# SPP Summer School: Planet formation

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**Planet formation framework:** This manuscript is intended to give a small overview of the important ingredients in planet formation simulations. A planet formation model needs (at a minimum) these ingredients:

- A disc model
- A planet growth (solid + gas) mechanism
- Planetary migration (type-I and type-II)

The intention of the following recipes and equations is to allow an easy manipulation of the parameters to test their influence on planet formation. Obviously all parts of the model listed below can be easily expanded with more detailed formulae and expressions that already exist in the literature and are listed within the text (see also Bitsch et al. 2015b). A recent review about planet formation is given in Drazkowska et al. (2022).

## 1 Disc model

The structure and evolution of the protoplanetary disc sets how planets grow and migrate, because the disc contains the available mass budget for planet formation. The evolution of the disc was long thought to be driven purely by the viscosity (Lynden-Bell & Pringle, 1974), using an $\alpha$ prescription (Shakura & Sunyaev, 1973). However, recently this picture has changed and it is thought that the disc evolution is mainly driven by disc winds that transport the angular momentum away (see Lesur et al. 2022 for a recent review). For simplicity we will follow here the classic Minimum Mass Solar Nebular prescript, which just follows a power law distribution in gas surface density $\Sigma_g$ and aspect ratio $H/r$. It's values are given as

$$\Sigma_g = \Sigma_0 \left(\frac{r}{\mathrm{AU}}\right)^{-\alpha_\Sigma} \quad \text{and} \quad H/r = 0.033 \left(\frac{r}{\mathrm{AU}}\right)^{f} \ . \tag{1}$$

Normally $\Sigma_0 = 1700\mathrm{g/cm}^2$, $\alpha_\Sigma = 3/2$ and $f = 1/4$ for the MMSN. But other power laws are also possible, depending on the disc profile.

This simple disc profile allows an easy analytical access as function of radial position. A more complex semi-analytical disc profile is described in Bitsch et al. (2015a). In principle all quantities in planet formation depend on the disc, implying that a planet formation model is only as good as its disc model.

## 2 Planet growth

### 2.1 Solid accretion

Planetary embryos can either grow via accretion of planetesimals (Pollack et al., 1996) or pebbles (see Johansen & Lambrechts 2017 for a review). While planetesimal accretion was long thought to be the dominant mechanism, there is increased questioning about the efficiency of planetesimal accretion, especially if planetesimals are large (ca. 100km in size) and if the accreting planet is far away from the star (Tanaka & Ida, 1999; Levison et al., 2010; Fortier et al., 2013; Johansen & Bitsch, 2019). Pebble accretion, on the other hand, allows the formation of planetary cores in wide orbits (Lambrechts & Johansen, 2014; Bitsch et al., 2015b), but it crucially depends on the pebble flux (Lambrechts et al., 2019; Bitsch et al., 2019). Here we will follow a simple pebble accretion recipe that depends on the pebble flux passing the planet. In particular we will make the assumption that the planet is already accreting in the efficient 2D Hill accretion regime, corresponding typically to 0.01 Earth masses. This implicitly assumes that the planetary Hill radius is already so large that it covers the full vertical distribution of the accreted pebbles. The pebble accretion rate onto a planetary core with mass $M_\mathrm{c}$ is then given by

$$\dot{M}_{\mathrm{c,2D}} = 2\left(\frac{\tau_\mathrm{f}}{0.1}\right)^{2/3} r_\mathrm{H} v_\mathrm{H} \Sigma_\mathrm{peb} \ , \quad (2)$$

where $r_\mathrm{H} = r[M_\mathrm{c}/(3M_\star)]^{1/3}$ is the Hill radius, $v_\mathrm{H} = \Omega_\mathrm{K} r_\mathrm{H}$ the Hill speed, and $\Sigma_\mathrm{peb}$ the pebble surface density. If the Stokes number of the particles $\tau_\mathrm{f}$ is larger than 0.1, the accretion rate is limited to

$$\dot{M}_{\mathrm{c,2D}} = 2 r_\mathrm{H} v_\mathrm{H} \Sigma_\mathrm{peb} \ , \quad (3)$$

because the planetary seed cannot accrete particles from outside its Hill radius (Lambrechts & Johansen, 2012). The pebble surface density is related to the pebble flux $\dot{M}_\mathrm{peb}$ via

$$\Sigma_\mathrm{peb} = \frac{\dot{M}_\mathrm{peb}}{2\pi r v_\mathrm{r}} \ , \quad (4)$$

where $v_\mathrm{r}$ is the radial velocity of the pebbles dependent on their Stokes number $\tau_\mathrm{f}$ via

$$v_\mathrm{r} = \frac{2\tau_\mathrm{f}}{\tau_\mathrm{f}^2 + 1} \eta_\mathrm{P} v_\mathrm{K} \ . \quad (5)$$

