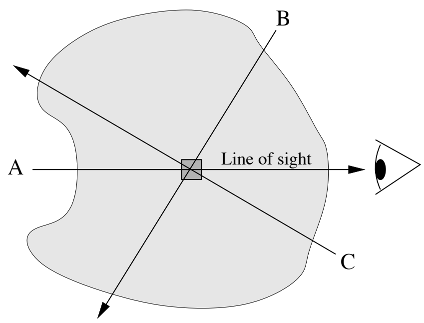
Chapter 4

What makes radiative transfer hard, and how to solve it - An introduction

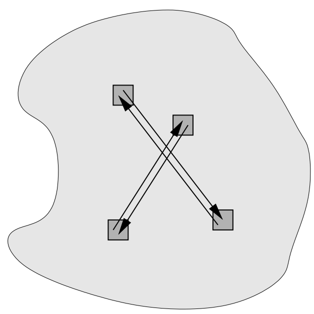
If we would, at all times and at all locations, know the values of j_ν and α_ν, then what we have learned in Chapter 3 would be enough to understand the topic of radiative transfer. Of course, some technical details still have to be refined, and we have to discuss all the "input physics" such as opacities and abundances of constituents of the medium. But those would be manageable. When you hear that radiative transfer is a very challenging topic, the reason is that in many cases we *do not know* the values of j_ν and/or α_ν in advance. The radiation field $I_\nu(\mathbf{x},\mathbf{n})$ that we wish to compute can affect the medium in such a way as to modify j_ν and α_ν. We are then faced with a "chicken or egg" effect: to compute $I_\nu(\mathbf{x},\mathbf{n})$ we need to know $j_\nu(\mathbf{x})$ and $\alpha_\nu(\mathbf{x})$, and to compute $j_\nu(\mathbf{x})$ and $\alpha_\nu(\mathbf{x})$ we need to know $I_\nu(\mathbf{x},\mathbf{n})$.

Ray coupling



And to make things worse: we cannot solve this problem for each ray separately, because a change in $j_\nu(\mathbf{x})$ will affect the formal transfer equation for all rays passing through $\mathbf{x}$, i.e. rays with different direction vectors $\mathbf{n}$. This is illustrated in the figure in the margin. For our observation we are interested in the formal radiative transfer along ray A, which we call the *line of sight*. We focus in this illustration on the j_ν and α_ν in the little cell at the center of the cloud. In addition to ray, A, also rays B and C pass through that cell. The intensity along those rays can therefore also affect j_ν and α_ν in the cell. This *ray coupling* effect means that we are forced to solve the radiative transfer problem for all rays at once. This is the true challenge of radiative transfer.

Radiative cell coupling



This challenge can also be expressed in terms of *radiative cell coupling*: the emission generated in one volume element of a cloud (a "cell") can travel to the other side of the cloud and affect the conditions in a cell there. Information is thus exchanged between regions of the cloud that are distant from each other. For example: the radiative cooling of one region can cause the radiative heating of another. While this may seem like a separate problem from the ray coupling problem, it is actually the same. The cell coupling and ray coupling problems are just two faces of the same problem.

4.1 The simplest example of ray coupling: Isotropic scattering

Let us consider the simplest radiative transfer problem in which such a ray coupling plays a role. Suppose we have a medium consisting of small dust particles that can scatter radiation in arbitrary directions. This process is called *isotropic scattering*, because the outgoing direction of a photon has, by assumption, no dependence on the direction of the photon before the scattering event. Let us also assume that the dust particles do not absorb nor emit any of the radiation, and let us focus on a single frequency ν (we omit any ν indices for notational convenience). Let us also assume that somewhere (either inside or outside of the cloud) there is a source of light, which we will treat as an initial value for the intensity at the start of rays emanating from that source.

The formal radiative transfer equation is then, as usual,

$$\mathbf{n}\cdot\nabla I(\mathbf{x},\mathbf{n}) = j(\mathbf{x}) - \alpha(\mathbf{x})I(\mathbf{x},\mathbf{n}) \tag{4.1}$$

The emissivity j is responsible for injecting radiation into the ray, which occurs through scattering. Since all photons that experience a scattering event have the same chance to be scattered into the direction $\mathbf{n}$, we only need to know how much radiation is being scattered per unit volume and time: we do not need to worry about the angular dependence of the incoming radiation. This means that the emissivity becomes:

$$j(\mathbf{x}) = \alpha(\mathbf{x})\frac{1}{4\pi}\oint I(\mathbf{x},\mathbf{n})\,d\Omega = \alpha(\mathbf{x})J(\mathbf{x}) \tag{4.2}$$

where in the last step we used the definition of the mean intensity J (Eq. 2.26). This allows us to write the formal transfer equation as

$$\mathbf{n}\cdot\nabla I(\mathbf{x},\mathbf{n}) = \alpha(\mathbf{x})\left[\frac{1}{4\pi}\oint I(\mathbf{x},\mathbf{n}')\,d\Omega' - I(\mathbf{x},\mathbf{n})\right] \tag{4.3}$$

or in more compact form:

