Chapter 6

Formation of structure in the Universe II: Formation of DM Halos

Once the density perturbations δ are no longer small, i.e. the condition $\delta \ll 1$ no longer holds, we must abandon the perturbation theory and treat the problem of the growth of structures in a non-linear way. A full treatment requires multi-dimensional numerical models such as dark matter (DM) structure formation simulations using N-body methods. The famous "Millennium Simulation" and "Mare Nostrum Simulation" are examples of very massive computations of this kind, and you are encouraged to google these simulations and see what kind of structure "real" simulations produce. What you will notice is that initially the structure produces a kind of "foam" structure with large voids, separated by walls, filaments and clumps of dark matter. In this chapter we will try to understand these structures using simplified approximate models. We will also try to make predictions for the statistical distribution masses of dark matter halos formed from these structures.

6.1 Zel'dovich approximation

Before the density perturbations go completely non-linear, there is a phase where the linear theory is no longer appropriate but a full non-linear treatment is not yet necessary. We can use an intermediate method: the Zel'dovich approximation. In this approximation we write the position of every dark matter particle $\vec{r}(t)$ as

$$\vec{r}(t) = a(t)\vec{x} + b(t)\vec{f}(\vec{x}) \tag{6.1}$$

The first term describes the usual expansion of the Universe and the second term describes the peculiar velocities. The vector field $\vec{f}(\vec{x})$ is very closely related to the peculiar velocities we derived in Section 5.1.6. The main new thing is that we now follow the motion of the particles for distances that are no longer small, i.e. we push this to the non-linear regime. It turns out that $b(t)$ can be written in terms of the $D_+(t)$ from that section:

$$\vec{r} = a\left[\vec{x} + D_+(a)\vec{f}\right] = a\left[\vec{x} + \frac{\vec{u}}{Hf(\Omega)}\right] \tag{6.2}$$

We therefore follow each dark matter particle along a straight line (since $\vec{f}$ does not change with time). Whereever the flow converges we get an increase of density, whereever it diverges the density goes down. It turns out that the result is the production of voids separated by walls of dark matter. Zel'dovich called them pancakes. This leads to a "foamy" structure. However, once the flow lines start crossing each other the Zel'dovich approximation breaks down, because it neglects the gravitational interaction between the particles which cross each other.

6.2 Full non-linear modeling of dark matter structure formation

To *really* model the growth of structure in the non-linear regime requires large scale 3-D models. The method of choice has been "N-body modeling" so far. This type of model computes the gravitional interactions between a huge set of point particles $i = 1 \cdots N$ that otherwise do not interact with each other. Each "particle" (which is just a computer representation of a portion of dark matter mass) has a position $\vec{r}_i(t)$ and velocity $\vec{v}_i(t) = d\vec{r}_i(t)/dt$ and a mass m_i. The equation of motion of particle i is

$$\frac{d^2\vec{r}_i(t)}{dt^2} = \sum_{k\neq i} \frac{Gm_k m_i}{|\vec{r}_k - \vec{r}_i|^3}(\vec{r}_k - \vec{r}_i) \tag{6.3}$$

i.e. it feels the gravitational pull of all the other particles. In practice this sum would require N^2 operations for each time step of the simulation. For large N this would be prohibitively computationally expensive. In practice various methods are used to approximate this sum for gravitational interactions between distance particles. Particle-Mesh (PM) methods compute the density of particles on a mesh and then use a Fourier transformation to solve the Poisson equation. An improved version of this, the Particle-Particle Particle-Mesh method (P^3M) allows particles that are near to each other to have direct interactions. A Tree Code handles distant gravitational coupling by grouping distant particles into groups.

A famous code for making such models is the GADGET code of V. Springel. Famous models are the Millennium Simulation and the Mare Nostrum simulation.

The results of these simulations show that, as predicted by the Zel'dovich approximation, large voids form, separated by dark-matter walls. At the intersection of the walls you get even denser dark matter ridges, and where the ridges meet you get even denser dark matter halos. These dark matter halos are the sites where galaxies and galaxy clusters are formed.

