Chapter 9

Inflation

9.1 Introduction

We already discussed inflation as a theory for the formation of the perturbations in the density of the early Universe. Inflation is in fact a convenient model for various other reasons as well:

- Observations of the CMB show that the Universe is approximately uniform at scales that are much larger than the particle horizon at the time of the CMB release. Regions that were cuasally detached still had the same properties. Inflation is a model by which these regions in fact *were* causally connected before the onset of inflation but got ripped apart by the inflation. This is a natural explanation for the homogeneity of the Universe at large scales.
- Inflation predicts that any pre-existing curvature of space (the Universe being open or closed) would be flattened out by the extreme expansion. This is a natural explanation for the observed flatness of the Universe today.
- The idea that the density perturbations in the universe originate from quantum fluctuations of an "inflaton field" not only naturally predicts the power spectrum correctly, but also predicts that the fluctuations are largely gaussian. This is because the fluctuations in the gravitational potential are caused by a linear sum of extremely many quantum fluctuations of the inflaton field. The *central limit theorem* then predicts that the result should be gaussian.
- The same theory also predicts the existence of slight deviations from gaussianity, because the fluctuations we observe in the CMB were the ones that were created shortly before the end of inflation. The end of the inflation period therefore naturally introduces non-linear effects.

9.2 Conditions for inflation

For inflation to be able to causally disconnect regions that were, before, in causal contact, the expansion must be so rapid that there exists an event horizon at a finite distance from any point. This is equivalent to saying that the Hubble radius in comoving coordinates must shrink in time:

$$\frac{d}{dt}\left(\frac{c}{aH}\right) < 0 \tag{9.1}$$

This implies, with $H = \dot{a}/a$, that

$$\frac{d}{dt}\left(\frac{c}{\dot{a}}\right) < 0 \tag{9.2}$$

which implies, in turn,

$$\ddot{a} > 0 \tag{9.3}$$

In other words: inflation requires accelerated expansion. With the second Friedmann equation without a cosmological constant

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{9.4}$$

and with the condition that $\rho \geq 0$, this directly implies that

$$p < -\frac{\rho c^2}{3} \tag{9.5}$$

for inflation to occur. We must therefore have a substance that has a negative pressure, and sufficiently negative.

9.3 The inflaton field

9.3.1 The basic idea of inflation using an inflaton scalar field

Suppose the Universe is permeated by a scalar field:

$$\phi(\vec{x}, t) \tag{9.6}$$

called the *inflaton field*. Suppose that at the Big Bang this field starts at a non-zero value:

$$\phi(\vec{x}, 0) = \phi_0 > 0 \tag{9.7}$$

(or at least at a value that is not the ground energy state). As we shall see below, this may lead to the condition of negative pressure, thus leading to a rapid, exponential expansion of the Universe ($\ddot{a} > 0$). At some point in time, however, the conditions for inflation cease to be fulfilled and inflation stops. Expansion is, from this time onward, non-accelerating $\ddot{a} < 0$, but still positive: $\dot{a} > 0$. It will proceed, from that point onward, according to the usual radiation-dominated expansion. It is *assumed* that around this time the inflaton field decays into other particles, which are the predecessors of the particles from which we are made. This is called *reheating of the Universe*. However, it is not known how this happens. In fact, even the Nature of the inflaton field is not known. At present its existence is purely speculation. However, the model of inflation works very well to explain the many questions posed in Section 9.1.

9.3.2 Lagrangian, energy density and pressure of the inflaton field

The Lagrangian of a scalar field is of the form:

$$\mathcal{L} = -\frac{1}{2}c^2\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{9.8}$$

(note that we have $g^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$) where $V(\phi)$ is some potential for the scalar field. A potential for a field typically gives the particles associated with that field a mass. In fact, the entire idea of the Higg boson is to provide, through its coupling to other particles, a potential to those particles and thus a mass. Likewise the Higgs gives a mass to itself. Early theories of inflation (including the original model of inflation by Alan Guth in 1980) in fact conjectured that the Higg field was the inflaton field, but this has later been put into doubt (though some recent work by Shaposhnikov & Bezrukov, arxiv/0710.3755, explains a possible way out of that problem).

