Exercises for
Introduction to Cosmology (WS2011/12)

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(Exercise 1 from Matthias Bartelmann's lecture)

Exercise sheet 4

1. **Necessity of the Big Bang**

   (a) [20 pt] Assume that there is pressure-less matter, radiation and a cosmological constant in the universe, and allow for curvature. Show that $\Omega_{\Lambda,0}$ must satisfy:

   $$\Omega_{\Lambda,0} = \frac{1 + \Omega_{r,0}(a_*^{-2} - 1) + \Omega_{m,0}(a_*^{-1} - 1)}{1 - a_*^2} \tag{11}$$

   for the scale factor to reach $\dot{a} = 0$ at a finite $0 < a_* < 1$.

   (b) [20 pt] Show that $\Omega_{\Lambda,0}$ must further obey the inequality

   $$\Omega_{\Lambda,0} \geq \frac{2\Omega_{r,0} + \Omega_{m,0}a_*}{2a_*^4} \tag{12}$$

   if this extremum of a is to be a minimum. In other words, $\Omega_{\Lambda,0}$ is constrained by Eqs. (11, 12) in a Friedmann model universe which avoids a Big Bang.

   (c) [15 pt] Combine Eqs. (11, 12) to show that

   $$\Omega_{r,0}(1 - a_*^2)^2 + \Omega_{m,0}a_*\left(\frac{1}{2} - \frac{3}{2}a_*^2 + a_*^3\right) \leq a_*^4 \tag{13}$$

   (d) [15 pt] Convince yourself that both terms on the left-hand side of Eq. (13) are positive. Use the relation between scale factor and redshift to derive the constraint

   $$\Omega_{m,0} \leq \frac{2}{z_*^2(z_* + 3)} \tag{14}$$

   Conclude from the existence of objects with redshifts $z > 6$ and the fact that the visible matter density contributes $\Omega_{m,0} \geq 0.05$ that a Big Bang was inevitable.

2. **Horizon scale on the map of the CMB**
   Until the CMB was released at $z \simeq 1100$ any signal travelling with the light speed had only had the chance to travel a limited distance since the end of inflation. This "particle horizon scale" can be projected on the map of the CMB as seen by us today. Let us calculate this in an approximate way: Assume, for simplicity, that the Universe was matter-dominated also well before $z = 1100$, i.e. ignore the radiation-dominated part. Assume that the Universe is flat.

   (a) [15 pt] Compute how far, in comoving coordinates (i.e. x), radiation could have travelled from $z = \infty$ to $z = 1100$.

   (b) [15 pt] The comoving distance to the CMB is $x \simeq 1.4 \times 10^{10}$ pc. Compute the angular scale corresponding to the particle horizon on the map of the CMB.