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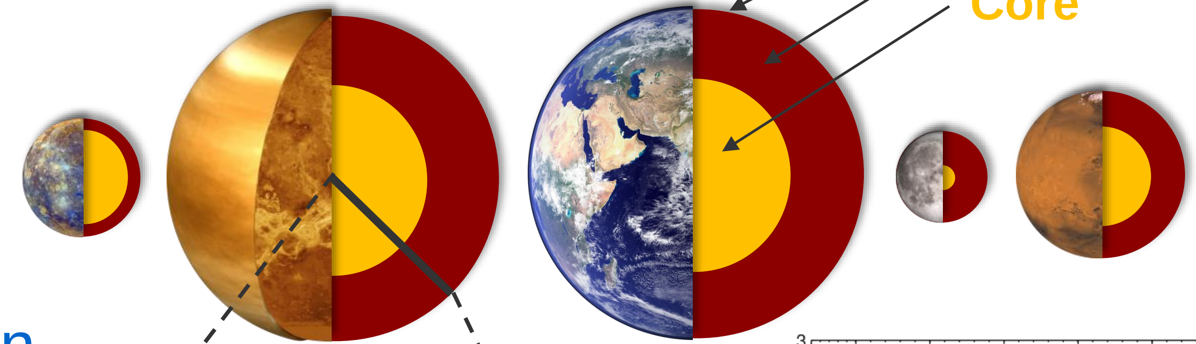
(Rocky) planet interior structure modeling

06.07.2022



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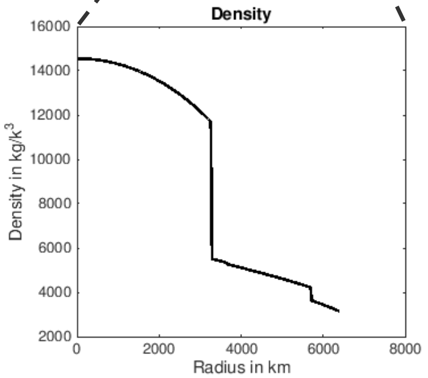
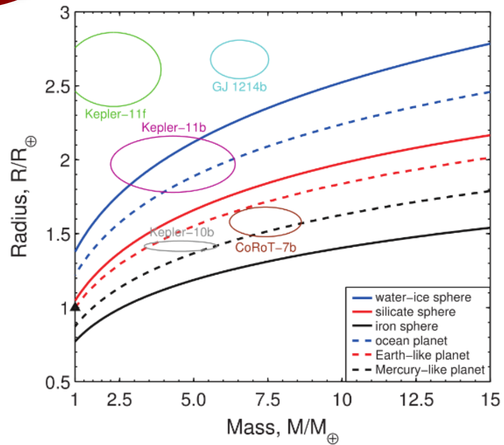
SPP1992 astrophysical summer school - July 4 to 8, 2022

