Entropy Wave

Problem sheet 3

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The dynamics of the ideal fluid is described by:

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{u}) &= 0 \qquad (1) \\ \partial_t (\vec{u}) + \vec{u} \cdot \nabla \vec{u} &= -\frac{\nabla P}{\rho} \qquad (2) \\ \partial_t (e_{int}) + \vec{u} \cdot \nabla e_{int} &= -\frac{P}{\rho} \nabla \cdot \vec{u}, \qquad (3) \end{aligned}$$

with ρ being the density, $\vec{u}$ velocity, P pressure and e_{int} the specific internal energy. We also learn that there exists three distinct eigenvalues:

$$\begin{aligned} \lambda_- &= u - C_s \qquad (4) \\ \lambda_0 &= u \qquad (5) \\ \lambda_+ &= u + C_s, \qquad (6) \end{aligned}$$

where C_s represents the adiabatic sound speed. The wave that travels with speed λ_0 is also called entropy wave. The following is the reason.

1. Define specific volume $v = 1/\rho$. Prove that $\nabla \cdot \vec{u}$ describes the changing rate of specific volume, i.e.,

$$\frac{1}{v} \frac{dv}{dt} = \nabla \cdot \vec{u} \qquad (7)$$

2. From the first law of thermaldynamics, eq. (1) and eq. (3) prove

$$D_t s = 0, \qquad (8)$$

   where s represents the specific entropy and operator $D_t = \partial_t + \vec{u} \cdot \nabla$. This means that eq. (3) can be replaced by eq. (8). Specific entropy serves as a natural dye or passive tracer which propagates with velocity u.

3. Explain that eq. (3) is nothing more than the first law of thermaldynamics $Tds = de_{int} + Pdv$ with T being the temperature.

4. Define voticity $\vec{\omega} = \nabla \times \vec{u}$. In 2D case, prove that for a impressible flow where $\nabla \cdot \vec{u} = 0$, vorticity also serves as a natural dye, i.e.,

$$D_t \vec{\omega} = 0.$$

5. Again in 2D case, but for a compressible flow, prove that $\vec{\omega}/\rho$ acts like a natural dye, i.e.,

$$D_t \left(\frac{\vec{\omega}}{\rho} \right) = 0.$$

The importance of seeking a natural dye is when we would like to plot the streamlines of a *smooth* fluid which has reached the steady state. Instead of integrating the velocity field, contour lines of these natural dye are actually the streamlines.