The non-linear evolution of the perturbations induces non-Gaussianity, even if the initial signal is gaussian. One can see this in the simplest way by realizing that δ is limited from below by 1 (because we cannot have negative densities) while it is unlimited from above (because we can increase the density by factors of many). This necessarily skews the probability distribution function for the density toward large densities, and thus breaks gaussianity.

6.3 Spherically symmetric model of DM halo formation

In Chapter 2 we made a model of the expanding Universe based on a spherically symmetric set of concentric shells of dark matter. It turns out that we can use this model almost 1-to-1 to model the formation of DM halos. Suppose that in an otherwise homogeneous Universe filled with DM there is a spherical patch of somewhat higher density. If we apply the model of Chapter 2 to this patch, we can simply ignore the Universe outside of this patch, since by the laws of Newtonian dynamics the matter inside the patch does not feel the matter outside the patch. The path is therefore like a little mini-Universe. Since we assumed it to be slightly denser than the surrounding, and we assume that the surrounding Universe has an exactly critical density, the patch is super-critical. We know that this must collapse, as would a closed Universe. This is a simple model of DM halo formation.

Conversely if the patch is assumed to be underdense, the expansion of the patch would be faster than the Universe surrounding it, and a void is created. This is a simple model for the formation of the voids seen in the full N-body simulations.

Let us focus on a collapsing (overdense) patch with homogeneous density, and radius $R(t)$ embedded in an Einstein de Sitter Universe. For the purpose of later use, we will cast this solution in a slightly adapted form, to embed it in an Einstein de Sitter Universe. We assume that we know when the "turn around" point occurs, i.e. when the patch has reached its maximum extent. The radius of this patch at this turn around time is called $R_{\rm ta}$, as in Chapter 2. The time of the turn around after the Big Bang is called $t_{\rm ta}$, and the scale factor at that time $a_{\rm ta}$. We have $H = H_0 a^{-3/2}$ for the Einstein de Sitter Universe, and we define $H_{\rm ta} = H_0 a_{\rm ta}^{-3/2}$. We now define three dimensionless quantities:

$$x := \frac{a}{a_{\rm ta}} \quad , \quad y := \frac{R}{R_{\rm ta}} \quad , \quad \tau := H_{\rm ta} t \tag{6.4}$$

We also define the overdensity parameter ξ as follows:

$$\xi := \left.\frac{\rho}{\rho_{\rm crit}}\right|_{a=a_{\rm ta}} \tag{6.5}$$

i.e. the density in the patch in units of the critical density, evaluated at the time of turn-around. One can show that:

$$\tau = \frac{2}{3}x^{3/2} = \frac{1}{\sqrt{\xi}}\left[\frac{1}{2}\arcsin(2y-1) - \sqrt{y-y^2} + \frac{\pi}{4}\right] \tag{6.6}$$

up to the turn around point. By expressing $\tau(y)$ we implicitly defined the function $y(\tau)$, which is actually what we want. An explicit analytical form of $y(\tau)$ does, however, not exist, so we will have to be content with this implicit form. The turn-around happens at $x = y = 1$ and $\tau = 2/3$. This implies that

$$\xi = \left(\frac{3\pi}{4}\right)^2 \simeq 5.55 \tag{6.7}$$

For $\tau > 2/3$ the solution is the reverse: $\tau \to \frac{4}{3} - \tau$. The solution collapses to a point at $\tau = 4/3$, which is at $x = x_c = 2^{2/3}$.

The overdensity at the center of the collapsing halo compared to the average density of the Universe can be written as:

$$\Delta = \left(\frac{x}{y}\right)^3 \xi \tag{6.8}$$

because the density of the background goes as $1/a^3 \sim 1/x^3$ while the density in the halo goes as $1/y^3$. In chapter 5 we introduced, instead, $\delta \equiv \rho/\rho_0$, which is related to Δ as

$$\delta = \Delta - 1 \tag{6.9}$$

Just as a consistency check: In the limit $x \to 0$ you can verify that $\Delta \to 1$, i.e. $\delta \to 0$, as expected.

At early times ($0 < x \ll 1$) we can linearize the solution (which is a bit cumbersome) to obtain the following expression for δ:

$$\delta = \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3} x = \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3} \frac{a}{a_{\rm ta}} \tag{6.10}$$

This expression confirms that δ is linearly proportional to a, as we already derived from linear perturbation theory.