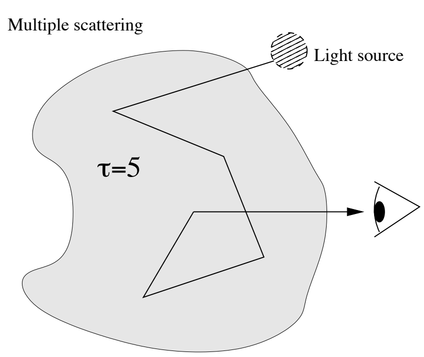
$$\mathbf{n}\cdot\nabla I(\mathbf{x},\mathbf{n}) = \alpha(\mathbf{x})\,[J(\mathbf{x}) - I(\mathbf{x},\mathbf{n})] \tag{4.4}$$

Eq. (4.4) clearly demonstrates the "chicken or egg" effect that makes radiative transfer so difficult: We need to know $J(\mathbf{x})$ before we can integrate Eq. (4.4) along any ray, but we need to know $I(\mathbf{x},\mathbf{n})$ for all directions $\mathbf{n}$ to compute $J(\mathbf{x})$.

4.1.1 The culprit: Multiple scattering

We can formulate this "chicken or egg" problem in another way by following light back to its source. The photons that we observe when we look at the cloud may have been scattered into the line of sight by a dust particle. Before that event, these photons moved along another ray. But they might have in fact be scattered into that ray by another scattering event, etc. Photons can scatter multiple times before they are scattered into the line of sight. This is called the *multiple scattering problem*.

Note: *The problem of isotropic multiple scattering can be considered a benchmark problem of radiative transfer. Understanding how to tackle this problem provides a solid basis for understanding the more complex radiative transfer problems in the next chapters. We will therefore spend considerable time on this admittedly fairly idealized problem.*



Multiple scattering can be regarded in terms of recursion: Each successive scattering event can be associated to one "chicken-egg" cycle: To compute J at some particular point $\mathbf{x}_0$ along the line of sight we would need to perform integrations of the formal transfer equation along all rays that go through $\mathbf{x}_0$, i.e. varying $\mathbf{n}$ all over 4π steradian. However, to be able to integrate the formal transfer equations along those rays we will need to know J at other locations $\mathbf{x} \neq \mathbf{x}_0$ along these rays, these involve again performing the transfer equation along all rays that go through $\mathbf{x}$, varying $\mathbf{n}$ all over 4π steradian, etc.

How to solve this?

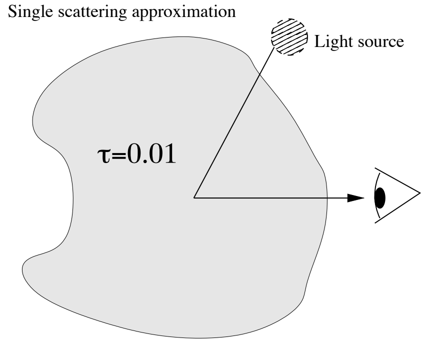
Exact analytical solutions to this problem are exceedingly rare. For a semi-infinite homogeneous plane-parallel atmosphere being irradiated from the top, a solution is given by Chandrasekhar's H-functions theory (from Chandrasekhar's book "Radiative Transfer", 1950/1960, Dover). However, for most practical cases a numerical approach is required, which is a challenge because of the high dimensionality of the problem.

In this lecture we will discuss several classes of numerical methods to tackle this and related problems. The main three classes of methods are: (1) *Monte Carlo methods*, which simulate the multiple scattering process directly, (2) *Discrete ordinate methods*, which solve the problem by dividing all coordinates, including the angles and the frequency, into discrete grid points or grid cells, and (3) *Moment methods*, including the diffusion method, which treat the angular and/or frequency domain in terms of its moments. We will discuss all these classes of methods in detail in later chapters, but we will already briefly introduce them in this chapter.