The energy-momentum tensor for such a scalar field is:

$$T_{\mu\nu} = c^2\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L} \tag{9.9}$$

Note that:

$$c\partial_0\phi = c\frac{\partial\phi}{\partial x^0} = c\frac{\partial\phi}{\partial(ct)} = \frac{\partial\phi}{\partial t} =: \dot{\phi} \tag{9.10}$$

We can write the energy density T_{00} as

$$\rho c^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + c^2(\nabla\phi)^2 \tag{9.11}$$

and the pressure T_{ii} as

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{1}{6}c^2(\nabla\phi)^2 \tag{9.12}$$

If we assume homogeneity and isotropy, the spatial derivatives vanish and we get

$$\rho c^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{9.13}$$

and

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{9.14}$$

9.3.3 Slow roll condition for inflation

Now, in Section 9.2 we saw that for inflation to occur we must have $p < -\frac{1}{3}\rho c^2$. In terms of the inflaton field this condition becomes

$$\frac{1}{2}\dot{\phi}^2 - V(\phi) < -\frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{9.15}$$

which leads to the condition:

$$\dot{\phi}^2 < V(\phi) \tag{9.16}$$

for inflation to occur. If ϕ is also the only (or dominant) form of energy in the universe at that time, then indeed the universe will expand in an accelerated way. Condition Eq. (9.16) is called the *slow roll condition*. We will study when and how the slow roll condition can be met. But let us first have a look at the equations governing the time-evolution of the inflaton field, and its effect on the Universe.

9.3.4 Dimensional analysis

Before we continue, let us first do some dimensional analysis, to get a better feeling for the numbers that we are dealing with:

$$[V(\phi)] = \frac{\text{erg}}{\text{cm}^3} \tag{9.17}$$

$$\left[\dot{\phi}^2\right] = \frac{\text{erg}}{\text{cm}^3} \tag{9.18}$$

$$[\phi] = \left(\frac{\text{ergs}^2}{\text{cm}^3}\right)^{1/2} = \left(\frac{\text{gram}}{\text{cm}}\right)^{1/2} \tag{9.19}$$

We can define a characteristic value for ϕ, the Planck value:

$$\phi_{\text{Planck}} = \sqrt{\frac{m_{\text{Planck}}}{l_{\text{Planck}}}} \tag{9.20}$$

with

$$m_{\text{Planck}} = \sqrt{\frac{\hbar c}{G}} = 2.17\times 10^{-5}\ \text{gram} \tag{9.21}$$

$$l_{\text{Planck}} = \sqrt{\frac{\hbar G}{c^3}} = 1.62\times 10^{-33}\ \text{cm} \tag{9.22}$$

we get

$$\phi_{\text{Planck}} = \frac{c}{\sqrt{G}} = 1.16\times 10^{14}\ \frac{\text{gram}^{1/2}}{\text{cm}^{1/2}} \tag{9.23}$$

This value will play a role lateron. Note that this characteristic field strength has no bearing anymore to $\hbar$, so it is, in a sense, no longer a "Planck"-strength.

9.4 Equations for the inflaton field coupled to Universe expansion

9.4.1 The equations

The expansion of the Universe is governed by the two Friedmann equations and the equation of adiabatic expansion (Eqs. 4.9, 4.10 and 4.11), where, as usual, one of these three equations is redundant. Let us, again as usual, drop the second Friedmann equation (Eq. 4.10). The remaining equations are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} \tag{9.24}$$

$$0 = \frac{d}{dt}(\rho a^3 c^2) + p\frac{d}{dt}(a^3) \tag{9.25}$$

If we insert Eqs. (9.13 and 9.14) into these equations we obtain

$$H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{Kc^2}{a^2} \tag{9.26}$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV(\phi)}{d\phi} \tag{9.27}$$

These two equations are sufficient to evolve both $\phi(t)$ and $a(t)$. Whether or not accelerated expansion happens may not be obvious from these equations, but it will follow automatically if one integrates them in time, given certain initial conditions.

