

# Chapter 11

## Smooth Particle Hydrodynamics Solvers

So far we have only looked at method for solving the equations of hydrodynamics based on a fixed pre-defined grid. These methods are very suitable for many circumstances, but there are cases where these methods become quite cumbersome. In particular when very strong and localized density enhancements (clumps) appear, for instance due to gravity in astrophysical problems, the fixed-grid methods fail to resolve all relevant scales in those clumps. Using *Adaptive Mesh Refinement* (AMR) one can then refine the grid around these clumps, but such methods are technically very complex. A method that is particularly useful for modeling problems with strong clumping, and yet is fairly simple to implement, is the method of *Smooth Particle Hydrodynamics* (SPH). In this method the to-be-modeled gas cloud is represented by a set of discrete blobs of gas. These SPH particles interact with neighboring particles through a repelling force that represents the gas pressure, but otherwise act like normal particles. The nice thing of this method is that the computational power is automatically focused there where the mass is. So if a clump forms, for instance due to self-gravity, the computational power is focused in this very small region where all the mass is concentrated. It is so-to-speak an automatic AMR scheme without technical complexities. For this reason this method is often used in astrophysics, where extreme density gradients are very common.

This chapter is based on the works by Monaghan (1992, ARA&A 30, 543) and Springel (2005, MNRAS 364, 1105).

### 11.1 Lagrange equations of hydrodynamics for SPH

In Smooth Particle Hydrodynamics a gas cloud of mass  $M_{\text{cloud}}$  is respresented by  $N$  ‘particles’, i.e.  $N$  gas subclouds of mass  $m = M_{\text{cloud}}/N$ . Each of these blobs moves according to the equations of Newtonian mechanics, including a pressure force. For each of these particles the equations are based on Lagrange form of the Euler equations (cf. Chapter 1) including the equation of propagation:

$$D_t \vec{x} = \vec{u} \quad (11.1)$$

$$D_t \rho = -\rho \vec{\nabla} \cdot \vec{u} \quad (11.2)$$

$$D_t \vec{u} = -\frac{\vec{\nabla} P}{\rho} \quad (11.3)$$

$$TD_t s = 0 \quad (11.4)$$

To derive expressions for  $\nabla P$  and  $\nabla \cdot \vec{u}$  we need to let go of the ‘particle’ picture and view the ‘particles’ instead as *comoving coordinates* rather than true particles. In this picture the flow is again described on a set of grid points, but this time the grid points are irregularly spaced (not in a mesh) and they move strictly along with the flow. If we now wish to compute numerical derivatives, we have no longer the luxury of a grid-like coordinate system in which it is obvious which gridpoints are neighbors of grid point  $i, j$  (namely  $i, j - 1; i, j + 1; i - 1, j; i + 1, j$ ), but an irregularly spaced set of neighbors.

## 11.2 The SPH Kernel

The way this practical problem is solved in the SPH method is by defining a *kernel*  $W(\vec{x} - \vec{x}', h)$  around every  $\vec{x}'$  location of an SPH particle. This is a function that is maximum at  $\vec{x} = \vec{x}'$  and falls off smoothly to zero with distance from  $\vec{x}'$ . Its value only depends on  $r \equiv |\vec{x} - \vec{x}'|$ . This function defines the shape of the SPH ‘blob’. The value  $h$  is the parameter which defines the ‘size’ of this blob and is called the *smoothing length*. The kernel is normalized to unity in the following way:

$$\int_V W(\vec{x} - \vec{x}', h) d\vec{x}' = 1 \quad (11.5)$$

and we also have

$$\lim_{h \rightarrow 0} W(\vec{x} - \vec{x}', h) = \delta(\vec{x} - \vec{x}') \quad (11.6)$$

Again, we should not regard this ‘blob’ as a true blob of gas. In fact, SPH blobs will overlap quite a bit, which would not be true for true balls of gas. We should regard the SPH kernel as defining a *region of influence* of the SPH particle. It will turn out that only SPH particles that lie within each others regions of influence will be able to interact with each other.