Here $v_\mathrm{K}$ denotes the Keplerian velocity $v_\mathrm{K} = \sqrt{GM_\odot/r}$ at the planets location and $\eta_\mathrm{P}$ describes the the sub-Keplerianity of the gas

$$\eta_\mathrm{P} = -\frac{1}{2}\left(\frac{H}{r}\right)^2 \frac{\partial \ln P}{\partial \ln r} \ . \quad (6)$$

Here, $\partial \ln P/\partial \ln r$ is the radial pressure gradient in the disc. The pressure $P$is given as

$$P = c_\mathrm{s}^2 \rho_\mathrm{g} \ , \quad (7)$$

where $c_\mathrm{s} = H\Omega$ is the sound speed and $\rho_\mathrm{g} \propto r^{-\alpha_\Sigma - 1}$ is the midplane gas density. The pressure gradient is then given as

$$\frac{\partial \ln P}{\partial \ln r} = 2f - \alpha_\Sigma - 2 \ . \quad (8)$$

In the following we will make the assumption that pebbles have a constant Stokes number as they drift through the disc (in agreement with detailed simulations of pebble growth and drift, e.g. Brauer et al. 2008; Birnstiel et al. 2012). Pebble accretion typically becomes efficient for Stokes numbers above 0.01.

A typical pebble flux of around 100 Earth masses over 3 Myr can lead to the formation of super-Earths (Lambrechts et al., 2019; Bitsch et al., 2019). Lower pebble fluxes might lead only allow the growth of Mars mass embryos, while even larger pebble fluxes will allow the formation of giant planet cores even in the outer disc. Detailed simulations have also shown that the pebble flux decays over time, because pebbles drift faster than the gas resulting in a depletion of the pebbles (Brauer et al., 2008; Birnstiel et al., 2012). We thus use the following pebble flux for the growth of the planet as function of time $t$:

$$\dot{M}_\mathrm{peb} = 10^{-4} \frac{\mathrm{M_E}}{\mathrm{yr}} \exp^{-t/2.5\mathrm{Myr}} \ . \quad (9)$$

Pebble accretion stops, once the planet starts to open a partial gap, resulting in a pressure bump exterior to the planetary orbit stopping inward flowing pebbles (Paardekooper & Mellema, 2006). The planetary mass at which this happens is the so-called pebble isolation mass $M_\mathrm{iso}$ (Morbidelli & Nesvorny, 2012; Lambrechts et al., 2014; Ataiee et al., 2018; Bitsch et al., 2018), where the accretion of pebbles stop. The pebble isolation mass depends on the disc's aspect ratio, the local pressure gradient and on the disc's viscosity as well as on the pebble size (see Ataiee et al. 2018; Bitsch et al. 2018 for detailed recipes), but we will here only use a simple expression depending on the disc's aspect ratio H/r:

$$M_\mathrm{iso} = 25\left(\frac{H/r}{0.05}\right)^3 M_\mathrm{E} \ . \quad (10)$$

When a planetary core reaches this mass, pebble accretion stops and gas accretion can start.

### 2.2 Gas accretion

While gas accretion on planetary cores is clearly a 3D problem (e.g. Lambrechts & Lega 2017; Cimerman et al. 2017; Schulik et al. 2019; Moldenhauer et al. 2022), 3D simulations have the disadvantage that they can not be integrated over the whole lifetime of the protoplanetary disc. To study the long term evolution, 1D models are used (Ikoma et al., 2000), which depend mainly on the core mass and the opacity inside the planetary envelope. In these recipes a simple Kelvin-Helmholtz contraction is assumed and we will use this here as well:

$$\dot{M}_{\rm KH} = \frac{M_{\rm planet}}{\tau_{\rm KH}} \quad , \quad \tau_{\rm KH} = \left(\frac{M_{\rm core}}{1{\rm M_E}}\right)^{-5/2} \left(\frac{\kappa_{\rm ross}}{1{\rm cm}^2/{\rm g}}\right) 10^8 {\rm yr} \ . \tag{11}$$