Something that is often done is to use this *linear* expression to estimate the density contrast δ at $t = t_{\rm ta}$, even though we would actually have to use the full non-linear solution for that. It would give, however, a reasonable (and much more easy-to-use) estimate of the density at turn-around:

$$\delta_{\rm lin,ta} = \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3} \simeq 1.04 \tag{6.11}$$

If we use the same method to estimate the density at the time of collapse, which is at $x = 2^{2/3}$:

$$\delta_{\rm lin,c} = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3} \simeq 1.69 \tag{6.12}$$

It turns out that we can turn the argument around and say that a halo is considered collapsed when, in the *linear* theory, the δ reaches a value of 1.69. It turns out that this is actually a good approximate criterion for collapse for any set of cosmological parameters, i.e. not only for the Einstein de Sitter model.

In the full non-linear model, of course, the density at the time and location of collapse should be infinite. In practice, however, the simple spherical halo model is not a good model of reality. We saw earlier that we have an entire spectrum of modes. Therefore, instead of going to infinite density, the halo will *virialize*. The trajectories of the DM particles will come very close to each other shortly before $a = a_{\rm c}$ and they will gravitationally swing by each other, thus converting systematic motion into random motion. This random motion can be regarded as a "temperature". In other words: after virialization we will, at a given spatial position in the DM halo, have DM particles moving in different directions. The DM is then no longer strictly cold anymore: It is hot, and has a temperature equal to the virial temperature.

We can estimate roughly what the size of such a virialized DM halo would be. According to the virial theorem, the kinetic energy of the particles must, on average, be equal to half the potential energy. Since the collapse converts systematic (collapse-) motion into random motions, we can estimate the size of the virialized DM halo by equating the kinetic velocity at some dimensionless radius y to the potential energy at that radius. This happens at $y = \frac{1}{2}$. The overdensity Δ_v at that time compared to the background density is given by Eq. (6.8) with $y = 1/2$ and $x = x_c = 2^{2/3}$ (the time of collapse):

$$\Delta_v = \left(\frac{2^{2/3}}{1/2}\right)^3 \times 5.55 \simeq 178 \tag{6.13}$$

This is the overdensity of the halo once it has collapsed and virialized. Note that this is *much* higher than the overdensity predicted from linear theory: $1 + \delta_c = 2.67$, but it is not infinity either.

While both δ_c and Δ_v are approximate estimates of the overdensity they can be used as reference for characterizing dark matter halo properties. We will use them in fact when we derive the Press-Schechter mass function below.

Note that this spherically collapsing DM halo model neglects the fact that (as one can show rigorously, see script of Matthias Bartelmann) a tiny ellipticity at the beginning will amplify, making the halo more elliptic as the collapse proceeds.

6.4 Press-Schechter mass function for halos

Although the spherical collapse model gives us some clues to the time scales of collapse, we still need to somehow link it to the initial perturbations we studied in Chapter 5. Only then will we be able to make estimates of the kind of DM halos that are formed. Dark matter halos are the non-linear "end"-product of the growth of DM perturbations. Of course, they are not true end-products, because at some point in time the halos start to be attracted to each other and they will merge. But let us postpone that for a later section and concentrate on the halos produced by direct collapse of the original perturbations. We wish to derive from the power spectrum of the density perturbations a sort of "initial mass function" for the halos before they attract each other and merge.

6.4.1 Relation between size scales and halo masses

Let us first define a relation between distance scales and mass. If we want to form a DM halo of mass M, we need to collect matter from a sphere of radius R such that

$$\frac{4\pi}{3}R^3\rho_0 = M \tag{6.14}$$

where ρ_0 is the background density. If we define ρ_0 to be the background density *at the present time*, then the above expression can also be used at $z > 0$ as long as we understand R to be a distance scale in comoving $\vec{x}$ coordinates (not in physical $\vec{r}$ coordinates). This will be convenient if we want to compare R to $1/k$.

6.4.2 Basic idea of the Press-Schechter model

The central idea of the Press-Schechter model we are going to describe here is that if we look at the *linear* density perturbation field $\delta(\vec{x}, t)$ at some time t, there may be regions where $\delta(\vec{x}, t) > \delta_c$. According to the simple spherical collapse model described above, by the time the *linear* perturbation exceeds δ_c, the *true* solution has already collapsed and virialized. We can therefore use the linear perturbation theory to predict which regions of space have already collapsed to form DM halos.

