag{5.21}$$

It is customary to define the symbol $\delta$ as

$$\delta := \frac{\delta\rho}{\rho_0} \tag{5.22}$$

This can be a bit confusing, because we also used $\delta$ as the symbol for "increment of something", but this is just the standard way of writing things. Before we start writing the above equations in terms of $\delta$, let us work out the following:

$$\dot{\delta} \equiv \frac{\partial\delta}{\partial t} = \frac{\partial(\delta\rho/\rho_0)}{\partial t} = \frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial t} - \frac{\delta\rho}{\rho_0^2}\frac{\partial\rho_0}{\partial t} = \frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial t} + 3H\frac{\delta\rho}{\rho_0} \tag{5.23}$$

$$\nabla\delta = \frac{\nabla\delta\rho}{\rho_0} \tag{5.24}$$

$$(\delta\vec{v}\cdot\nabla)\vec{v}_0 = (\delta\vec{v}\cdot\nabla)(H\vec{r}) = H\delta\vec{v} \tag{5.25}$$

The continuity equation then becomes

$$\dot{\delta} + \nabla\cdot\delta\vec{v} + \vec{v}_0\cdot\nabla\delta = 0 \tag{5.26}$$

The Euler equation becomes

$$\frac{\partial\delta\vec{v}}{\partial t} + H\delta\vec{v} + (\vec{v}_0 \cdot \nabla)\delta\vec{v} + c_s^2\nabla\delta + \nabla\delta\Phi = 0 \tag{5.27}$$

where we write the pressure as $\nabla p = c_s^2\nabla\rho$ with $c_s$ the adiabatic sound speed defined by $c_s^2 = \partial p/\partial\rho$. Finally the Poisson equation becomes

$$\nabla^2\delta\Phi = 4\pi G\rho_0\delta \tag{5.28}$$

Now let us use comoving coordinates:

$$\vec{x} := \frac{\vec{r}}{a(t)} \tag{5.29}$$

and define a comoving velocity perturbation

$$\vec{u} := \frac{\vec{\delta}v}{a(t)} \tag{5.30}$$

We must now be careful because the partial derivatives now change. The $\nabla_r$ operator in $\vec{r}$ coordinates must be replaced somehow by a $\nabla_x$ operator in $\vec{x}$ coordinates:

$$\nabla_r = \frac{1}{a(t)}\nabla_x \tag{5.31}$$

Also the partial time derivative changes, because up till now

$$\frac{\partial}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{\vec{r}=\text{const}} \tag{5.32}$$

while from now on

$$\frac{\partial}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{\vec{x}=\text{const}} \tag{5.33}$$

where the $\partial$ is (for the moment) used to distinguish this new partial time derivative from the original one. The new time derivative is a time derivative comoving with the Hubble Flow:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla_r = \frac{\partial}{\partial t} + H\vec{r} \cdot \nabla_r = \frac{\partial}{\partial t} + H\vec{x} \cdot \nabla_x \tag{5.34}$$

The continuity equation then becomes

$$\dot{\delta} + \nabla \cdot \vec{u} = 0 \tag{5.35}$$

For the Euler equation is it useful to first work out the following:

$$\frac{\partial\delta\vec{v}}{\partial t} = \frac{\partial\delta\vec{v}}{\partial t} - H\vec{x} \cdot \nabla_x\delta\vec{v} = a\frac{\partial\vec{u}}{\partial t} + aH\vec{u} - aH\vec{x} \cdot \nabla_x\vec{u} \tag{5.36}$$

The Euler equation then becomes

$$\begin{aligned} 0 &= a\frac{\partial\vec{u}}{\partial t} + aH\vec{u} - aH\vec{x} \cdot \nabla_x\vec{u} + aH\vec{u} + (\vec{v}_0 \cdot \nabla_x)\vec{u} + \frac{c_s^2}{a}\nabla_x\delta + \frac{1}{a}\nabla_x\delta\Phi \\ &= a\left(\frac{\partial\vec{u}}{\partial t} + 2H\vec{u}\right) + \frac{1}{a}\left(c_s^2\nabla_x\delta + \nabla_n\delta\Phi\right) \end{aligned} \tag{5.37}$$

This brings us to the final form of the perturbation equations:

$$\dot{\delta} + \nabla \cdot \vec{u} = 0 \tag{5.38}$$

$$\dot{\vec{u}} + 2H\vec{u} = -\frac{c_s^2\nabla\delta + \nabla\delta\Phi}{a^2} \tag{5.39}$$

$$\nabla^2\delta\Phi = 4\pi G\rho_0 a^2\delta \tag{5.40}$$

where we tacitly assume that the $\partial/\partial t$ and $\nabla$ operators are the new ones.