Here $M_{\rm planet}$ is the total planetary mass, $M_{\rm core}$ is the mass of the planetary core, while $\tau_{\rm KH}$ represents the Kelvin-Helmholtz contraction time, including the opacity in the planetary envelope $\kappa_{\rm ross}$. Typically this opacity is of the order of $0.05{\rm cm}^2/{\rm g}$ (Movshovitz & Podolak, 2008), but it clearly depends on the grain sizes and amounts in the planetary atmosphere (Mordasini, 2014; Bitsch & Savvidou, 2021; Brouwers et al., 2021).

However, this gas accretion rate can lead to unphysically large values, as there is no dependency on the local environment (e.g. the planet could accrete more gas than is available). As soon as the planet opens a gap (see below), the planetary gas accretion rate becomes limited by what the disc can provide:

$$\dot{M}_{\rm disc} = 3\pi\alpha H^2 \Omega_{\rm K} \Sigma_{\rm g} \ , \tag{12}$$

where $\alpha$ is the viscosity parameter, typically in the range of $10^{-4}$ to $10^{-2}$. The real gas accretion rate onto the planet is thus the minimum between these two rates:

$$\dot{M}_{\rm gas} = \min\left(\dot{M}_{\rm KH}, 0.8\dot{M}_{\rm disc}\right) \ . \tag{13}$$

We limit the maximum accretion rate to 80% of the disc's accretion rate onto the star, because gas can flow through the gap, even for high mass planets (Lubow & D'Angelo, 2006).

## 3 Planet migration

### 3.1 Type-I migration

We focus here on the fully unsaturated torques following (Paardekooper et al., 2010), which corresponds to the isothermal limit of the full torque expression (Paardekooper et al., 2011). We only make this choice for *simplicity* and a more sophisticated planet formation models *needs* to take the saturation effects into account. While we follow here the approach of (Paardekooper et al., 2011), a similar recipe by Jiménez & Masset (2017) exists, yielding similar results regarding the final planetary positions (Baumann & Bitsch, 2020).

The total torque acting on a low-mass planet consists of two main contributions the Lindblad torque, $\Gamma_{\rm L}$, plus the corotation torque, $\Gamma_{\rm c}$

$$\Gamma_{\rm tot} = \Gamma_{\rm L} + \Gamma_{\rm c} \ . \tag{14}$$

The Lindblad torque is caused by the action of the induced spiral arms and is given as (Paardekooper & Papaloizou, 2008)

$$\gamma\Gamma_{\rm L}/\Gamma_0 = -2.5 - 1.7\beta + 0.1\alpha_\Sigma \ , \tag{15}$$

where $\alpha$ denotes the negative slope of the surface density profile $\Sigma \propto r^{-\alpha_\Sigma}$, $\beta$ refers to the slope of the temperature profile $T \propto r^{-\beta}$, and $\gamma$ is the adiabatic index of the gas (normally set to 7/5, but it can also vary to 5/3, which can change the migration behavior of planets, Bitsch et al. 2013). The slope of the temperature profile relates to the slope of the disc's aspect ratio $f$ via:

$$\beta = 1 - 2f \ . \tag{16}$$

It is important to note that all torques listed here are normalized to

$$\Gamma_0 = \left(\frac{q}{h}\right)^2 \Sigma_{\rm P} r_{\rm P}^4 \Omega_{\rm P}^2 \ ,$$

with $q$ the planet/star mass ratio and $h = H/r$. The subscript $P$ denotes that the corresponding quantity is evaluated at the position of the planet.

The corotation torque consists of a contribution of the barotropic and entropy-related part. Here we also make the assumption that we are in the *isothermal* regime. In this case the corotation torque is given by the barotropic, non-linear horseshoe drag plus the linear, entropy-related corotation torque:

$$\Gamma_{\rm c} = \Gamma_{\rm hs,baro} + \Gamma_{\rm c,lin,ent} \ .$$

These torques are defined as follows

$$\gamma\Gamma_{\rm hs,baro}/\Gamma_0 = 1.1(1.5 - \alpha_\Sigma) \quad , \quad \gamma\Gamma_{\rm c,lin,ent}/\Gamma_0 = \big(2.2 - \frac{1.4}{\gamma}\big)\xi, \tag{17}$$

where $\xi = \beta - (\gamma - 1.0)\alpha_\Sigma$ is the negative of the power-law index of the entropy. For a more complete approach the saturation effects have to be taken into account complicating the picture (Paardekooper et al., 2011), especially due to its dependence on the disc's viscosity.