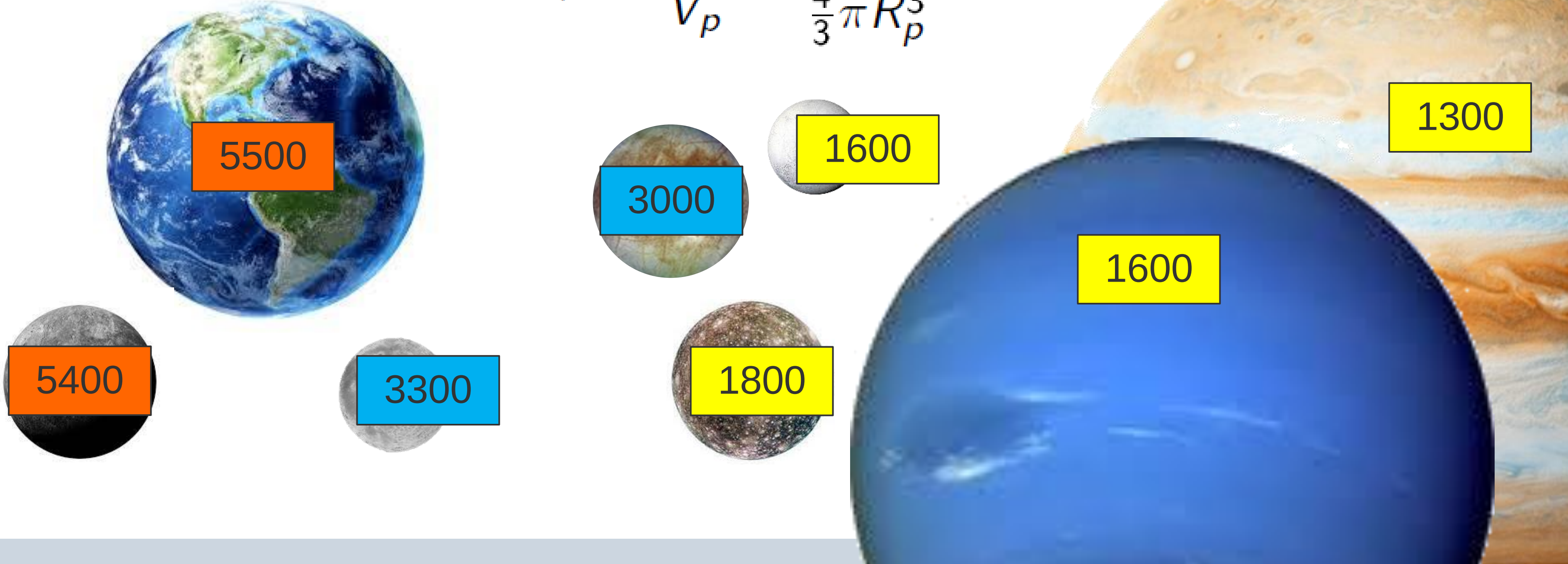




Characterization of planets via average density

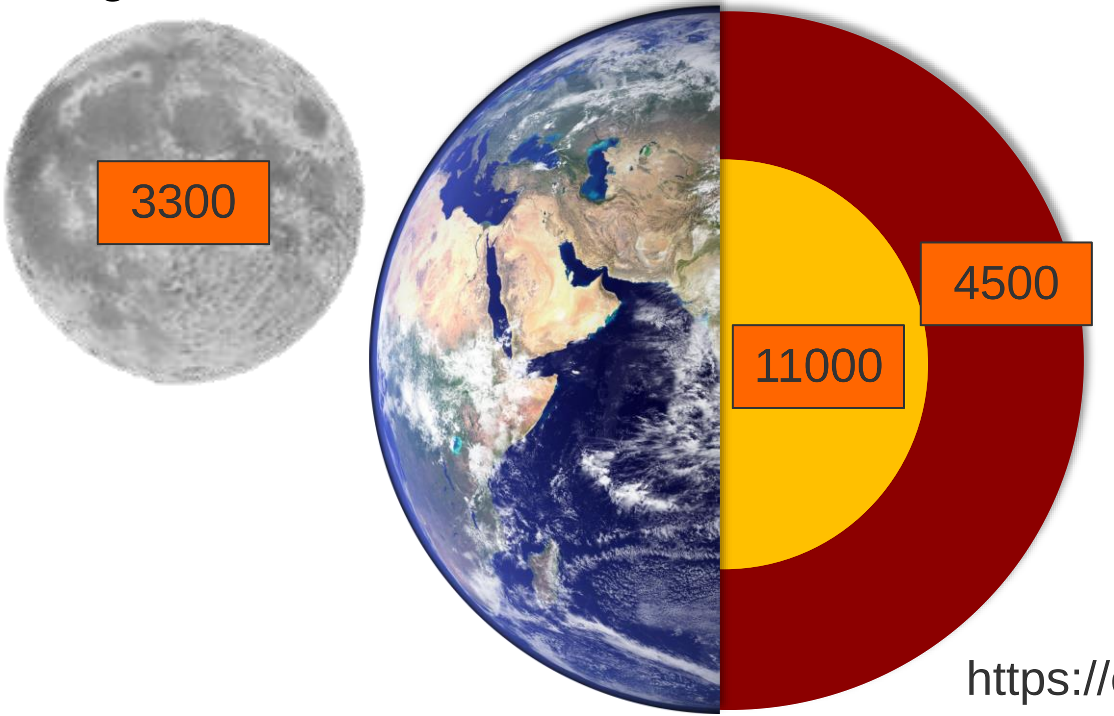
If we know the mass M_p and radius R_p of a planet, we can calculate the average density ρ by dividing the planet mass by the volume V_p :

$$\rho = \frac{M_p}{V_p} = \frac{M_p}{\frac{4}{3}\pi R_p^3}$$



A hollow Moon?

