Appendix A

Fourier transforms

Fourier transforms (named after Jean Baptiste Joseph Fourier, 1768-1830, a French mathematician and physicist) are an essential ingredient in many of the topics of this lecture. Therefore let us review the basics here. We assume, however, that the reader is already mostly familiar with the concepts.

A.1 Fourier integrals in infinite space: The 1-D case

Let us start in 1-D. Given a function $f(x)$ on the domain $\langle -\infty, +\infty\rangle$, where x is always assumed to be real, the Fourier transform $g(u)$ is

$$g(u) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i u x} dx \tag{A.1}$$

Note that there are several conventions taken here that may differ from other literature! First, the sign of the exponent is chosen as negative for $ux > 0$. Secondly, there is a 2π in the exponent, which is useful for the normalization of the Fourier transform, for with this factor in the exponent, the Fourier transform of $g(u)$ is again $f(x)$:

$$\begin{aligned}
\int_{-\infty}^{+\infty} g(u) e^{-2\pi i x u} du &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x') e^{-2\pi i u(x+x')} du dx' && \text{(A.2)}\\
&= \int_{-\infty}^{+\infty} f(x') \left[\int_{-\infty}^{+\infty} e^{-2\pi i u(x+x')} du\right] dx' && \text{(A.3)}\\
&= \int_{-\infty}^{+\infty} f(x')\delta(x+x') dx' && \text{(A.4)}\\
&= f(-x) && \text{(A.5)}
\end{aligned}$$

The Fourier transform of a Fourier transform is again the original function, but mirrored in x. A couple of properties (Pinski 2002, "Introduction to Fourier Analysis and Wavelets"):

- Linearity: The Fourier transform of $f_1(x)+f_2(x)$ is the sum of the Fourier transforms of $f_1(x)$ and $f_2(x)$.
- Translation in x leads to a winding in u: The Fourier transform of $f(x-x_0)$ is $e^{-2\pi i u x_0} g(u)$.

- Winding in x leads to a shift in u: The Fourier transform of $e^{2\pi i u_0 x} f(x)$ is $g(u - u_0)$.
- The Fourier transform of $f(ax)$ where a is a non-zero real number is $g(u/a)/|a|$.
- The Fourier transform of $f^*(x)$ (the complex conjugate) is $g^*(-u)$.
- If $f(x)$ is real, then $g(-u) = g^*(u)$ (i.e. the Fourier transform of a real function is not necessarily real, but it obeys $g(-u) = g^*(u)$).
- If we have two functions $f_1(x)$ and $f_2(x)$ which we convolve: $(f_1 * f_2)(x) \equiv \int_{-\infty}^{+\infty} f_1(x') f_2(x - x') dx'$, then the Fourier transform is simply the product of the two Fourier transforms: $g_1(u) g_2(u)$.

Let us list a couple of elementary Fourier transforms in 1-D:

f(x)	g(u)
$\delta(x - x_0)$	$\exp(-2\pi i u x_0)$
$\exp(2\pi i u_0 x)$	$\delta(u - u_0)$
$\mathrm{box}(ax)$	$(1/a)\mathrm{sinc}(u/a)$
$\mathrm{sinc}(ax)$	$(1/a)\mathrm{box}(u/a)$
$\exp(-\pi a^2 x^2)$	$(1/a)\exp(-\pi u^2/a^2)$

where $\mathrm{box}(x) = 1$ for $-1/2 \leq x \leq 1/2$ and 0 elsewhere, $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$, and $a > 0$.