9.4.2 Simplification of the equations during inflation

Suppose we start at some time $t = t_{\text{start}} > 0$ with a situation where the Kc^2/a^2 is not negligible compared to the ϕ-dependent terms. Then, either the Universe collapses (if K is sufficiently negative), in which case we would not be there. Or the Universe expands, in which case the Kc^2/a^2 term becomes progressively smaller. If we start with $\dot{\phi} = 0$, then we have clearly (see above) the condition for accelerated expansion met. *If*, over a time scale by which a substantially changes, the ϕ-field does not change much, and thereby $V(\phi)$ does not change much, then the Kc^2/a^2 term quickly becomes negligible compared to the other terms. Then Equation (9.26) quickly reduces to

$$H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{9.28}$$

to very good approximation. *If* the slow roll condition $\dot{\phi}^2 \ll V(\phi)$ is also met, then we obtain

$$H^2 = \frac{8\pi G}{3c^2}V(\phi) \tag{9.29}$$

Also, *if* the slow roll condition is met, we can derive (by taking the time derivative) that

$$\ddot{\phi} \ll \frac{dV(\phi)}{d\phi} \tag{9.30}$$

In that case Equation (9.27) reduces to

$$3H\dot{\phi} = -\frac{dV(\phi)}{d\phi} \tag{9.31}$$

9.4.3 A-posteriori check of conditions

Whether we can use Eqs. (9.29,9.31) depends on whether the conditions mentioned above are met. What one can do is solve Eqs. (9.29,9.31) and verify a-posteriori whether the conditions are indeed met at all times. If not, then we must fall back to the original equations, Eqs. (9.26,9.27).

If we take the square of Eq. (9.31) we obtain

$$9H^2\dot{\phi}^2 = (V'(\phi))^2 \tag{9.32}$$

where $V' \equiv dV/d\phi$. If we then insert Eq. (9.29) we get

$$\frac{24\pi G}{c^2}\dot{\phi}^2 V(\phi) = (V'(\phi))^2 \tag{9.33}$$

or equivalently

$$\dot{\phi}^2 = \frac{c^2}{24\pi G}\frac{(V'(\phi))^2}{V(\phi)} \tag{9.34}$$

The slow roll condition $\dot{\phi}^2 \ll V(\phi)$ can thus be written as

$$\frac{c^2}{24\pi G}\frac{(V'(\phi))^2}{V(\phi)} \ll V(\phi) \tag{9.35}$$

which reduces to

$$\frac{c^2}{24\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \epsilon \ll 1 \tag{9.36}$$

So what about the $\ddot{\phi} \ll V'(\phi)$ condition? If we take $\dot{\phi}$ from Eq. (9.31) and we insert this into:

$$\frac{d}{dt}\dot{\phi} = \frac{d}{dt}\left(-\frac{V'(\phi)}{3H}\right) \tag{9.37}$$

then we can work this out to:

$$\frac{d}{dt}\dot{\phi} = -\frac{1}{3H}V''(\phi)\dot{\phi} + \frac{V'(\phi)}{3}\frac{\dot{H}}{H^2} \tag{9.38}$$

Now let us take the time derivative of Eq. (9.29)

$$\frac{d}{dt}H^2 = 2H\dot{H} = \frac{8\pi G}{3c^2}V'(\phi)\dot{\phi} = H^2\frac{V'(\phi)}{V(\phi)}\dot{\phi} \tag{9.39}$$

so that we can write

$$\frac{\dot{H}}{H^2} = \frac{1}{2H}\frac{V'(\phi)}{V(\phi)}\dot{\phi} \tag{9.40}$$