### 5.1.5 Analysis of perturbations

If we take the divergence of Eq. (5.39) we obtain

$$\nabla \cdot \dot{\vec{u}} + 2H\nabla \cdot \vec{u} = -\frac{c_s^2\nabla^2\delta + \nabla^2\delta\Phi}{a^2} \tag{5.41}$$

We have dropped $\delta\nabla c_s^2$ terms because they are of second order. We can now use Eqs. (5.38, 5.40) to eliminate $\vec{u}$ and $\delta\Phi$ altogether:

$$\ddot{\delta} + 2H\dot{\delta} = \frac{c_s^2\nabla^2\delta}{a^2} + 4\pi G\rho_0\delta \tag{5.42}$$

We can now analyze this by decomposing $\delta$ in plane waves:

$$\delta(\vec{x}, t) = \frac{1}{(2\pi)^3}\int \hat{\delta}(\vec{k}, t)e^{-i\vec{k}\cdot\vec{x}}d^3k \tag{5.43}$$

This yields:

$$\ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} = \left(4\pi G\rho_0 - \frac{c_s^2k^2}{a^2}\right)\hat{\delta} \tag{5.44}$$

We can now write also the time part as

$$\hat{\delta}(\vec{k}, t) = \delta(\vec{k})e^{i\omega t} \tag{5.45}$$

This gives the following dispersion relation

$$\omega^2 - i\omega 2H = -\left(4\pi G\rho_0 - \frac{c_s^2k^2}{a^2}\right) \tag{5.46}$$

Let us define the so-called "Jeans length" $\lambda_J$

$$\lambda_J \equiv \frac{2\pi}{k_J} := \frac{2\pi c_s}{a\sqrt{4\pi G\rho_0}} \tag{5.47}$$

If we have a static background $H = 0$, then we can easily distinguish two scenarios:

$$\begin{cases} k > k_J \quad (\lambda < \lambda_J) \quad \rightarrow \quad \omega^2 > 0 & \text{Oscillation / wave} \\ k < k_J \quad (\lambda > \lambda_J) \quad \rightarrow \quad \omega^2 < 0 & \text{Exponential decay and growth} \end{cases} \tag{5.48}$$

This means that perturbations on a small scale tend to oscillate like waves on the sea. But perturbations on a large scale can be classified into exponentially decaying and exponentially growing modes. The decaying modes are, in our context, not interesting. The growing modes grow as

$$\hat{\delta}(\vec{k}, t) = \delta(\vec{k})e^{\gamma t} \tag{5.49}$$

with

$$\gamma = \sqrt{4\pi G\rho_0 - \frac{c_s^2k^2}{a^2}} \qquad \text{for} \qquad k < k_J \tag{5.50}$$

The typical time scale of growth of a perturbation by a factor of $e$ ("e-folding time scale") is $\tau_{\rm growth} = 1/\gamma$. For perfectly cold gas ($c_s = 0$) we get $\tau_{\rm growth} = 1/\sqrt{4\pi G\rho_0}$. The larger the density $\rho_0$ the faster the growth of the perturbations.

For an expanding universe this growth is not exponential, in particular not for a critically expanding Universe. Let us focus on scales much larger than the Jeans length, so that we can ignore the term proportional to $k^2$. Let us also assume (which is reasonable for most part of the matter dominated phase) that $\rho_0(t) = \rho_{\rm crit}(t) = 3H(t)^2/8\pi G$. The growth equation is then:

$$\ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} = \frac{3}{2}H^2\hat{\delta} \tag{5.51}$$

During this matter dominated phase we also know that

$$H = \frac{2}{3t} \tag{5.52}$$

This gives

$$\ddot{\hat{\delta}} + \frac{4}{3}\frac{\dot{\hat{\delta}}}{t} = \frac{2}{3}\frac{\hat{\delta}}{t^2} \tag{5.53}$$

Now take as an Ansatz

$$\hat{\delta} \propto t^n \tag{5.54}$$

so that we obtain

$$n^2 + \frac{1}{3}n = \frac{2}{3} \tag{5.55}$$

which has as solutions:

$$n = -1 \qquad \text{and} \qquad n = 2/3 \tag{5.56}$$

The decaying solution is not interesting, because those modes disappear. Of interest to us is the growing mode:

$$\hat{\delta} \propto t^{2/3} \propto a \tag{5.57}$$

This shows that the growth of the amplitude of perturbations in an expanding matter-dominated universe is not exponential with time, but a powerlaw of time, and in particular: linear with the scale function $a(t)$. Important is that the *growth of the amplitude is independent of the wavelength*, at least for wavelength much larger than the Jeans length.

It turns out that this linear growth with $a$ remains a good approximation even if we add a cosmological constant $\Lambda$ to our equations and/or if we have a sub/supercritical universe. We must then, however, replace the growth equation with

$$\hat{\delta}(a) = \hat{\delta}_0 D_+(a) \tag{5.58}$$

with in excellent approximation

$$D_+(a) = \frac{5a}{2}\Omega_m \left[\Omega_m^{4/7} - \Omega_\Lambda + \left(1 + \frac{1}{2}\Omega_m\right)\left(1 + \frac{1}{70}\Omega_\Lambda\right)\right]^{-1} \tag{5.59}$$

There is one caveat in the above derivations for the growth of perturbation amplitudes. We treated the dark matter and the baryonic matter in the same manner. While the baryonic matter is clearly a gas, the cold dark matter is better described by a set of collisionless particles. It is the basic assumption of cold dark matter that the particles have no random velocities on top of their local average flow. However, it may be that there are small random velocities; let's write them as $\Delta v$. This would be "warm dark matter". Also, after dark matter halos form, the velocities of the CDM particles randomize to a certain extent, which also induces a $\Delta v$. It turns out that, in spite of the fact that CDM is not strictly speaking a gas (because of it being collisionless), the description of the growth of perturbations we derived holds surprisingly well. One can even define an effective Jeans length, by replacing $c_s$ in Eq. (5.47) with $\langle \Delta v \rangle$.