A.2 Fourier integrals in infinite space: The 2-D case

Fourier integrals in 2-D are simple extensions of those in 1-D:

$$g(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-2\pi i (ux + vy)} dx dy \tag{A.6}$$

Like before we have the Fourier transform of $g(u, v)$ to be:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(u, v) e^{-2\pi i (xu + yv)} du dv = f(-x, -y) \tag{A.7}$$

All the same properties hold in 2-D as in 1-D. Some particular examples:

f(x,y)	g(u,v)
$\delta(x - x_0)\delta(y - y_0)$	$\exp(-2\pi i(ux_0 + vy_0)$
$\exp(2\pi i(u_0 x + v_0 y)$	$\delta(u - u_0)\delta(v - v_0)$
$\mathrm{circ}(\sqrt{x^2 + y^2}/a)$	$J_1(2\pi a\sqrt{u^2 + v^2})/\sqrt{u^2 + v^2}$
$\exp(-\pi[a^2 x^2 + b^2 y^2])$	$(1/\lvert ab\rvert)\exp(-\pi[u^2/a^2 + v^2/b^2])$

where $\mathrm{circ}(r) = 1$ for $r \leq 1$ and 0 for $r > 1$, and $J_1(u)$ is the Bessel function of the first kind of order 1.

A.3 FT on a limited domain: The continuous case

Suppose we have a function $f(x)$ but for some reason we can analyze only a limited domain of this function. This is a common issue, because typically we do not have a measurement of any physical quantity over an infinite domain (either in space or in time). So we somewhat arbitrarily choose some domain, say $x \in [-L/2, L/2]$, and wish to analyze this with a Fourier integral. The Fourier transform $g(u)$ of this function, limited to the domain $[-L/2, L/2]$ is then

$$g(u) = \int_{-L/2}^{L/2} f(x) e^{-2\pi ux} dx \tag{A.8}$$

Let us define the block function $\text{box}(x)$ as 1 for $|x| \leq 1/2$ and 0 for $|x| > 1/2$. We can then rewrite the above restricted Fourier transform as:

$$g(u) = \int_{-\infty}^{+\infty} f(x) \text{box}(x/L) e^{-2\pi iux} dx = \mathcal{F}[f] * L\text{sinc}(Lu) \tag{A.9}$$

where in the last step we used that the Fourier transform of the product of two functions is the convolution of their Fourier transforms in Fourier space. We then also used that the Fourier transform of the block function is the sinc function. In other words: if we sample only a limited domain of the function $f(x)$, we obtain the Fourier transform of the function, convolved with the sinc function. This means we lost information, in particular for $|u| \lesssim 1/L$. So, as long as we take L large enough, the sinc function will be narrow enough that we don't lose too much information. This formulation of Fourier transforms is closer to real applications.

A.4 FT on a limited domain: The continuous periodic case

If a signal $f(x)$ is periodic, then we automatically deal with a limited domain: that of one period. We do not need to explicitly integrate over all of space, because the analysis of one period is sufficient to know all there is to know about the function and its Fourier transforms. Let us assume that a signal has a period L and take the domain $[-L/2, L/2]$ as the representative part. Let us define the complex Fourier coefficients g_n as

$$g_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-2\pi inx/L} dx \tag{A.10}$$

for any integer value of n (either positive or negative). The inverse can be shown to be

$$f(x) = \sum_{n=-\infty}^{\infty} g_n \, e^{2\pi inx/L} \tag{A.11}$$

This is called the *Fourier series*. One sees that a continuous function on a restricted domain (assuming periodicity) can be uniquely described by an infinite discrete set of numbers.

A.5 FT on a limited domain: The discrete case

A special version of the periodic Fourier transform and the Fourier series is the *discrete Fourier transform*. This is applicable if instead of a smooth function $f(x)$ we have a discretized (sampled) function f_k for $0 \leq k \leq N-1$, where $N > 0$ is some number of sampling points. We encounter such situations often when we perform numerical analyses of physical problems: We do not have infinite information. N denotes the amount of information we have. We can define now the Fourier components g_n as

$$g_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-2\pi i n k/N} \tag{A.12}$$

where also $0 \leq n \leq N-1$ (both sets contain equal amount of information). The inverse is

$$f_k = \sum_{n=0}^{N-1} g_n e^{2\pi i n k/N} \tag{A.13}$$

Note that there are different conventions used. The normalization can be different in different conventions. Some put the $1/N$ in the reverse Fourier transform (Eq. A.13), and some use a *unitary* normalization ($1/\sqrt{N}$) for both equations. Also the domain can be shifted, e.g. from $-(N-1)/2 \leq k \leq (N-1)/2$, which is the discrete version of the domain choice in Section A.4.