It is important to understand that this analysis cannot be done for each wave mode $\vec{k}$ separately. It is the sum of the modes at one specific location $\vec{x}$ that may, or may not, exceed δ_c. Also, we know from the previous chapter that if we do not introduce a window function $W_R(\vec{x}-\vec{x}')$ to smooth-out wave modes of $k \gg 1/R$, then the variance σ^2 diverges. This makes an analysis meaningless. We therefore have to work with a smoothed version of $\delta(\vec{x}, t)$:

$$\delta_R(\vec{x}, t) = \int \delta(\vec{x}', t)W_R(\vec{x} - \vec{x}')d^3x' \tag{6.15}$$

Now let us start with large R, so that all small-scale structures are washed out. Suppose that at early enough times the $\delta_R(\vec{x}, t)$ will virtually nowhere exceed δ_c. We interpret this that no DM halos of mass $(4\pi/3)R^3\rho_0$ have yet formed. Now let us gradually decrease R. This means that new (higher $\vec{k}$) modes are added to the already existing density perturbation. Now remember that for large enough k we have $P(k) \propto 1/k^3$, and remember that this means that not only is this spectrum scale-free, but it also has equal power per order of magnitude in k. This means that every time we decrease R by a factor 10 we add perturbations of equal variance as the previous contribution. If at the largest R the $\delta_R(\vec{x}, t)$ was not *too* small, then as you decrease R there will come a point when $\delta_R(\vec{x}, t)$ will reach δ_c at some points in space. We interpret this that at those points in space dark matter halos of typical size R and mass $M = (4\pi/3)R^3\rho_0$ are formed.

As we continue to decrease R we will obtain more and more points in space where $\delta_R(\vec{x}, t)$ exceeds δ_c. In other words: smaller halos have also already formed, and the smaller the halo size, the more of them we find.

6.4.3 Non-linear mass

At a given time there will be a typical size scale R_* that just becomes, on average, non-linear and forms DM halos of mass M_*. Halos of smaller mass have already been formed and halos of larger mass have not yet been formed. This typical mass M_* is called the *non-linear mass*. It is defined such that the standard deviation squared $\sigma^2_{R_*}$ at length scale R_* corresponding to mass M_* equals δ_c^2:

$$\sigma^2_{R_*} = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k)\hat{W}^2_{R_*}(k) = \delta_c^2 \tag{6.16}$$

6.4.4 Derivation of the Press-Schechter mass function

If we assume that $\delta_R(\vec{x}, a)$ is a Gaussian random field at the time given by the scale factor a, then we can write the probability of finding $\delta_R(\vec{x}, t)$ between some value δ and $\delta + d\delta$:

$$p_R(\delta, a)d\delta = \frac{1}{\sqrt{2\pi}\sigma_R(a)} \exp\left(-\frac{\delta^2}{2\sigma_R^2(a)}\right) d\delta \tag{6.17}$$

Since $\sigma_R(a)$ will grow with a, this probability distribution function changes with time.

According to the model assumption by Press & Schechter, the fraction $F(M, a)$ of the cosmic volume filled with halos of masses M *or larger* is given by the fraction of the cosmic volume that has the linear filtered density $\delta_R(\vec{x}, a)$ above δ_c. In formulae:

$$F(M, a) = \int_{\delta_c}^{\infty} p_R(\delta, a)d\delta = \frac{1}{2}\text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R(a)}\right) \tag{6.18}$$

where erfc is the *complementary* error function. The function $F(M, a)$ is some sort of cumulative distribution function (though see a discussion on this slightly lateron).