Now let us do the above analysis for the radiation-dominated phase. We start from Eqs. (5.13, 5.14, 5.15) and do the entire derivation again. If we assume that we focus on scales much larger than the Jeans length (which is, by the way, much larger for radiation-dominated matter, see later), we can ignore the pressure gradient, but not the $p/c^2 = \rho/3$ terms. After the whole procedure we end up with the radiation-dominated equivalent of Eq. (5.51):

$$\ddot{\hat{\delta}} + 2H\dot{\hat{\delta}} = 4H^2\hat{\delta} \tag{5.60}$$

For the radiation-dominated era we have

$$H(t) = \frac{1}{2t} \tag{5.61}$$

so that we obtain with $\hat{\delta} \propto t^n$

$$n^2 = 1 \tag{5.62}$$

with solutions

$$n = -1 \qquad \text{and} \qquad n = 1 \tag{5.63}$$

Of interest to us is the growing mode:

$$\hat{\delta} \propto t \propto a^2 \tag{5.64}$$

### 5.1.6 Peculiar velocities

The (small) deviations that distant galaxies have from a pure Hubble Flow are called "pecular velocities". While the linear perturbation analysis of Section 5.1.5 showed how the *density* perturbations grow, it did not give information about the $\vec{u}$ field, because we eliminated $\vec{u}$ in favor of $\delta$. Let us therefore return to Eq. (5.39) but without pressure (i.e. looking at large scales):

$$\dot{\vec{u}} + 2H\vec{u} = -\frac{\nabla\delta\Phi}{a^2} \tag{5.65}$$

A full solution may require numerical methods, but this equation strongly suggests that the velocity field $\vec{u}$ will point in the direction of $-\nabla\delta\Phi$ and that the proportionality constant will be time dependent but not space-dependent (because, as we have seen earlier, the growth rate of perturbations does not depend on spatial scale). Thus let us try a form of $\vec{u}$ of:

$$\vec{u}(\vec{x}, t) = -u(t)\nabla\delta\Phi \tag{5.66}$$

If we evalutate the divergence of this assumed velocity field we get

$$\nabla \cdot \vec{u}(\vec{x}, t) = -u(t)\nabla^2\delta\Phi \tag{5.67}$$

Now with Eq. (5.40) we find

$$\nabla \cdot \vec{u}(\vec{x}, t) = -4\pi G\rho_0 a^2 u(t)\delta \tag{5.68}$$

Now, the divergence of $\vec{u}$ is also, with Eq. (5.38) equal to

$$\nabla \cdot \vec{u} = -\dot{\delta} = -\dot{a}\frac{\partial\delta}{\partial a} = -H\frac{\partial\delta}{\partial \ln a} \tag{5.69}$$

This gives (with $4\pi G\rho_0 = \frac{3}{2}H^2\Omega_m$)

$$\frac{\partial\delta}{\partial \ln a} = \frac{3}{2}H\Omega_m a^2 u(t)\delta \tag{5.70}$$

If we write

$$\delta = \delta_0 D_+(a) \tag{5.71}$$

we obtain

$$\frac{\partial \ln D_+}{\partial \ln a} = \frac{3}{2}H\Omega_m a^2 u(t) \tag{5.72}$$

which means we can write $u(t)$ as

$$u(t) = \frac{2f(\Omega_m)}{3H\Omega_m a^2} \tag{5.73}$$

with $f(\Omega_m)$

$$f(\Omega_m) := \frac{\partial \ln D_+}{\partial \ln a} \tag{5.74}$$

It turns out that an excellent approximation of $f(\Omega_m)$ is $f(\Omega_m) \simeq \Omega_m^{0.6}$. We now have the solution for the peculiar motions, at least in the linear regime:

$$\delta\vec{v} = a\vec{u} = -\frac{2f(\Omega_m)}{3H\Omega_m a}\nabla\delta\Phi \tag{5.75}$$

### 5.1.7 The same analysis for the radiation-dominated era

All the analysis we have done above for the growth of perturbations in the matter-dominated era of the Universe can also be derived for the radiation-dominated era. We start from Eqs. (5.13, 5.14, 5.15), and set $p = \rho c^2/3$. After some work we arrive at

$$\dot{\delta} + \tfrac{4}{3}\nabla \cdot \vec{u} = 0 \qquad (5.76)$$

$$\dot{\vec{u}} + 2H\vec{u} = -\frac{\tfrac{1}{4}c^2\nabla\delta + \nabla\delta\Phi}{a^2} \qquad (5.77)$$

$$\nabla^2\delta\Phi = 8\pi G\rho_0 a^2\delta \qquad (5.78)$$

These are the radiation-dominated versions of Eqs. (5.38, 5.39, 5.40). The radiation-dominated version of Eq. (5.42) is then

$$\frac{3}{4}\ddot{\delta} + \frac{3}{2}H\dot{\delta} = \frac{c^2\nabla^2\delta}{4a^2} + 8\pi G\rho_0\delta \qquad (5.79)$$

In Fourier space we get the following radiation-dominated version of Eq. (5.46):

$$\frac{3}{4}\omega^2 - i\omega\frac{3}{2}H = -\left(8\pi G\rho_0 - \frac{c^2k^2}{4a^2}\right) \qquad (5.80)$$

This gives the following Jeans length:

$$\lambda_J = \frac{2\pi}{k_J} = \frac{c\sqrt{\pi}}{2a\sqrt{2G\rho_0}} \qquad (5.81)$$

(compare to Eq. 5.47 for the non-relativistic case). If we take for the average density $\rho_0$ the critical density $\rho_{\rm crit} = 3H^2/8\pi G$ (which is the case in the radiation-dominated era) we obtain

$$\lambda_J = \frac{\pi}{\sqrt{3}}\frac{1}{a}\frac{c}{H} = \frac{\pi}{\sqrt{3}}\frac{1}{a}r_H \qquad (5.82)$$

This means that the Jeans length in the radiation-dominated era is roughly the Hubble radius. This means that wave moves larger than the particle horizon are unstable (i.e. they amplify), but wave modes that are smaller than the particle horizon are just oscillating and do not amplify.