Discrete Fourier transformations can be seen as the discrete version of the periodic Fourier transform in Section A.4. One can see the values f_k as samplings of a "true" function $f(x)$ at locations $x_k = x_0 + k\Delta x$, where Δx is some sampling width. The points x_k can be seen as the *grid points* of a *regular grid* on which the function $f(x)$ is sampled. Since computers can seldom deal with analytic functions, such *discretized representations* of these functions on such regular grids can be used to cast a physical problem into a numerical problem that can be solved by a computer. A Fourier transform of the type discussed in Section A.4 is then represented approximately as a discrete Fourier transform (but be careful of the normalizations!).

Discrete Fourier transforms can be computed extremely efficiently using *Fast Fourier Transform (FFT)* algorithms. The most common one is the *Cooley-Tukey* algorithm, which is a *divide and conquer* type algorithm that recursively divides the domain into two subdomains to solve the problem. Usually this is "the" FFT algorithm. FFT subroutines are available in many mathematical libraries, so they are in very broad use. When using such a subroutine, it is important to know which conventions are used. Note that they work best if N is a power of 2.

Important note: One drawback of FFT routines is that they assume periodicity. Strictly speaking one cannot use FFT routines for computing the numerical equivalents of the Fourier transforms of Sections A.1 and A.3. However, if the actual signal f is limited to only a small fraction of the entire domain in k, and the rest is "padded to zero", then the effect of periodicity is negligibly small, and the FFT gives good results. But always keep this caveat in mind!

A.6 Nyquist sampling and aliasing

As mentioned above, the discrete Fourier transform is often used for cases in which a continuous function $f(x)$ is sampled on a grid $x_k = x_0 + k\Delta x$, where Δx is some sampling width. By the process of sampling on a regular grid we throw away information about the original function $f(x)$. It is intuitively clear that we in fact lose "high frequency information", i.e. information about "wiggles" in the function $f(x)$ that have a wavelength smaller than $2\Delta x$. There is in fact a rigorous mathematical theory behind this: the Nyquist sampling theory (named after Harry Nyquist, 1889-1976, a Swedish-American engineer working for AT&T and Bell Laboratories).

Consider a function $f(x)$, defined on the entire domain $\langle -\infty, \infty \rangle$. If this function is *band-limited* to frequencies u smaller than the *Nyquist frequency*,

$$u_c = \frac{1}{2\Delta x} \tag{A.14}$$

(i.e. all Fourier components $g(u)$ with $|u| > u_c$ are 0), then Nyquist's sampling theorem says that the function $f(x)$ is entire determined by a sampling of $f(x)$ at regular intervals of Δx. In other words, a function that is band-limited to below the Nyquist frequency contains exactly the same amount of information as its sampled version.

One can reconstruct the function $f(x)$ from f_k with the formla

$$f(x) = \Delta x \sum_{k=-\infty}^{+\infty} f_k \frac{\sin(2\pi u_c [x - k\Delta x])}{\pi [x - k\Delta x]} \tag{A.15}$$

Just to verify: you can indeed see that at all sampling points you retrieve $f(x_k) = f_k$, because $\lim_{x\to 0} \sin(x)/x = 1$.

The Nyquist sampling theorem can also be applied to the Fourier transforms of periodic functions. This shows the relation between the Fourier series and the discrete Fourier transform.

What it also says is that if we have a function $f(x)$ which is *not* band-limited to below the Nyquist frequency, then any Fourier components with $|u|$ above u_c will be *aliased* to smaller u. This is a *stroboscope effect* that leads to *Moiré patterns*. Examples are for instance the seemingly backward rotating cartwheels in old movies, or the strange effects when taking low-resolution digital photos of regular patterns.