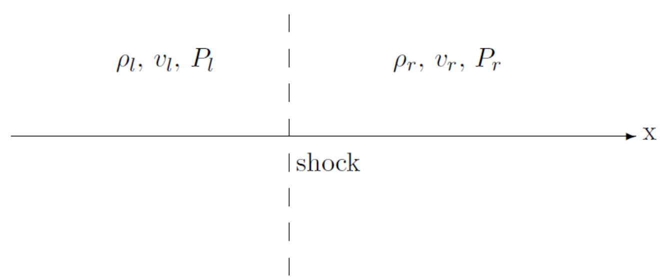
Isothermal Shock

Problem sheet 2

05/05/2009

Assume a shock (the flow to the left and to the right side differs in density, velocity and pressure) in a 1D-isothermal flow of an ideal gas:



Calculate a relation between the two different velocities, which depends only on the isothermal sound speed $c_s = \sqrt{\frac{kT}{\mu}}$, with k being Boltzmann's constant, T temperature and μ molecular weight.

Hint: Apply Rankin-Hugoniot jump condition over the shock front (conservation law of mass and momentum).

Continuity Equation

Problem sheet 2

05/05/2009

Discretization of Space

1. Begin with the general continuity equation of hydrodynamics.
2. Switch to 1D case.
3. Assume a constant (space and time independent) velocity $v(x,t) = v_0$.
4. Discretize the time derivative operator ∂_t as done in the first exercise, but stay continuous in space so far, and calculate the resulting expression for $\rho^{n+1}(x)$ in explicit form.
5. The discretized value of the density (at time $t = n\Delta t$) at location $x = i\Delta x$ will be denoted as ρ_i^n. Now substitute the spatial spatial derivative operator at grid point i with $\partial_x \rho^n(x) = ...$

   (a) $\frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}$

   (b) $\frac{\rho_i^n - \rho_{i-1}^n}{\Delta x}$

   (c) $\frac{\rho_{i+1}^n - \rho_{i-1}^n}{2\Delta x}$

6. Solve the resulting equations (a)$\sim$(c) numerically:
   Code guidance:

   - Choose the time evolution from $t_{min} = 0$ to $t_{max} = 50$
   - Choose a grid from $x_{min} = 0$ to $x_{max} = 100$, $\Delta x = 1$
   - Choose the following step function as the initial density distribution: $\rho(x, t = t_{min}) = \Theta(x - 45)\Theta(55 - x)$, where the theta-function is defined as

   $$\Theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}.$$

   - Choose spatial boundary values $\rho(0,t) = \rho(x_{max}, t) = 0$.

- Choose $v_0 = 1$ and $v_0 = -1$.

Task to do:

- Determine the intrinsic upper timestep limit Δt_{crit} for this specific setup (Hint: For $\Delta t > \Delta t_{crit}$ all discretization schemes are unconditionally unstable for both velocity $v_0 = \pm 1$.
  Explain this critical timestep and derive an analytic expression for it.
- From now on use $\Delta t = 0.8\Delta t_{crit}$. Check for $v_0 = \pm 1$ the stability of the different discritization schemes:

Scheme	$v_0 = 1$	$v_0 = -1$
(a)		
(b)		
(c)		

Extra work:

From the exercises above, we realize that catching the correct information from the correct direction, i.e., upwind, is both physically and numerically important. In real life, fluid does not necessarily flow in just one direction. This exercise accomodates your code to a more general situation.

- Starting from the result of problem 4 replace v_0 with an arbitrary space-dependent velocity distribution $v(x)$. Now modify your code to get a generally stable method.
- Use for testing $v(x) = (-1)^{\Theta(50-x)} = 2\Theta(x-50) - 1$. Before running the code, can you imagine what the exact solution looks like? Continuity equation states that total mass should be conserved if there is no inflow or outflow from the boundary. In this case, is the total mass conserved? Why is this stable discretization method still not a solution of the continuity equation?
- Does the differential form of continuity equation fail in this case?