## 5.2 Statistical analysis of perturbations

Now that we know how *seed perturbations* grow in an expanding universe, we next need to know how these perturbations are created in the first place. To get an understanding of this, we must first review some of the statistical tools that are used to analyse fluctuations. We will do this analysis in "normal" space $\vec{x}$ as well as in Fourier space $\vec{k}$, and show that they are related to each other.

### 5.2.1 The power spectrum

If we start from a perturbation field $\delta(\vec{x})$ with $\langle\delta\rangle = 0$, we can decompose this in Fourier componets:

$$\delta(\vec{x}, t) = \frac{1}{(2\pi)^3}\int \hat{\delta}(\vec{k}, t)e^{-i\vec{k}\cdot\vec{x}}d^3k \qquad (5.83)$$

We can compute these Fourier components through:

$$\hat{\delta}(\vec{k}, t) = \int \delta(\vec{x}, t)e^{i\vec{k}\cdot\vec{x}}d^3x \qquad (5.84)$$

Because $\langle\delta\rangle = 0$ we have $\hat{\delta}(0, t) = 0$. The *power spectrum*[1] is simply the amplitude-squared of the Fourier components:

$$P(\vec{k}) = \hat{\delta}^*(\vec{k})\hat{\delta}(\vec{k}) = |\hat{\delta}(\vec{k})|^2 \tag{5.85}$$

Since in our case we expect that there is isotropy, we are only interested in $P(|\vec{k}|)$, i.e. in the direction-averaged power spectrum. Define $k := |\vec{k}|$. We then get

$$P(k) = \frac{1}{4\pi}\oint |\hat{\delta}(\vec{k})|^2 d\Omega \tag{5.86}$$

This power spectrum function tells us how much "power" there is in perturbations of spatial scales of $\lambda = 2\pi/k$.

There is a subtlety in this definition of the power spectrum. We have cheated a bit there... If we have an infinite space, and the perturbation $\delta(\vec{x})$ is present everywhere, then the integrals Eqs. (5.83, 5.84) diverge. What we have to do instead is devide space into square boxes of $L \times L \times L$ in size, i.e. with volume $V = L^3$. The boxes must be larger than any of the scales of interest, the larger the better. For each of the boxes we then define the periodic Fourier decomposition:

$$\delta(\vec{x}, t) = \frac{1}{V}\sum_{l,m,n} \hat{\delta}_{l,m,n}(t) e^{-2\pi i(lx+my+nz)/L} \tag{5.87}$$

The inverse is:

$$\hat{\delta}_{l,m,n}(t) = \int_V \delta(\vec{x}, t) e^{2\pi i(lx+my+nz)/L} d^3x \tag{5.88}$$

This gives a discrete fourier transform for every box. The sums and integrals are now well-defined by virtue of the finite sizes of the boxes. We can make a link to Eqs. (5.83, 5.84) by associating each discrete Fourier component $\hat{\delta}_{l,m,n}(t)$ with a little volume in $\vec{k}$-space:

$$\vec{k} = (2\pi/L)\begin{pmatrix} l \\ m \\ n \end{pmatrix} \qquad d^3k = \left(\frac{2\pi}{L}\right)^3 = \frac{(2\pi)^3}{V} \tag{5.89}$$

We can then associate the continuous and discrete Fourier amplitudes:

$$\hat{\delta}(\vec{k}, t) = \hat{\delta}_{l,m,n}(t) \qquad \text{with } \vec{k} \text{ given by Eq. (5.89)} \tag{5.90}$$

We can also associate the power spectrum with the discrete form:

$$P(\vec{k}) = \hat{\delta}^*_{l,m,n}\hat{\delta}_{l,m,n} \tag{5.91}$$

Now, for each volume we will of course get slightly different values of $\hat{\delta}_{l,m,n}(t)$ because, after all, the perturbation $\delta(\vec{x}, t)$ is a random fluctuation. By averaging over many of these volumes we can obtain the *expectation value* of $\hat{\delta}^*(\vec{k})\hat{\delta}(\vec{k})$. This leads us to the better definition of power spectrum:

$$P(\vec{k}) = \langle\hat{\delta}^*(\vec{k})\hat{\delta}(\vec{k})\rangle = \langle|\hat{\delta}(\vec{k})|^2\rangle \tag{5.92}$$

$$P(k) = \frac{1}{4\pi}\oint\langle|\hat{\delta}(\vec{k})|^2\rangle d\Omega \tag{5.93}$$

This is the definition we shall use from now on. In fact, this contains all the statistical information there is to know about the perturbation field $\delta(\vec{x}, t)$. The different modes must be entirely uncorrelated:

$$\langle\hat{\delta}^*(\vec{k})\hat{\delta}(\vec{k}')\rangle = (2\pi)^3\delta_D(\vec{k} - \vec{k}')P(\vec{k}) \tag{5.94}$$

[1] A better definition of power spectrum follows later.

where $\delta_D(\vec{k}-\vec{k}')$ is the Dirac delta-function. If they were not uncorrelated, then we would break homogeneity and/or isotropy of $\delta(\vec{x},t)$.