The planetary migration timescale $\tau_{\rm m}$ is now defined as

$$\tau_{\rm m} = 0.5\frac{L}{\Gamma_{\rm tot}} \ , \tag{18}$$

where $L = M_{\rm P} r_{\rm P}^2 \Omega_P$ is a planet's orbital angular momentum. The planetary position is then updated via

$$r_{\rm new} = r_{\rm old} + \frac{r_{\rm old}}{\tau_{\rm m}} {\rm dt} \ , \tag{19}$$

where dt is the chosen time step (e.g. 500 years).

### 3.2 Type-II migration

Planets that have reached their pebble isolation mass (eq. 10) start to accrete gas and grow even further until they finally open a gap inside the disc. A gap can be opened (with $\Sigma_{\rm Gap} < 0.1\Sigma_{\rm g}$), when

$$\mathcal{P} = \frac{3}{4}\frac{H}{r_{\rm H}} + \frac{50}{q\mathcal{R}} \leq 1 \ , \tag{20}$$

where $r_H$ is the Hill radius, $q = M_{\rm P}/M_\star$, and $\mathcal{R}$ the Reynolds number given by $\mathcal{R} = r_{\rm P}^2\Omega_{\rm P}/\nu$ (Crida et al., 2006). If the planet becomes massive enough to fulfill this criterion, it opens up a gap in the disc, and it migrates in the type-II regime. The gap-opening process splits the disc in two parts, which both repel the planet towards the centre of the gap, meaning that the migration time scale of the planet is the accretion time scale of the disc, $\tau_\nu = r_P^2/\nu$. $\nu$ corresponds here to the viscosity, $\nu = \alpha H^2\Omega$. However, if the planet is much more massive than the gas outside the gap, it will slow down the viscous accretion. This happens if $M_{\rm P} > 4\pi\Sigma_{\rm g} r_{\rm P}^2$, which leads to the migration time scale of

$$\tau_{\rm II} = -\tau_\nu \times \max\left(1, \frac{M_{\rm P}}{4\pi\Sigma_{\rm g} r_{\rm P}^2}\right) \ , \tag{21}$$

resulting in slower inward migration for massive planets (Baruteau et al., 2014).

This classical approach of type-II migration has been put into question recently (e.g. Dürmann & Kley 2015; Robert et al. 2018), where also a correlation between the gap depth that the planet generates and the type-I migration rate has been proposed (Kanagawa et al., 2018). It is obvious that the planet migration speed shapes the final planetary system. Here the different migration recipes for type-II can play a role, but most important is the viscosity of the protoplanetary disc. For example, a smaller viscosity allows an easier transition into type-II migration (gap opening is easier) and at the same time a reduced type-II migration speed (eq. 21). In reality also gas accretion can help (or prevent) opening a gap (Bergez-Casalou et al., 2020), resulting in a different outcome in planet formation simulations (Ndugu et al., 2021).

# 4 Exercises

- Implement the simple planet formation model by starting with the disc and accretion (pebble + gas) model. For the initial mass of the embryo use 0.01 Earth masses and plot the mass growth as a function of time.
- Investigate how the different parameters related to the disc (power law index of $\Sigma$ and H/r, total surface density $\Sigma_0$) influence planet formation.
- Investigate how a change in the pebble flux or Stokes number can change the outcome for your growing planet.
- Implement the simple planet migration model and investigate how a change in the disc's viscosity influences planet growth and migration (for gas giants). NOTE: the planet migration recipe is very simple and might lead to *very* fast inward migration, so do not be surprised if the planet reaches the inner edge in a short time!