Now let us, for a brief moment, do some math with the function  $W(\vec{x} - \vec{x}', h)$ , forgetting any reference to discrete SPH particles. Suppose that we have some function  $A(\vec{x}')$  defined at every position in space. We can now define the convolution of  $A(\vec{x}')$  with the kernel  $W(\vec{x} - \vec{x}', h)$ :

$$\tilde{A}(\vec{x}) = \int_V A(\vec{x}') W(\vec{x} - \vec{x}', h) d\vec{x}' \quad (11.7)$$

In this way the function  $A(\vec{x}')$  is blurred using the kernel  $W(\vec{x} - \vec{x}', h)$  as some kind of point spread function. If we now return to the discrete SPH picture, the values of  $\vec{x}'$  are now the set of discrete positions of the SPH particles, and the integral must be replaced by a sum in some way. To do this we must find a discrete expression for the integration volume  $d\vec{x}'$ . It turns out that the way to express  $\tilde{A}(\vec{x})$  is

$$\tilde{A}(\vec{x}) = \sum_{i=1, N} \frac{m}{\rho_i} A_i W(\vec{x} - \vec{x}_i, h) \quad (11.8)$$

where  $\rho_i$  is the density of gas corresponding to SPH particle  $i$ , for which we will derive an expression later, and  $A_i$  is the value of the function  $A$  corresponding to SPH particle  $i$ . Let us look carefully at this expression. The ratio  $m/\rho_i$  gives the volume that SPH particle  $i$  takes up. So if you have a situation where 100 SPH particles are spread over a large volume but another 100 SPH particles are jammed within a clump with volume much less than  $h^3$ , then the ratio  $m/\rho_i$  makes sure that the 100 particles in the clump together count for only the volume they occupy, which is much less than the volume occupied by the other 100 SPH particles. The ratio

$m/\rho_i$  therefore makes sure that the sum over SPH particles in fact represents an integral over *volume* and not over mass. Therefore, Eq. (11.8) is the discrete representation of Eq. (11.7).

The usefulness of Eq. (11.8) now lies in the fact that we have produced a smooth function  $\tilde{A}(\vec{x})$  from a function only known at discrete points  $A_i$ . It is not guaranteed that  $\tilde{A}(\vec{x}_i) = A_i$ , but it will also not be too far from this value. Later we will replace  $A$  by the pressure  $P$  or the density  $\rho$  so that we can use the above expressions to create a smooth pressure and density function from the discrete one. But let us for now stick to the abstract function  $A$ . Now that we have this smooth function we can define the derivative:

$$\nabla \tilde{A} = \sum_{i=1,N} \frac{m}{\rho_i} A_i \nabla W(\vec{x} - \vec{x}_i, h) \quad (11.9)$$

where we see that the derivative operator eventually works only on the kernel. Since we know the shape of the kernel analytically, we know the derivatives analytically.

The two most common kernels are a Gaussian kernel

$$W(r, h) = \frac{1}{h\sqrt{\pi}} e^{-r^2/h^2} \quad (11.10)$$

and the spline kernel <sup>1</sup>

$$W(r, h) = \frac{8}{\pi h^3} \begin{cases} 1 - 6 \left(\frac{r}{h}\right)^2 + 6 \left(\frac{r}{h}\right)^3, & 0 \leq \frac{r}{h} \leq \frac{1}{2} \\ 2 \left(1 - \frac{r}{h}\right)^3, & \frac{1}{2} < \frac{r}{h} \leq 1 \\ 0, & \frac{r}{h} > 1 \end{cases} \quad (11.11)$$

The Gaussian Kernel may be more natural but has the disadvantage that strictly speaking the region of influence of each SPH particle is infinite, even though the kernel may have very low values at large distances. The spline kernel has a very strictly defined outer edge, so that we can a-priori ignore anything that is further away from an SPH particle than a distance of  $h$ . For a computer implementation of the SPH method this property will save enormous amounts of computing time, because the sum over all particles then reduces to the sum over all particles that have a distance less than  $h$  to the point  $\vec{x}_i$ .

→ **Exercise:** The expression for the spline kernel, Eq. (11.11), has a normalization constant  $8/\phi h^3$  that was chosen such that the kernel integrates to unity in 3-D. Derive the normalization for the case of SPH in 1-D.

### 11.3 Some expressions in SPH-form

Let us write down some expressions that will be useful for later. First of all the smoothed form of the density  $\tilde{\rho}$  at a given position  $\vec{x}$ :

$$\tilde{\rho}(\vec{x}) = \sum_{i=1,N} m W(\vec{x} - \vec{x}_i, h) \quad (11.12)$$

This has the desired effect that the more SPH particles are located in the same small region with size  $L < h$ , the higher is the density. However, if the SPH particles are further away from each

<sup>1</sup>Note: This is the definition used by Springel (2005, MNRAS 364, 1105). Monaghan (1992, ARA&A 30, 543) uses a formula that is equal to this one if his  $h$  is replaced by  $h/2$ .

other than  $h$ , then the SPH kernels do not overlap and the gas density around each SPH particle simply reflects the kernel of this particle. We see here that SPH (at least with a globally fixed value of  $h$ ) has the unfortunate property that it cannot model regions of very low density. It will instead model such regions as being filled with localized blobs of gas separated by vacuum. A variable smoothing length  $h \rightarrow h_i$  can solve this problem, but let us save this for later.