There is a very interesting relation between the power spectrum and the *autocorrelation function* defined by

$$A_\delta(\vec{y}) := \langle \delta^*(\vec{x})\delta(\vec{x}+\vec{y})\rangle \tag{5.95}$$

Here the expectation value symbols $\langle$ and $\rangle$ can be meant in two different, yet equivalent, ways. It can mean an average over the volume $V$:

$$A_\delta(\vec{y}) = \frac{1}{V}\int \delta^*(\vec{x})\delta(\vec{x}+\vec{y})d^3x \tag{5.96}$$

But it can also mean averaging over a large number of volumes for a given $\vec{x}$ (where $\vec{x}$ now is a coordinate relative to the box):

$$A_\delta(\vec{y}) = \frac{1}{N}\sum_{i=1}^{N}[\delta^*(\vec{x})\delta(\vec{x}+\vec{y})]_{\text{volume}\,i} \tag{5.97}$$

The homogeneity and isotropy of the Universe implies that these two definitions lead to the same answer. Mathematically put: the perturbation is *ergodic*.

From this point onward we tacitly assume that integrals "over infinity" are actually integrals over a large volume $V$ and instead of writing the discrete $\hat{\delta}_{l,m,n}(t)$ we write the continuous $\hat{\delta}(\vec{k},t)$, with Eq. (5.89) as the way to associate them.

The relation between the power spectrum and the autocorrelation function is given by:

$$\begin{aligned}
\langle \delta^*(\vec{x})\delta(\vec{x}+\vec{y})\rangle &= \left\langle \int \frac{d^3k'}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\hat{\delta}^*(\vec{k}',t)\hat{\delta}(\vec{k},t)e^{i\vec{k}'\cdot\vec{x}}e^{-i\vec{k}\cdot(\vec{x}+\vec{y})}\right\rangle \\
&= \int \frac{d^3k'}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\left\langle \hat{\delta}^*(\vec{k}',t)\hat{\delta}(\vec{k},t)\right\rangle e^{i\vec{x}\cdot(\vec{k}'-\vec{k})-i\vec{k}\cdot\vec{y}} \\
&= (2\pi)^3\int \frac{d^3k'}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\delta(\vec{k}-\vec{k}')P(\vec{k})e^{i\vec{x}\cdot(\vec{k}'-\vec{k})-i\vec{k}\cdot\vec{y}} \\
&= \int \frac{d^3k}{(2\pi)^3}P(\vec{k})e^{-i\vec{k}\cdot\vec{y}}
\end{aligned} \tag{5.98}$$

or equivalently:

$$P(\vec{k}) = \int \langle \delta^*(\vec{x})\delta(\vec{x}+\vec{y})\rangle e^{i\vec{k}\cdot\vec{x}}d\vec{x} \tag{5.99}$$

This is known as the *Wiener-Khinchin theorem*: The power spectrum is the fourier transform of the autocorrelation function.

It is useful to write Eq. (5.98) in terms of an integral over $P(k)dk$ (instead of the more complicated integral over $P(\vec{k})d^3k$):

$$\begin{aligned}
\langle \delta^*(\vec{x})\delta(\vec{x}+\vec{y})\rangle &= \int \frac{k^2 dk \sin\theta d\theta d\phi}{(2\pi)^3}P(k)e^{-iky\cos\theta} \\
&= 2\pi\int \frac{k^2dk}{(2\pi)^3}P(k)\cdot\int_0^\pi \sin\theta e^{-iky\cos\theta}d\theta \\
&= 2\pi\int \frac{k^2dk}{(2\pi)^3}P(k)\cdot\int_{-1}^{1} e^{-iky\mu}d\mu \\
&= 4\pi\int \frac{k^2dk}{(2\pi)^3}P(k)\frac{\sin ky}{ky}
\end{aligned} \tag{5.100}$$

where $y = |\vec{y}|$, the angle $\theta$ is defined by $ky\cos\theta = \vec{k}\cdot\vec{y}$ and $\phi$ is the angle around the axis of $\vec{y}$.

### 5.2.2 Variance and the window function

The variance $\sigma$ of $\delta(\vec{x}, t)$ is the autocorrelation function at $\vec{y} = 0$:

$$\sigma^2 = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \tag{5.101}$$

As we shall see later, the fluctuations in $\delta(\vec{x}, t)$ are at all spatial scales. While at large scales we are limited by our volume $V$ (see Section 5.2.1), we are not limited toward small scales. To make sense of $\sigma^2$ it is useful to introduce a *window function* $W_R(x)$, which is essentially a kind of "smoothing kernel" with which we can convolve the $\delta(\vec{x}, t)$ to

$$\delta(\vec{x}, t) = \int \delta(\vec{y}, t) W_R(|\vec{x} - \vec{y}|) d^3y \tag{5.102}$$

The window function $W_R(x)$ is a function that is non-zero for $x \lesssim R$ but decreases to zero for $x \gg R$ and is normalized as

$$\int W_R(|\vec{y}|) d^3y = 4\pi \int W_R(y) y^2 dy = 1 \tag{5.103}$$

It could be, for example, a Gaussian:

$$W_R(y) = \frac{1}{(2\pi)^{3/2} R^3} \exp\left(-\frac{y^2}{2R^2}\right) \tag{5.104}$$

This means that in $\delta(\vec{x}, t)$ all scales smaller than about $R$ are smeared out, so that we expect that all Fourier components with $k \gtrsim 2\pi/R$ are suppressed. In Fourier space the convolution becomes a simple multiplication:

$$\hat{\delta}(\vec{k}, t) = \hat{\delta}(\vec{k}, t) \hat{W}_R(k) \tag{5.105}$$

The power spectrum for $\delta(\vec{x}, t)$ is then

$$P(k) = P(k) \hat{W}_R^2(k) \tag{5.106}$$

and the variance thus becomes

$$\sigma_R^2 = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \hat{W}_R^2(k) \tag{5.107}$$

Often the variance on a scale of $8h^{-1}$ Mpc is used, and is conventionally written as $\sigma_8$.