# References


Ataiee, S., Baruteau, C., Alibert, Y., & Benz, W. 2018, A&A, 615, A110
Baruteau, C., Crida, A., Paardekooper, S. J., et al. 2014, in Protostars and Planets VI, arXiv:1312.4293
Baumann, T. & Bitsch, B. 2020, A&A, 637, A11
Bergez-Casalou, C., Bitsch, B., Pierens, A., Crida, A., & Raymond, S. N. 2020, arXiv e-prints, arXiv:2010.00485
Birnstiel, T., Klahr, H., & Ercolano, B. 2012, A&A, 539, id.A148
Bitsch, B., Boley, A. C., & Kley, W. 2013, A&A, 550, id.A52
Bitsch, B., Izidoro, A., Johansen, A., et al. 2019, A&A, 623, A88
Bitsch, B., Johansen, A., Lambrechts, M., & Morbidelli, A. 2015a, A&A, 575, id.A28
Bitsch, B., Lambrechts, M., & Johansen, A. 2015b, A&A, 582, id.A112
Bitsch, B., Morbidelli, A., Johansen, A., et al. 2018, A&A, 612, id.A30
Bitsch, B. & Savvidou, S. 2021, A&A, 647, A96
Brauer, F., Dullemond, C., & Henning, T. 2008, A&A, 480, pp.859
Brouwers, M. G., Ormel, C. W., Bonsor, A., & Vazan, A. 2021, A&A, 653, A103
Cimerman, N. P., Kuiper, R., & Ormel, C. W. 2017, MNRAS, 471, 4662
Crida, A., Morbidelli, A., & Masset, F. 2006, Icarus, 181, 587
Drazkowska, J., Bitsch, B., Lambrechts, M., et al. 2022, arXiv e-prints, arXiv:2203.09759
Dürmann, C. & Kley, W. 2015, A&A, 574, id.A52
Fortier, A., Alibert, Y., Carron, F., Benz, W., & Dittkrist, K. M. 2013, A&A, 549, A44
Ikoma, M., Nakazawa, K., & Emori, H. 2000, ApJ, 537, pp. 1013
Jiménez, M. A. & Masset, F. S. 2017, MNRAS, 471, 4917
Johansen, A. & Bitsch, B. 2019, A&A, 631, A70
Johansen, A. & Lambrechts, M. 2017, AREP, 45
Kanagawa, K. D., Tanaka, H., & Szuszkiewicz, E. 2018, ApJ, 861, id.140
Lambrechts, M. & Johansen, A. 2012, A&A, 544, id.A32
Lambrechts, M. & Johansen, A. 2014, A&A, 572, id.A107
Lambrechts, M., Johansen, A., & Morbidelli, A. 2014, A&A, 572, id. A35
Lambrechts, M. & Lega, E. 2017, A&A, 606, A146
Lambrechts, M., Morbidelli, A., Jacobson, S. A., et al. 2019, arXiv e-prints
Lesur, G., Ercolano, B., Flock, M., et al. 2022, arXiv e-prints, arXiv:2203.09821
Levison, H. F., Thommes, E., & Duncan, M. J. 2010, AJ, 139, pp.1297
Lubow, S. H. & D'Angelo, G. 2006, ApJ, 641, pp.526
Lynden-Bell, D. & Pringle, J. E. 1974, MNRAS, 168, 603
Moldenhauer, T. W., Kuiper, R., Kley, W., & Ormel, C. W. 2022, A&A, 661, A142
Morbidelli, A. & Nesvorny, D. 2012, A&A, 546, id.A18
Mordasini, C. 2014, A&A, 572, id.A118
Movshovitz, N. & Podolak, M. 2008, Icarus, 194, p.368
Ndugu, N., Bitsch, B., Morbidelli, A., Crida, A., & Jurua, E. 2021, MNRAS, 501, 2017
Paardekooper, S., Baruteau, C., Crida, A., & Kley, W. 2010, MNRAS, 401, 1950+
Paardekooper, S. J., Baruteau, C., & Kley, W. 2011, MNRAS, 410, 293
Paardekooper, S.-J. & Mellema, G. 2006, A&A, 453, pp.1129
Paardekooper, S.-J. & Papaloizou, J. C. B. 2008, A&A, 485, 877

Pollack, J. B., Hubickyj, O., Bodenheimer, P., et al. 1996, Icarus, 124, 62
Robert, C. M. T., Crida, A., Lega, E., Méheut, H., & Morbidelli, A. 2018, A&A, 617, id.A98
Schulik, M., Johansen, A., Bitsch, B., & Lega, E. 2019, A&A, 632, A118
Shakura, N. I. & Sunyaev, R. A. 1973, A&A, 24, 337
Tanaka, H. & Ida, S. 1999, ICARUS, 139, 350