4.1.2 For $\tau \ll 1$: The single scattering approximation

For the case when the optical depth of the cloud is very low we can make an approximation that makes the problem solvable either analytically or at least numerically with little computational cost: We can then ignore the effect of multiple scattering, and assume that every photon that scatters into the line of sight experienced no scattering evens before that. This is the *single scattering approximation*. This approximation becomes better the lower the optical depth of the cloud is. Since, as we showed above, each successive scattering event is associated with one "chicken-egg" cycle of Eqs. (4.1, 4.2), the single scattering approximation allows us to limit our efforts by:

Single scattering approximation



1. integrating the formal transfer equation for all rays connecting the light source to any of the points along the line of sight,
2. computing the j_ν along the line of sight
3. integrating the formal transfer equation along the line of sight to the observer.

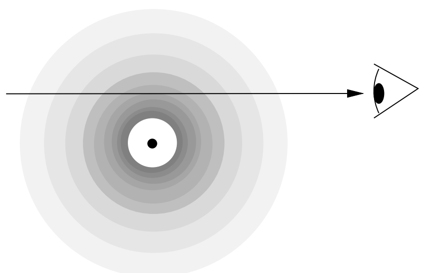
While the procedure of integrating the transfer equation along all rays connecting source and line of sight may still be difficult analytically, or require quite a bit of computation time numerically, the effort is manageable.

4.1.3 A worked-out example of single scattering

To get a better feeling for the practical issues, let us work out a concrete example of single scattering. Let us assume that we have a star of radius R_* and temperature T_* that radiates as a perfect blackbody emitter. Surrounding it is a spherically symmetric cloud of dust. The density of the cloud of dust is given by the following formula:

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-2} \qquad \text{for} \quad r \geq r_0 \tag{4.5}$$

Spherical envelope around a star



and zero inside of r_0. The scattering opacity is assumed to be independent of frequency: $\kappa_\nu = \kappa$ and independent of density or temperature. We will take it constant. We assume that the radial optical depth between the star and a point at distance r is small enough that the single scattering approximation can be safely made:

$$\tau_\nu(r) = \kappa_\nu \int_{r_0}^{r} \rho(r')dr' \ll 1 \tag{4.6}$$

Let us assume that $r_0 \gg R_*$ so that in good approximation we can treat the star as a point source. The flux from the star is:

$$F_\nu(r) = \frac{L_\nu}{4\pi r^2} \qquad \text{with} \qquad L_\nu = 4\pi R_*^2 \pi B_\nu(T_*) \tag{4.7}$$

For the computation of the scattering emissivity j_ν we need the mean intensity J_ν, which is for this case:

$$J_\nu = \frac{F_\nu}{4\pi} \qquad \text{(for exactly outward-pointing radiation)} \tag{4.8}$$

so that

$$j_\nu(r) = \alpha_\nu \frac{F_\nu}{4\pi} = \frac{1}{(4\pi)^2}\,\kappa_\nu L_\nu \rho_0 r_0^2 \frac{1}{r^4} \tag{4.9}$$

Now we must integrate this emissivity along a line of sight. Let us choose a line of sight with an impact parameter $b > r_0$. Let us choose our coordinate s along the ray such that $s = 0$ is the closest approach. We can then write

$$r = \sqrt{b^2 + s^2} \tag{4.10}$$

The integral of the formal transfer equation along the line of sight then becomes:

$$\begin{aligned} I_\nu^{\mathrm{obs}}(b) &= \frac{1}{(4\pi)^2}\,\kappa_\nu L_\nu \rho_0 r_0^2 \int_{-\infty}^{+\infty} \frac{ds}{(b^2+s^2)^2} \\ &= \frac{1}{(4\pi)^2}\,\kappa_\nu L_\nu \rho_0 \frac{r_0^2}{b^3} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} \\ &= \frac{1}{32\pi}\,\kappa_\nu L_\nu \rho_0 \frac{r_0^2}{b^3} \end{aligned} \tag{4.11}$$

assuming no background intensity. On an image we will thus see the scattered light of the envelope decay as $1/b^3$ away from the star. Since the $\rho \propto 1/r^2$ density profile is what you would expect from a stellar wind (ballistic outflow), this is in fact a reasonably realistic model for reflection nebulae around stars with dusty stellar winds. In reality, as we shall see in Chapter 6, the isotropic scattering approximation is not always a good approximation for light scattering off dust particles. But the $1/b^3$ decay of scattered light is, also in the case of anisotropic scattering, not a bad approximation.

4.1.4 Including absorption and thermal emission

While the multiple scattering problem formulated so far is an extremely challenging problem to solve, it is somewhat idealized because we assumed that the dust particles do not absorb any radiation (they only scatter) nor do they thermally emit any radiation. For water droplet clouds in the Earth's atmosphere at optical wavelengths this is a reasonable approximation. But there are many cases where some amount of thermal absorption and emission is present in addition to the scattering. In the Earth's atmosphere this is, for instance, the case for aerosols. In astrophysics there are many situations where both absorption/emission and scattering play a role. The dust in the interstellar medium has this property, and so does the dust in circumstellar disks.