To get the actual distribution, we must take the derivative of $F(M, a)$ with respect to M:

$$\frac{\partial F(M, a)}{\partial M} = \frac{1}{2}\frac{\partial}{\partial M}\text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R(a)}\right) \tag{6.19}$$

The M-dependence is hidden in $\sigma_R(a)$. We can write

$$\frac{\partial}{\partial M} = \frac{d\sigma_R}{dM}\frac{\partial}{\partial \sigma_R} \tag{6.20}$$

Using

$$\frac{d}{dx}\text{erfc}(x) = -\frac{2}{\sqrt{\pi}}e^{-x^2} \tag{6.21}$$

we find

$$\frac{\partial F(M, a)}{\partial M} = \frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma_R(a)}\frac{d\ln\sigma_R}{dM}\exp\left(-\frac{\delta_c^2}{2\sigma_R^2(a)}\right) \tag{6.22}$$

With the definition

$$\sigma_R(a) = \sigma_R D_+(a) \tag{6.23}$$

(where σ_R is the variance measured today) we get the slightly more familiar form:

$$\frac{\partial F(M, a)}{\partial M} = \frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma_R D_+(a)}\frac{d\ln\sigma_R}{dM}\exp\left(-\frac{\delta_c^2}{2\sigma_R^2 D_+^2(a)}\right) \tag{6.24}$$

Before we can convert this into a number density of halos we must become aware of a subtlety. Because $F(M, a)$ is a cumulative distribution function, we might expect it to be 0 for $M = 0$ and to go to 1 for $M \to \infty$. However, from Eq. (6.18) we can see that

$$\lim_{M\to 0} F(M, a) = \frac{1}{2} \quad \text{and} \quad \lim_{M\to\infty} F(M, a) = 0 \tag{6.25}$$

This means that at most half of our volume will be filled with halos of *any* mass. The reason for this is because $\langle\delta\rangle = 0$ by definition. So for every excursion above δ_c there must be a compensating excursion below $-\delta_c$.

As a result of this strange factor 1/2, in order to convert this into a comoving number density (number density per $\vec{x}$-volume) for halos between mass M and dM we must divide the above expression by *half* the mean volume of such halos: M/ρ_0 with ρ_0 being the DM density today:

$$N(M, a)dM = \sqrt{\frac{2}{\pi}}\frac{\rho_0\delta_c}{\sigma_R D_+(a)}\frac{d\ln\sigma_R}{dM}\exp\left(-\frac{\delta_c^2}{2\sigma_R^2 D_+^2(a)}\right)\frac{dM}{M} \tag{6.26}$$

This is the famous Press-Schechter mass function. It turns out that it holds surprisingly well up to full scale N-body calculations.

6.4.5 Hierarchical structure formation

From the previous analysis we see that small halos form first, and then larger halos form. But often the small-scale density perturbation are on top of a larger-scale one. In fact, only in that way can they become non-linear in the first place. So after the small-scale perturbations have become non-linear and formed small DM halos, at some point the larger-scale perturbation on top of which the small scale perturbation "stood" also becomes non-linear and virializes. In practice what happens is that the small halos merge to form a bigger one. You can pictographically represent this with a "merger tree". According to large scale simulations, the virialization is, however, not expected to lead to perfectly smooth halos: the original small-scale DM halo structures are expected to remain present as "clumps" inside the larger DM halo. That is the case, at least, if the DM is indeed cold. If the DM has some prior temperature, this could lead to smoother halos.

6.4.6 Halo formation as a random walk

As we already noted, for the power spectrum shape of $P(k) \propto 1/k^3$ there is equal power in each equal interval in $\ln(k)$. If we start from a smoothed-out δ at large R (small k) and we increase k with equal factors of, say, 2, then we essentially add perturbations of similar amplitude in each step. If we look at a fixed position $\vec{x}$ then this procedure resembles a 1-D random walk in δ.

During the procedure of gradually decreasing R the δ at some given point could, at some point $\vec{x}$, exceed δ_c. In that case we call $\vec{x}$ part of a halo of the mass M corresponding to R. Let us call this mass M_c for later reference. If we now continue our random walk, it could happen that δ drops below δ_c again. This is because in our more massive halo we now get substructure, some of which may drop below δ_c. For the sake of consistency, however, we still want our point $\vec{x}$ to belong to the halo of mass M_c. Moreover, although point $\vec{x}$ may be *also* part of a sub-halo of mass $M \ll M_c$ that is contained inside the halo of mass M_c, we consider the largest halo, i.e. that of mass M_c, to be the one that counts. To account for this in the random walk picture we stop the random walk essentially when it enters the $> \delta_c$ region. It is therefore some sort of "absorbing barrier".