### 5.2.3 Gaussianity, non-gaussianity

The power spectrum tells only part of the story of the fluctuations. One could say that $P(\vec{k}, t)$ gives the variance of $\delta(\vec{x}, t)$ for a given $\vec{k}$. If we denote the probability distribution function for finding $\hat{\delta}(\vec{k})$ at some value with small-letter $p_k$ we thus get

$$P(\vec{k}, t) = \langle \hat{\delta}^2(\vec{k}, t) \rangle = \int p_k(\hat{\delta}, t) \hat{\delta}^2 d\hat{\delta} \tag{5.108}$$

where $p_k(\hat{\delta}, t) d\hat{\delta}$ is the probability of finding $\hat{\delta}$ between $\hat{\delta}$ and $\hat{\delta} + d\hat{\delta}$. In $\vec{x}$-space, if we introduce a window function $W_R(k)$, we can write equivalently:

$$\sigma_R^2 = \langle \delta^2(\vec{x}, t) \rangle = \int p_R(\delta, t) \delta^2 d\delta \tag{5.109}$$

where $p_R(\delta, t) d\delta$ is the probability of finding $\delta$ between $\delta$ and $\delta + d\delta$; the $R$ stands for the smoothing radius of the window function.

The power spectrum $P(k)$ and the variance $\sigma_R^2$, however, do not give any information about the *shape* of $p_k(\hat{\delta}, t)$ and $p_R(\delta, t)$. A reasonable guess is that the shape is *Gaussian*:

$$p_R(\delta, t) = \frac{1}{\sigma_R \sqrt{2\pi}} \exp\left(-\frac{\delta^2}{2\sigma_R^2}\right) \tag{5.110}$$

It turns out that CMB data confirms gaussianity to high precision, at least so far. But deviations from gaussianity would be important to find, as they would be able to distinguish between models of inflation.

## 5.3 The power spectrum of the seed perturbations

### 5.3.1 Harrison-Zel'dovich-Peebles spectrum

The perturbations $\delta(\vec{x}, t)$ that eventually grow into clusters of galaxies, galaxies, stars and planets originate in all likelihood during the era of inflation, during the first $10^{-32}$ second of the Universe. We will discuss inflation in more detail later. Here we ask ourselves what happens to inevitable quantum fluctuations of the inflaton field during this era of ultra-rapid expansion of the Universe. Under non-expanding circumstances these fluctuations would quickly die out. But the expansion during inflation is so rapid that before they die out they have already been "frozen in" by the expansion. To express this more visually: if the field at two points $\vec{x}_1$ and $\vec{x}_2$ become slightly different as a result of a quantum fluctuation, then before this contrast can be removed again, points $\vec{x}_1$ and $\vec{x}_2$ have already moved out of each others event horizon. The quantum fluctuation has thus been preserved, albeit beyond the horizon. Long after inflation, however, the points $\vec{x}_1$ and $\vec{x}_2$ can come back into each other's horizon. By this time, however, the spatial scale associated to $|\vec{x}_2 - \vec{x}_1|$ has grown to huge proportions, and is no longer at the quantum scale. The perturbation, which started as a quantum fluke during inflation, is now a perturbation on macroscopic scale.

This simple picture makes a far-reaching prediction for the spectrum of these seed fluctuations. During inflation the Hubble radius $r_H \equiv c/H$ remains approximately constant. The scale factor $a(t)$ goes as

$$a(t) = a(t_{\mathrm{infl},0}) e^{H(t - t_{\mathrm{infl},0})} \tag{5.111}$$

where $t_{\mathrm{infl},0}$ is defined as the time of the start of inflation. The time $\Delta t$ it takes for a quantum fluctuation of size $\lambda_{\mathrm{quant}}$ to freeze out is given by the time this fluctuation takes to expand to the Hubble radius $r_H$:

$$\frac{a_{\mathrm{freeze}}}{a_{\mathrm{quant}}} \lambda_{\mathrm{quant}} = r_h = \frac{c}{H} \tag{5.112}$$

This gives

$$\Delta t = \frac{1}{H} \ln\left(\frac{a_{\mathrm{freeze}}}{a_{\mathrm{quant}}}\right) = \frac{1}{H} \ln\left(\frac{c}{H\lambda_{\mathrm{quant}}}\right) \tag{5.113}$$

During inflation $H$ remains approximately constant, and it is reasonable to assume that $\lambda_{\mathrm{quant}}$ does as well. This means that $\Delta t$ remains constant.

If during the inflation period perturbations are generated with a given rate (events per time), then because of the exponential expansion this yields a certain number of perturbations per *logarithmic* interval in spatial scale. Together with the constant $\Delta t$ given above, we see that perturbations of identical nature are continuously being sent beyond the horizon. This process continues for *many* expansion e-folding times $1/H$. This must give a scale-free power spectrum. More precisely: the power spectrum of perturbations soon after the inflation period $P_i(k)$ must be a powerlaw: $P_i(k) \propto k^n$.