Inserting this into Eq. (9.38) gives

$$\frac{d}{dt}\dot{\phi} = -\frac{1}{3H}V''(\phi)\dot{\phi} + \frac{1}{H}\frac{(V'(\phi))^2}{6V(\phi)}\dot{\phi} \tag{9.41}$$

Now we use this in the condition Eq. (9.30),

$$-\frac{1}{3H}V''(\phi)\dot{\phi} + \frac{1}{H}\frac{(V'(\phi))^2}{6V(\phi)}\dot{\phi} \ll V'(\phi) = -3H\dot{\phi} \tag{9.42}$$

By dividing out $-\dot{\phi}$ (because we assume here that $\dot{\phi} < 0$) and dividing by $3H$ we get

$$\frac{1}{3H^2}V''(\phi) - \frac{1}{H^2}\frac{(V'(\phi))^2}{6V(\phi)} \ll 3 \tag{9.43}$$

Now replace H^2 using Eq. (9.29) again so that we obtain

$$\frac{c^2}{8\pi G}\frac{V''(\phi)}{V(\phi)} - \frac{3}{2}\epsilon \ll 3 \tag{9.44}$$

Since $\epsilon \ll 1$ we thus get

$$\frac{c^2}{8\pi G}\frac{V''(\phi)}{V(\phi)} \equiv \eta \ll 1 \tag{9.45}$$

(where we replaced the 3 with a 1 because that is equivalent).

The two slow-roll conditions are thus:

$$\epsilon := \frac{c^2}{24\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll 1 \qquad \text{and} \qquad \eta := \frac{c^2}{8\pi G}\frac{V''(\phi)}{V(\phi)} \ll 1 \tag{9.46}$$

9.5 Inflaton field with a quadratic potential

There are many possible forms of the potential $V(\phi)$ that one may try out to see if inflationary behavior can be obtained. It is not known what the "correct" form should be. The simplest form would be a quadratic potential, which is what we discuss here.

9.5.1 General behavior

Let us take the following potential (in units of erg/cm^3):

$$V(\phi) = q\phi^2 \tag{9.47}$$

In quantum field theory, the parameter q can be associated with the mass of the corresponding particle through:

$$m^2 = \frac{\hbar^2}{c^4}q \tag{9.48}$$

But let us, for simplicity, stick to the q. We then have

$$\frac{V'(\phi)}{V(\phi)} = \frac{2q\phi}{q\phi^2} = \frac{2}{\phi} \qquad \text{and} \qquad \frac{V''(\phi)}{V(\phi)} = \frac{2q}{q\phi^2} = \frac{2}{\phi^2} \tag{9.49}$$

The slow roll conditions can then be written (using $\phi_{\text{Planck}} = c/\sqrt{G}$) as

$$\phi \gg \frac{1}{\sqrt{6\pi}}\phi_{\text{Planck}} \qquad \text{and} \qquad \phi \gg \frac{1}{\sqrt{4\pi}}\phi_{\text{Planck}} \tag{9.50}$$

respectively. These two conditions are, in this case, equivalent. One sees that this condition is independent of q.

Inflation proceeds according to Eqs. (9.29,9.31) until ϕ drops below $\phi_{\text{Planck}}/\sqrt{4\pi}$. During the inflation H *slowly* decreases because $V(\phi)$ slowly decreases. In other words, the variation time scale for H obeys

$$\tau_{\text{vary}} = \left(\frac{d\ln H}{dt}\right)^{-1} \gg \frac{1}{H} := t_{\text{exp}} \tag{9.51}$$

where t_{exp} is the typical expansion time scale $1/H$. Equivalently this can be written as $|\dot{H}| \ll H^2$. This means that for a time scale substantially smaller than τ_{vary} we can regard H as approximately constant. This leads, to very good approimation, to exponential growth of the Universe:

$$a(t) \propto e^{Ht} \tag{9.52}$$

Let us write the Hubble constant at the end of inflation: H_{infl}. When inflation ends, $H(t)$ will drop from H_{infl} in the usual way of a radiation-dominated Universe:

$$H(t) = \frac{1}{2(t-t_0)} = \frac{H_{\text{infl}}}{2(t-t_{\text{infl}})H_{\text{infl}} + 1} \tag{9.53}$$

where t_0 is defined as

$$t_0 = t_{\text{infl}} - \frac{1}{2H} \tag{9.54}$$

9.5.2 Some consistency checks and calculations

If we want $\dot{\phi}^2 \ll V(\phi)$, then this gives a lower limit on the *duration* τ_{infl} of inflation, assuming that during the inflation process ϕ goes from some initial ϕ_0 an appreciable way toward $\phi \to 0$. Very roughly this yields the condition that $\tau_{\text{infl}} \gg 1/\sqrt{q}$ (or, given an assumed time scale τ_{infl} this puts a lower limit on $q \gg 1/\tau_{\text{infl}}^2$).

Interestingly, the same limit on $\tau_{\rm infl}$ can be obtained in another way, by using Eq. (9.29) and putting $\phi = \phi_{\rm Planck}/\sqrt{4\pi}$ for the end of inflation:

$$H^2 = \frac{8\pi G}{3c^2} q\phi^2 = \frac{2G}{3c^2} q\phi^2_{\rm Planck} = \frac{2}{3} q \tag{9.55}$$

The Hubble constant defines the expansion e-folding time scale $\tau_{\rm exp} = 1/H$. For succesful inflation we want

$$\tau_{\rm infl} \gg \tau_{\rm exp} \tag{9.56}$$

i.e. we want inflation to expand the Universe by many e-folding times. Inserting $\tau_{\rm exp} = 1/H = \sqrt{3/2q}$, and taking the square, we obtain

$$q \gg \frac{3}{2}\frac{1}{\tau^2_{\rm infl}} \tag{9.57}$$

Right after the end of inflation the inflaton field should, by definition, have an energy density corresponding to the critical energy density:

$$\rho c^2 = \rho_{\rm crit} c^2 = \frac{3H^2 c^2}{8\pi G} = \frac{3H^2}{8\pi}\phi^2_{\rm Planck} \tag{9.58}$$

Let us do this consistency test. If we take $\rho c^2 \simeq V(\phi) = q\phi^2$ and insert again $\phi = \phi_{\rm Planck}/\sqrt{4\pi}$ for the end of inflation, we obtain at the end of inflation:

$$\rho c^2 \simeq \frac{q}{4\pi}\phi^2_{\rm Planck} \tag{9.59}$$

Inserting this into Eq. (9.58) we obtain

$$H^2 \simeq \frac{2}{3} q \tag{9.60}$$

This leads to the same equation as Eq. (9.55), showing that, as expected, at the end of inflation the inflaton field has a density equal to the critical density at that time.

Finally, let us calculate the mass of the inflaton particle, assuming that inflation ends at $\tau_{\rm infl} = 10^{-32}$ seconds. From $q \gg 1/\tau^2_{\rm infl}$ we obtain $q \gg 10^{64}$ sec^{-2}. This gives

$$mc^2 = \hbar\sqrt{q} = 6.5 \times 10^7 \text{GeV} = 6.5 \times 10^4 \text{TeV} \tag{9.61}$$

This is well beyond what can be detected by the Large Hadron Collider, which is of the order of 7 TeV. The field might thus be all around us today, but we would not notice it.

9.6 Flatness of the Universe: Condition on inflation

Now that we have a model for inflation, let us see whether we can solve the flatness problem of the Universe with this. Let us go back to Eq. (9.26), but assume the slow roll condition, so that we obtain

$$H^2 = \frac{8\pi G}{3c^2} V(\phi) - \frac{Kc^2}{a^2} \tag{9.62}$$

Since $V(\phi)$ is approximately constant in time during inflation, the ratio of the curvature term to the potential term goes as $1/a^2$. After inflation, the $V(\phi)$ turns into the energy density ρc^2 of matter through the assumed “reheating” phenomenon (the decay of the scalar field particles into matter particles). This radiative matter ρ goes, however, as $1/a^4$. The ratio of the curvature to the radiative energy density term now goes as a^2.