Now one may raise the question, what is the density  $\rho_i$  associated with each SPH particle, which need not necessarily be equal to  $\tilde{\rho}(\vec{x}_i)$ ? One way to define this is using the following expression:

$$\frac{4\pi}{3}h^3\rho_i = N_{h,i}m \quad (11.13)$$

where  $N_{h,i}$  is the number of nearest neighboring particles that happen to lie within a radius  $h$  of particle  $i$ .

→ **Exercise:** Argue that both Eq. (11.12) and Eq. (11.13) produce the same result for the case of regularly spaced SPH particles with inter-spacing distance  $\Delta x \ll h$ .

Now what about  $\nabla \cdot \vec{u}$ ? We have

$$\tilde{u}(\vec{x}) = \sum_{i=1,N} \frac{m_i}{\rho_i} W(\vec{x} - \vec{x}_i, h) u_i \quad (11.14)$$

So we get

$$\nabla \cdot \tilde{u}(\vec{x}) = \sum_{i=1,N} \frac{m_i}{\rho_i} u_i \cdot \nabla W(\vec{x} - \vec{x}_i, h) \quad (11.15)$$

But we can also derive an alternative expression. Let us first write

$$\nabla \cdot \tilde{u} = \frac{1}{\tilde{\rho}} \left[ \nabla \cdot (\tilde{\rho} \tilde{u}) - \tilde{u} \cdot \nabla \tilde{\rho} \right] \quad (11.16)$$

We have

$$\nabla \cdot (\tilde{\rho} \tilde{u}) = \sum_{i=1,N} m_i u_i \cdot \nabla W(\vec{x} - \vec{x}_i, h) \quad (11.17)$$

$$\nabla \tilde{\rho} = \sum_{i=1,N} m_i \nabla W(\vec{x} - \vec{x}_i, h) \quad (11.18)$$

So we obtain:

$$\rho_k (\nabla \cdot \tilde{u})_k = \sum_{i=1,N} m_i (\vec{u}_i - \vec{u}_k) \cdot \nabla_k W(\vec{x}_k - \vec{x}_i, h) \quad (11.19)$$

where  $\nabla_k$  is the derivative operator with respect to  $x_k$ . Now a little check: If we insert the Gaussian kernel we obtain

$$\rho_k (\nabla \cdot \tilde{u})_k = \sum_{i=1,N} \frac{2m_i}{h^2} (\vec{u}_k - \vec{u}_i) \cdot (\vec{x}_k - \vec{x}_i) W(\vec{x}_k - \vec{x}_i, h) \quad (11.20)$$

For each pair of SPH particles this expression indeed gives a positive contribution when these particles move away from each other, as it should be.

## 11.4 The SPH equations of motion

With the expressions we derived above, and with the exercise we have obtained in manipulating derivatives with SPH, we can now attempt to write down the final set of SPH equations.

### 11.4.1 Momentum equation

The SPH version of the momentum equation starts from

$$D_t \vec{u}_k = - \left( \frac{\vec{\nabla} P}{\tilde{\rho}} \right)_k \quad (11.21)$$

Following the above examples, we can write  $\nabla \tilde{P}$  in various forms. For instance

$$(\nabla \tilde{P})_k = \sum_{i=1, N} \frac{m_i}{\rho_i} P_i \nabla_k W(\vec{x}_k - \vec{x}_i, h) \quad (11.22)$$

or

$$(\nabla \tilde{P})_k = \frac{1}{\rho_k} \sum_{i=1, N} m_i (P_i - P_k) \nabla_k W(\vec{x}_k - \vec{x}_i, h) \quad (11.23)$$

Neither of these expressions conserves momentum. But if we write

$$\frac{\nabla P}{\rho} = \nabla \left( \frac{P}{\rho} \right) + \frac{P}{\rho^2} \nabla \rho \quad (11.24)$$

then we obtain

$$(\nabla \tilde{P})_k = \rho_k \sum_{i=1, N} m_i \left( \frac{P_i}{\rho_i^2} + \frac{P_k}{\rho_k^2} \right) \nabla_k W(\vec{x}_k - \vec{x}_i, h) \quad (11.25)$$

The momentum equation now becomes

$$D_t \vec{u}_k = - \sum_{i=1, N} m_i \left( \frac{P_i}{\rho_i^2} + \frac{P_k}{\rho_k^2} \right) \nabla_k W(\vec{x}_k - \vec{x}_i, h) \quad (11.26)$$

This has the desirable property that it obeys the ‘action is minus reaction’ of Newtonian mechanics: the fact that momentum is conserved. To see this we write the momentum equation as a force equation:

$$D_t (m_k \vec{u}_k) = - \sum_{i=1, N} m_k m_i \left( \frac{P_i}{\rho_i^2} + \frac{P_k}{\rho_k^2} \right) \nabla_k W(\vec{x}_k - \vec{x}_i, h) \quad (11.27)$$

We see for each pair of SPH particles that the gain of momentum of one is the gain of opposite momentum of the other, exactly conserving momentum as desired.