There are several arguments that put $n = 1$. One argument goes as follows. If the scalar field we perturb is (related to) the gravitational potential field $\Phi$, and if the *amplitude* of the perturbations we induce in $\Phi$ during inflation are always of the same order of magnitude, then not only must $P_{\Phi,i}(k)$ be a powerlaw spectrum: It must in fact be $P_{\Phi,i}(k) \propto 1/k^3$. This is because scale-invariance with keeping the amplitude constant means that if we scale $k' \leftarrow \psi k$ (for some scale factor $\psi$) then $P_{\Phi,i}(k')d^3k' = P_{\Phi,i}(k)d^3k$. With the power spectrum of the potential being $P_{\Phi,i}(k) \propto 1/k^3$, the power spectrum of the density must be $P_{\rho,i}(k) \propto k$, because of the Poisson equation (Eq. 5.3) where we replace (in Fourier space) the $\nabla^2$ by $k^2$.

Let us assume that this is right, and we thus have the following initial power spectrum of density perturbations that seeds the structure formation in the Universe:

$$P_i(k) \propto k \tag{5.114}$$

This is called the *Harrison-Zel'dovich-Peebles spectrum*. The WMAP results show that for small $k$ this spectrum quite accurately agrees with the observations. The tiny deviation measured with WMAP can be well understood by the finite duration of the inflation period.

### 5.3.2 Power spectrum entering the horizon

Inflation sends perturbations beyond the horizon. Once inflation ends, however, we enter the radiation-dominated era in which the horizon expands again, so that points that originally went "out of sight" come back in sight: i.e. they "enter the horizon" again.

Consider a perturbation of *comoving* wavelength $\lambda = 2\pi/k$. It enters the horizon when a light signal emitted at very early times from one end of the wave reaches the other end. We have calculated horizons in Section 4.10, cf. Eq. (4.80). Here, however, we are interested in the particle horizon at some time before the present time. We wish to find out how far light has travelled in the comoving distance from $a \simeq 0$ up to some $a$. So we go back to Eq. (4.64) but integrate not to $a = 1$ but up to some $a < 1$:

$$x_{\rm hori}(a) = \frac{c}{H_0}\int_0^a \frac{da'}{a'^2 E(a')} \tag{5.115}$$

Let us focus on the radiation-dominated era, in which case we have

$$E(a) = \sqrt{\Omega_{r,0}}\frac{1}{a^2} \tag{5.116}$$

Inserting this into Eq. (5.115) yields

$$x_{\rm hori}(a) = \frac{c}{H_0\sqrt{\Omega_{r,0}}}\int_0^a da' = \left(\frac{c}{H_0\sqrt{\Omega_{r,0}}}\right)a \tag{5.117}$$

So the comoving wavelength $\lambda$ enters the horizon when $x_{\rm hori} \simeq \lambda = 2\pi/k$, i.e.

$$a \simeq \left(\frac{H_0\sqrt{\Omega_{r,0}}}{c}\right)\lambda \tag{5.118}$$

Not surprisingly, large scale perturbations enter the horizon later than small scale perturbations.

Perturbations that have not yet entered the horizon can nevertheless grow in amplitude. This may seem surprising: Shouldn't they be frozen in? A rigorous general relativisitc treatment shows that growth can proceed even "beyond the horizon". This can be understood because the growth of the perturbations depends on the *local* divergence of the velocity field, which depends, in turn, on the *local* gradient of the gravitation potential. What happens beyond the horizon is of no concern for the growth of the amplitude of the perturbation through gravity. Therefore, in the radiation-dominated era, $\hat{\delta}(k)$ grows as $a^2$ (cf. Eq. 5.64) and thus $P(k)$ grows as $a^4$. Since large scale perturbations enter the horizon later, they have had more time to grow before they enter the horizon. If we compare two modes $k_1$ and $k_2$ with $k_2 < k_1$, then mode $k_1$ enters the horizon first, and $k_2$ later. According to Eq. (5.118) the ratio of the values of $a$ at which the modes enter the horizon is $a_2/a_1 = k_1/k_2$. The ratio of the powers $P(k_1, a = a_1)$ to $P(k_2, a = a_2)$ is, including the $(a_2/a_1)^4$ factor growth that mode $k_2$ experiences between $a = a_1$ and $a = a_2$:

$$\frac{P(k_2, a = a_2)}{P(k_1, a = a_1)} = \frac{k_2}{k_1}\left(\frac{a_2}{a_1}\right)^4 = \left(\frac{k_1}{k_2}\right)^3 \tag{5.119}$$

In other words the total power entering the horizon is always the same:

$$P_{\rm enter}(k)k^3 = \text{constant} \tag{5.120}$$

One can redo the calculation for the matter-dominated era and one will find the same. Historically the Harrison-Zel'dovich-Peebles spectrum was derived from the *assumption* that the power entering the horizon is always the same. Nowadays this is usually argued via the model of inflation, as we did.