Now let us calculate the probability $p_s(\delta, a)$ that δ is reached via a path that *never crossed the absorbing barrier*, i.e. an entirely "allowed" path. To calculate this we define the "mirror point" of δ:

$$\delta_m = \delta_c + (\delta_c - \delta) = 2\delta_c - \delta \tag{6.27}$$

As soon as a path reaches the absorbing barrier, it has 50% chance to return below the barrier and 50% chance to continue above the barrier. Suppose it goes above the barrier. Once we are an infinitesimal bit above the barrier the distance to δ is still the same as to δ_m, so there is still 50% chance to go back to δ as to δ_m. So to study the probability to reach δ from that point is identical to studying the probability to reach δ_m. However, to reach δ_m one must pass through "forbidden" territory. In other words: Finding the probability to reach δ_m is equivalent to finding the probability to reach δ via a path that *at least once* passed beyond the absorbing barrier.

So if we want to find the probability to reach δ *without ever passing the barrier* we must compute the probability of reaching δ via *any* path, minus the probability of reaching δ_m:

$$p_s(\delta) = \frac{1}{\sqrt{2\pi}\sigma_R}\left[\exp\left(-\frac{\delta^2}{2\sigma_R^2}\right) - \exp\left(-\frac{(2\delta_c - \delta)^2}{2\sigma_R^2}\right)\right] \tag{6.28}$$

We have now the probability of reaching δ without ever exceeding δ_c on *any* scale down to R. To find the probability of δ to *exceed* δ_c on *some* scale is thus

$$1 - P_s = 1 - \int_{\delta_c}^{\infty} d\delta p_s(\delta) = \text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma_R}\right) \tag{6.29}$$

Here we do not run into the problematic factor 1/2. The rest of the derivation of the Press-Schechter mass function is the same as before.

This method of a random walk is a powerful tool for further analysis of the statistics of DM halos. One can analyze merger histories, for example. Please refer to the lecture script of Matthias Bartelmann for further details.

6.5 Halo density profiles

The structure of DM halos can be complex. But some simple models can be made. One simple approximation is to regard the virialized DM as a fluid with a pressure. Let us make a spherical model of a DM halo in that approximation. We then have the pressure equilibrium equation:

$$\frac{dp}{dr} = -\frac{GM(r)}{r^2}\rho \tag{6.30}$$

where

$$M(r) = 4\pi \int_0^r \rho(r')r'^2 dr' \tag{6.31}$$

and

$$p = \frac{\rho}{m} kT \tag{6.32}$$

with m the DM particle mass. If we now take T=constant, then the equation of hydrostatic equilibrium can be brought into the form

$$\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -\frac{4\pi Gm}{kT} r^2 \rho \tag{6.33}$$

One solution is the singular isothermal sphere:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2} \qquad \text{with} \qquad \sigma_v^2 = \frac{kT}{m} \tag{6.34}$$

Another is a flat-core profile (Bonnor-Ebert sphere), which can e approximated by:

$$\rho(r) = \frac{\rho_0}{1 + (r/r_0)^2} \tag{6.35}$$

which, by the way, becomes equal to the isothermal sphere for large r. Note that the mass of this sphere diverges as $r \to \infty$. Therefore this model can only describe the inner parts of DM halos.

From numerical simulations it turns out that a more accurate profile approximation is:

$$\rho(r) = \frac{\rho_s}{x(1+x)^2} \qquad \text{with} \qquad x := \frac{r}{r_s} \tag{6.36}$$

This is the so-called Navarro-Frenk-White profile (NFW profile). It goes as $1/r^3$ for large r and flattens to $1/r$ for small r.

The virial radius is often defined as the radius enclosing a mass with 200 times overdensity over the background. The 200 is just a convenient "approximation" of the number 178 we derived earlier.

$$r_{200} = \left(\frac{GM}{100H^2}\right)^{1/3} \tag{6.37}$$

The ratio

$$c = \frac{r_{200}}{r_s} \tag{6.38}$$

is called the "concentration" of the halo.

Although the NFW-like profiles follow directly from N-body numerical simulations, it is not clear from fundamental principles why they have this shape.