### 11.4.2 Energy equation

The energy equation can be written in the form

$$T D_t s = 0 \quad (11.28)$$

where  $s$  is the entropy. Basically this means that each SPH particle has a specific entropy, and it keeps this entropy as the particle moves along. Another way of saying this is that if the density  $\rho_i$  is known, the pressure follows from

$$P_i = K_i \rho_i^\gamma \quad (11.29)$$

where  $K_i$  is another way of writing the entropy (see Eq. 1.22 of Chapter 1). So, for each SPH particle  $K_i$  remains constant in time. Of course, if a shock appears, then the entropy must increase, which is discussed in Section 11.6.

### 11.4.3 Propagation equation

The next thing to do is to integrate the motion of the SPH particles, i.e. to compute the  $\vec{x}_i(t)$ . In doing so we must keep in mind that we would like elementary conserved quantities to remain conserved as much as possible. Mass is obviously conserved, so that is no problem. Total energy is complicated to keep globally conserved, but specific entropy is conserved for each SPH particle. Momentum conservation requires that if two SPH particles exchange momentum through their mutual pressure force, the one must receive exactly what the other loses. A simple first order integration scheme can be built using operator splitting. The spatial advection operator is:

$$x_i \rightarrow x_i + \Delta t u_i \quad (11.30)$$

This is a *drift operator*. Then we do the pressure forces:

$$u_i \rightarrow u_i + \Delta t f_i \quad (11.31)$$

where  $f_i$  is the pressure force computed using the methods described above, expressed at the new location  $x'_i$ . To get slightly better accuracy one often does:

$$x_i \rightarrow x_i + \frac{1}{2} \Delta t u_i \quad (11.32)$$

$$u_i \rightarrow u_i + \Delta t f_i \quad (11.33)$$

$$x_i \rightarrow x_i + \frac{1}{2} \Delta t u_i \quad (11.34)$$

This is called the *leapfrog* method. It has the advantage that it is very simple and it has the benign property that it is a so-called *symplectic integrator*. Such an integrator is particularly well suited for equations based on a Hamiltonian, and conserves the Poincare integral invariants. In particular for systems with gravity this is useful, for instance when integrating the orbit of a particle around a star. This method conserves the orbital elements much better than ordinary integrators such as Runge-Kutta. We refer to Springel (2005, MNRAS 364, 1105) and references therein for an in-depth discussion of the leapfrog method, symplectic integrators in general and their properties.

## 11.5 Variable smoothing length

As mentioned above, one of the problems of taking a fixed smoothing length for all SPH particles is that at low densities the gas is not well represented: it is represented by isolated blobs. There is also an opposite problem with a constant smoothing length: in regions of very dense clustering of SPH particles, where hundreds or thousands of SPH particles are all within a region the size of  $h$ , the spatial resolution can not become any better than  $h$ , and much computing power is wasted on these thousands of particles. A variable smoothing length is therefore of absolute necessity if problems of high density contrast are to be modeled.

One simple way to define the variable smoothing length of particle  $i$  is to take it such that there are always  $N_{h,i} = N_{\text{nb}}$  nearest neighbors within a radius  $h_i$  from that position, and where  $N_{\text{nb}}$  is a constant for the simulation. This constant can be typically taken to be about 60, so that in each direction there are roughly two ‘layers’ of particles within the radius  $h_i$ .

With variable smoothing length one must reconsider some of the equations in which the interaction between two SPH particles with potentially different smoothing lengths is computed.