## 5.4 The CDM power spectrum at $z = z_{\rm eq}$

Between the end of inflation (when $P_i(k) \propto k$) and the decoupling of the radiation from the baryons (the CMB release at $z \simeq 1100$) the perturbations grow. But some of them also undergo a phase of stalled (suppressed) growth. The end result, by $z = 1100$, is a power spectrum that differs considerably in shape from the initial power spectrum $P_i(k)$. Since $z_{\rm eq} = 3232$ is not much larger than $z_{\rm CMB} = 1100$ let us take $z_{\rm eq}$ as a reference redshift.

### 5.4.1 Suppression of growth during the radiation-dominated era

As we derived in Section 5.1.7, if a mode enters the horizon during the radiation-dominated era, its growth will cease. Instead, the modes will oscillate due to the radiation pressure. To be more precise: the *radiation-baryon "fluid"* will oscillate. What happens to the CDM will be discussed below. These oscillations of the radiation-baryon fluid are called *baryonic acoustic oscillations*, and we will study them later. Before the mode entered the horizon, though, information could not travel fast enough to prevent collapse. Therefore a mode grows as $a^2$ before entering the horizon and stays at the same amplitude after it enters the horizon.

Once the radiation decouples from the baryons (at $z \simeq 1100$) the radiation no longer behaves like a fluid, and the oscillations cease. In fact, already slightly before that, when matter starts dominating the energy density, the radiation pressure will become less effective at preventing the growth of modes.

If a mode enters the horizon *after* matter starts dominating the Universe, then there will be no period of stalled growth.

### 5.4.2 Effect of stalled growth on the CDM power spectrum

The stalling of growth and the formation of oscillations is a property of the radiation-baryon fluid. Of more interest to us is, however, the behavior of the cold dark matter (CDM), because dark matter halos will later be the birthplace of galaxies. During the radiation-dominated era it is the radiation fluid that produces the growth of modes. The CDM only interacts with the radiation fluid through gravity. And this interaction only goes one way: The CDM reacts to the gravitational potential of the radiation, but not vice versa, because radiation dominates the mass. The density of the CDM can increase at some point simply because CDM starts streaming into the gravitational well produced by a perturbation in the radiation fluid.

If a mode enters the horizon during the radiation-dominated era and the radiative fluid starts oscillating, this does not necessarily mean that the CDM behaves in the same way. In fact, on small enough scales the effect of the fluctuation in the gravitational potential on the CDM can be regarded as averaged out. The CDM perturbations can now only grow through their own gravity, which is much weaker than the radiation-driven perturbations through which the CDM perturbations grew before. This effectively stalls the growth of the CDM perturbations. A bit of growth still happens through the CDM self-gravity, but the growth goes with $\log(a)$. Let us, for simplicity, ignore this little bit of growth and consider the growth as being stalled. Once matter starts to dominate the Universe, growth proceeds linearly with $a$.

This stalling of the growth has a very strong consequence for the power spectrum of CDM density perturbations at the time of the CMB decoupling as well as for the power spectrum of the CMB anisotropies we see on the sky. Basically one expects for all $k$ which enter the horizon after $a_{\rm eq}$ (the scale factor $a$ at the time of matter-radiation equilibrium) to have a power spectrum similar in shape to the initial one ($P(k) \propto k$), but grown by a factor of $(a_{\rm eq}/a_i)^4$ since the end of inflation (where $a_i$ is the scale factor $a$ at the end of inflation):

$$P_{\rm eq}(k) = \left(\frac{a_{\rm eq}}{a_i}\right)^4 P_i(k) \qquad \text{for} \quad k < k_{\rm enter-eq} \tag{5.121}$$

where the subscript "eq" to $P$ means: power spectrum at the time of matter-radiation equilibrium; and where $k_{\rm enter-eq}$ is the $k$ for the mode which enters the horizon exactly at $a = a_{\rm eq}$.

However, for $k$ that enter the horizon before $a_{\rm eq}$, the growth proceeded only by a factor $(a_{\rm enter}(k)/a_i)^4$, where $a_{\rm enter}(k) \propto 1/k$ is the scale factor at the time when the mode with wavenumber $k$ entered the horizon.

$$P_{\rm eq}(k) = \left(\frac{a_{\rm enter}(k)}{a_i}\right)^4 P_i(k) \qquad \text{for} \quad k > k_{\rm enter-eq} \tag{5.122}$$

With $a_{\rm enter}(k) \propto 1/k$ this becomes $P_{\rm eq}(k) \propto k^{-3}$. In summary, the power spectrum at $z = a_{\rm eq}$ has roughly the following shape:

$$P_{\rm eq}(k) \propto \begin{cases} k & \text{for } k \ll k_{\rm enter-eq} \\ k^{-3} & \text{for } k \gg k_{\rm enter-eq} \end{cases} \tag{5.123}$$

For $k \simeq k_{\rm enter-eq}$ the curve smoothly passes from the $k$ to the $k^{-3}$ shape. The power $P_{\rm eq}(k)$ is maximal around $k \simeq k_{\rm enter-eq}$.

The way to interpret Eq. (5.123) is that very large modes (small $k$) still represent the shape (though of course not the amplitude) of the power spectrum at the end of inflation, while for small scales the power spectrum is modified due to the corresponding stalled growth.

The $k^{-3}$-dependence of the density power spectrum for small scale structures means that for $k \gg k_{\rm enter-eq}$ not only are the density perturbations scale-free (because of the powerlaw dependence), but also the amplitude of the perturbations is the same at all scales. We will use this property of the spectrum later on.

Now that we have an idea of the linear perturbations, the next step is to follow the subsequent non-linear behavior leading to the formation of dark matter halos, galaxies etc. This is the topic of the next chapter.