When we include absorption, then at every frequency ν we have two kinds of opacity: absorption opacity and scattering opacity:

$$\alpha_\nu = \alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{scat}} \tag{4.12}$$

We define the *albedo* as:

$$\eta_\nu = \frac{\alpha_\nu^{\mathrm{scat}}}{\alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{scat}}} \tag{4.13}$$

In some fields of physics a symbol α is used for albedo, but we already reserved that for the extinction coefficient, in accordance with stellar atmosphere radiative transfer conventions. Conversely we can define the *photon destruction probability* as:

$$\epsilon_\nu = \frac{\alpha_\nu^{\mathrm{abs}}}{\alpha_\nu^{\mathrm{abs}} + \alpha_\nu^{\mathrm{scat}}} \tag{4.14}$$

This quantity is often used in non-LTE radiative transfer theory in stellar atmospheres. We clearly have

$$\epsilon_\nu = 1 - \eta_\nu \tag{4.15}$$

Also the emissivity j_ν can be seen as consisting of two contributions:

$$j_\nu = j_\nu^{\rm emis} + j_\nu^{\rm scat} \tag{4.16}$$

where $j_\nu^{\rm emis}$ is the emissivity corresponding to the absorption coefficient $\alpha_\nu^{\rm abs}$. Note that in this case no special symbols are defined for their ratios.

The source function is

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{j_\nu^{\rm emis} + j_\nu^{\rm scat}}{\alpha_\nu^{\rm abs} + \alpha_\nu^{\rm scat}} \tag{4.17}$$

We can rewrite this into the form

$$\begin{aligned} S_\nu &= \epsilon_\nu \frac{j_\nu^{\rm emis}}{\alpha_\nu^{\rm abs}} + \eta_\nu \frac{j_\nu^{\rm scat}}{\alpha_\nu^{\rm scat}} \\ &= \epsilon_\nu S_\nu^{\rm abs} + \eta_\nu S_\nu^{\rm scat} \end{aligned} \tag{4.18}$$

For isotropic scattering we have $j_\nu^{\rm scat}/\alpha_\nu^{\rm scat} = J_\nu$. If the emission is thermal emission at some temperature T, then we have $j_\nu^{\rm emis}/\alpha_\nu^{\rm abs} = B_\nu(T)$. Then we can write the source function as

$$\begin{aligned} S_\nu &= \epsilon_\nu B_\nu(T) + \eta_\nu J_\nu \\ &= \epsilon_\nu B_\nu(T) + (1 - \epsilon_\nu) J_\nu \end{aligned} \tag{4.19}$$

where the latter way of writing is the standard used in the community of stellar atmospheres. The transfer equation remains (cf. Eq. 3.13):

$$\frac{dI_\nu}{ds} = \alpha_\nu [S_\nu - I_\nu] \tag{4.20}$$

(where the explicit reference to s- and $\mathbf{n}$-dependency is omitted for notational convenience) which now becomes, if we insert Eq. (4.19):

$$\frac{dI_\nu}{ds} = \alpha_\nu [\epsilon_\nu B_\nu(T) + (1 - \epsilon_\nu) J_\nu - I_\nu] \tag{4.21}$$

For $\epsilon_\nu = 1$ we retrieve Eq. (3.11). For $\epsilon_\nu = 0$ we retrieve Eq. (4.4). Equation (4.21) is thus a true mix of the thermal emission/absorption and the scattering problem.

How does this change the nature of the problem? Clearly, if $\epsilon_\nu = 1$, assuming that we know what the temperature T is everywhere, then there is no "chicken or egg" problem. The problem is most profound for $\epsilon_\nu = 0$. So the problem is of moderate complexity for $0 < \epsilon_\nu < 1$. If $\epsilon_\nu = 0.5$, for instance, a photon can scatter not more than a few times before it will be destroyed by absorption. Information can thus be transported, on average, not farther than a few mean free paths before the radiation field "forgets" that information. Whereas for $\epsilon_\nu = 0$ a photon will scatter as long as it takes to escape, and may thus traverse macroscopic distances through the cloud, for $\epsilon_\nu = 0.5$ radiative information travels only a few mean free paths from its origin before it is deleted. As we shall see in later sections and chapters, the closer ϵ_ν is to 0, the harder the radiative transfer problem gets.

4.2 Monte Carlo methods (Introduction)

One of the main methods for solving the multiple scattering problem is called the *Monte Carlo* method. It is named after the town of Monte Carlo, famed for its Grand Casino. As with gambling, Monte Carlo methods are based on chance. The idea is to follow the path of a photon from one scattering event to the next, and to use random