For the momentum equation Springel & Hernquist (2002, MNRAS 333, 649) derive the following expression:

$$D_t \vec{u}_k = - \sum_{i=1, N} m_i \left( f_i \frac{P_i}{\rho_i^2} \nabla_k W(\vec{x}_k - \vec{x}_i, h_i) + f_k \frac{P_k}{\rho_k^2} \nabla_k W(\vec{x}_k - \vec{x}_i, h_k) \right) \quad (11.35)$$

with  $f$  defined as

$$f_i = \left( 1 + \frac{h_i}{3\rho_i \frac{\partial \rho_i}{\partial h_i}} \right)^{-1} \quad (11.36)$$

## 11.6 Shocks: artificial viscosity

As mentioned above, using the energy equation in the form of  $Tds = 0$  guarantees perfect conservation of specific entropy for each SPH blob. However, if the gas moves through a shock, then physically it is clear that entropy *must* be generated. It is therefore clear that if we do not include some form of artificial viscosity, such as the von Neumann-Richtmyer viscosity, then our solutions are bound to fail miserably. The way this artificial viscosity is implemented in Gadget-2 (see Section 11.8) is

$$(D_t \vec{u}_k)_{\text{visc}} = - \sum_{i=1, N} m_i \Pi_{ki} \nabla_k \bar{W}(\vec{x}_k - \vec{x}_i) \quad (11.37)$$

(see Springel 2005, MNRAS 364, 1105), where  $\Pi_{ik} \geq 0$  is only non-zero when particles approach each other. The associated increase in ‘entropy’  $K_i$  is

$$(D_t K_k)_{\text{visc}} = \frac{1}{2} \frac{\gamma - 1}{\rho_i^{\gamma-1}} \sum_{i=1, N} m_i \Pi_{ki} (\vec{u}_k - \vec{u}_i) \cdot \nabla_k \bar{W}(\vec{x}_k - \vec{x}_i) \quad (11.38)$$

where  $\bar{W}$  is the arithmetic average of the two kernels. An example of a recipe for the artificial viscosity is that of Monaghan & Gingold (1983) and Balsara (1995):

$$\Pi_{ki} = \begin{cases} (-\alpha c_{ki} \mu_{ki} + \beta \mu_{ki}^2) / \rho_{ki} & \text{if } \vec{u}_{ki} \cdot \vec{x}_{ki} < 0 \\ 0 & \text{otherwise} \end{cases} \quad (11.39)$$

with

$$\mu_{ki} = \frac{h_{ki} \vec{u}_{ki} \cdot \vec{x}_{ki}}{|\vec{x}_{ki}|^2} \quad (11.40)$$

where  $\vec{u}_{ki} = \vec{u}_k - \vec{u}_i$ ,  $\vec{x}_{ki} = \vec{x}_k - \vec{x}_i$ , and  $h_{ki}$ ,  $\rho_{ki}$  and  $c_{ki}$  are the arithmetic mean of  $h$ ,  $\rho$  resp. the sound speed of both particles. Typically the values of  $\alpha$  and  $\beta$  are taken to be  $\alpha \simeq 0.5 - 1.0$  and  $\beta = 2\alpha$ .

## 11.7 Some thoughts about SPH

SPH is a flexible method for hydrodynamics. One can relatively easily build in various source terms and forces such as for instance gravity. For very large simulations it becomes a challenge to manage the particles and to make sure that the nearest neighbors are quickly found. Moreover, to manage gravitational forces over both short and long ranges requires rather sophisticated

schemes such as *tree codes* which organize the particles in a way that near particles are quickly found. A nice aspect of SPH is that it is Lagrangian. One can easily include for instance chemistry of the gas, as long as no mixing between gas in neighboring SPH particles is required. If that is necessary, a special recipe for mixing must be used, which is a bit more complicated. Another nice aspect of the method is that it has no numerical bulk viscosity. But one has to pay a price: the numerical shear viscosity can be enormous. If too few SPH particles are used, the intrinsic clumpiness of SPH particles causes a strong shear viscosity which is usually bigger than intrinsic physical shear viscosity of the system to be modeled. The shear viscosity can be suppressed only by going to very large numbers of SPH particles, but that is numerically costly. It is sometimes reported that SPH is intrinsically too diffusive for certain problems. Also it is very difficult to build in physics like radiative transfer into the SPH method.

All in all the method is very useful for many problems of astrophysics, as long as the many inaccuracies and potential problems are kept in mind.

## **11.8 The code GADGET-2**

There are many SPH and N-body codes in the literature, and many can be downloaded from the web. A code that has gained particular popularity in recent years in the cosmological community and elsewhere is the code GADGET-2 by Volker Springel (Max-Planck-Institut für Astrophysik in Garching, Germany). The code is well documented, well tested, is easy to install and use and it is accompanied by a scientific paper describing the methodology of the code in detail (Springel 2005, MNRAS 364, 1105). It allows, among many things, for SPH gas dynamics and at the same time for N-body dynamics, including gravitational interactions between all bodies. The code can be downloaded from: <http://www.mpa-garching.mpg.de/gadget/>.