Knowing from what we know today ($a = 1$), i.e. that the curvature term $|\Omega_K| \ll 0.01$, we can now calculate backwards to the end of inflation at $t \simeq 10^{-32}$ seconds with

$a \simeq 10^{-26}$. Since $\Omega_K \propto a^2$ ever since the end of inflation, it must have been as small as $\Omega_K(t_{\rm infl}) \ll 10^{-54}$. Assuming that at the start of inflation Ω_K could have been of the order of unity, the inflation must have increased a by *at least* $10^{27} = e^{62}$. This implies an inflation of

$$\tau_{\rm infl} \gtrsim 62\tau_{\rm exp} = 62\frac{1}{H_{\rm infl}} \tag{9.63}$$

Let us put in some numbers. The radius of the currently visible Universe is 1.43×10^{10} parsec. At the end of inflation at $t \simeq 10^{-32}$ seconds, i.e. $a \simeq 10^{-26}$, this (comoving) region corresponded to a size of the order of a few meters in diameter (though keep in mind: the part of the Universe that was visible *at that time* was much smaller!). With the *minimal* amount of inflation (62 e-folding times) this region started inflation at a diameter of a few $\times10^{-25}$ cm (less than 10^{-11} times the proton radius!). So, all that we see today was, at the start of inflation, as tiny as 10^{-11} times the proton radius. The Planck length is $l_{\rm Planck} = 1.62 \times 10^{-33}$ cm, so this corresponded to a few $\times10^8$ Planck lengths.

What about the curvature radius? Remember Eq. (4.34)

$$\Omega_{K,0} = -K\frac{c^2}{H_0^2} = -Kr_{H0}^2 = -\left(\frac{r_{H0}}{R_{\rm curv}}\right)^2 \tag{9.64}$$

From WMAP observations we know that the curvature radius must be much larger than the Hubble radius, hence $|\Omega_{K,0}| \ll 1$. How does this change with a? In comoving coordinates the curvature radius stays constant: $X_{\rm curv}$ =constant. This means that if the current curvature radius is, say, 1000 times the radius of the visible Universe, then it was (according to the above estimates) $1000\times10^8 = 10^{11}$ Planck lengths at the start of inflation and 1000×400 cm = 4 km at the end of inflation. What you see is that the curvature radius scales in the same way as all other scales. The problem with the flatness of the Universe today is not one of scale, but one of energy. And inflation can solve this energy problem, as we showed above.

9.7 Homogeneity of the Universe: Condition on inflation

To solve the homogeneity problem of the Universe, we must make sure that there is sufficient time before inflation starts. We need that time to ensure that information can propagate at the light speed for typical distances of what later (=today) becomes the visible Universe. And preferably we need many times that amount of time. Only in this way the Universe can homogenize sufficiently well before inflation rips apart regions that were before causally connected.

If we take $\sim 4 \times 10^{-25}$ cm as the size of the to-become-visible-Universe at the start of inflation, then we see that we need *at least*

$$\tau_{\rm before} \gtrsim \frac{4 \times 10^{-25}}{c} \simeq 10^{-35}\ \text{sec} \tag{9.65}$$

before the onset of inflation to allow homogenization, but preferably more.

From demanding at least 62 e-foldings of expansion to occur over a time period of 10^{-32} seconds, we see that $\tau_{\rm exp} \simeq 10^{-34}$. The required time to homogenize is still 10 times shorter, so we could in principle even start inflation straight away and still have time to homogenize a sufficiently large region before it is ripped apart. To say this in another way: If we calculate the Hubble radius corresponding to $\tau_{\rm exp} \simeq 10^{-34}$, this is $r_H = 3 \times 10^{-24}$ cm, which is still about 10 times larger than what will become the visible Universe. Note, however, that these estimate are all fairly rough.