Starting with the science fiction novel by H. G. Wells on „The First Men in the Moon“, speculations have appeared and re-appeared about the Moon being too low-dense and therefore hollow/an alien spacecraft invention

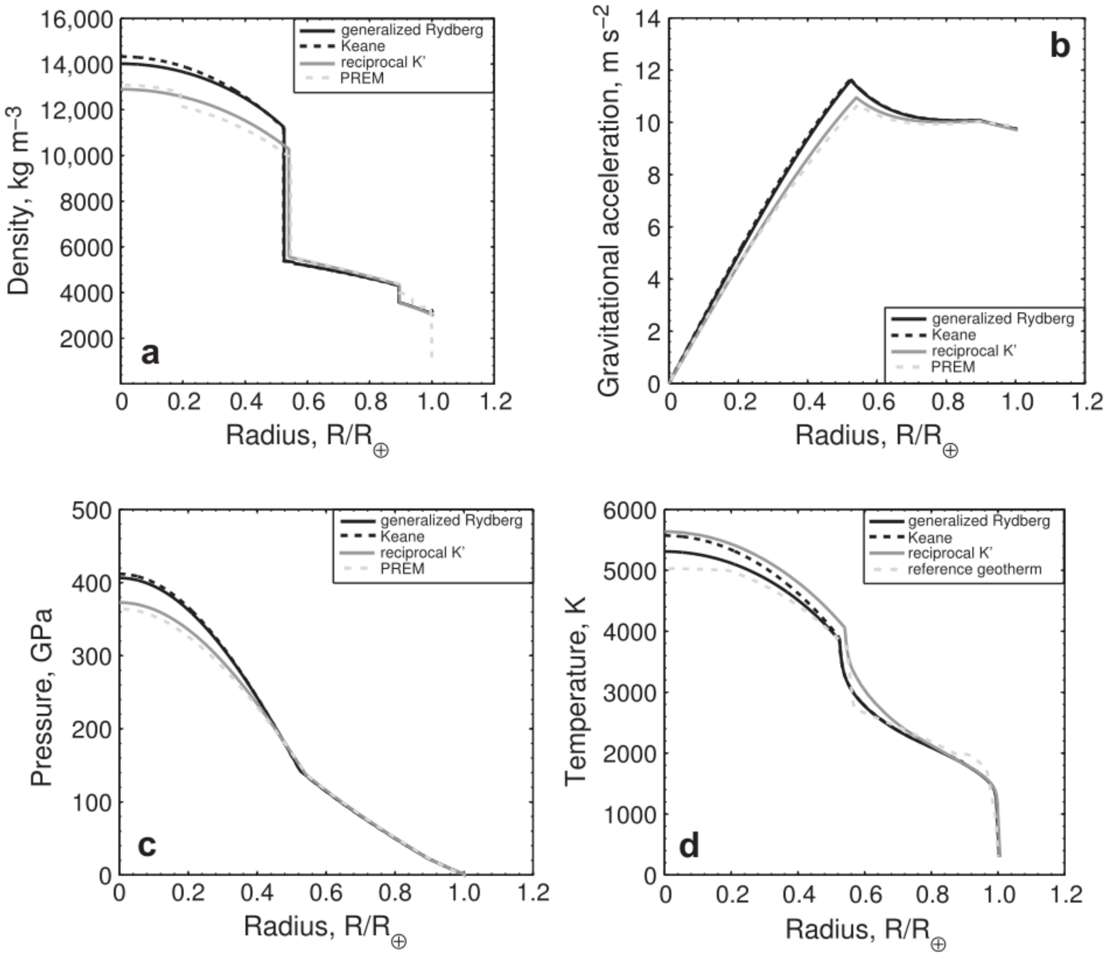


https://en.wikipedia.org/wiki/Hollow_Moon

Interior structure model

Wagner et al., 2011

All materials
are
compressible!



Interior structure model

An interior structure model is used to

- assess the radius of a planet or moon for given mass and composition
- to determine the depth-dependent pressure, density and other thermodynamic properties for a specific temperature profile

<u>1st assumption</u>: Hydrostatic equilibrium, which means that the material is in static equilibrium - the weight of each element is entirely supported by the pressure difference across the element

<u>2nd assumption</u>: planet is differentiated into Fe core and silicate mantle consisting of Mg-Si-O (based on olivine and high-pressure equivalents)

How to build an interior structure model

1) Split the planet in several thin shells

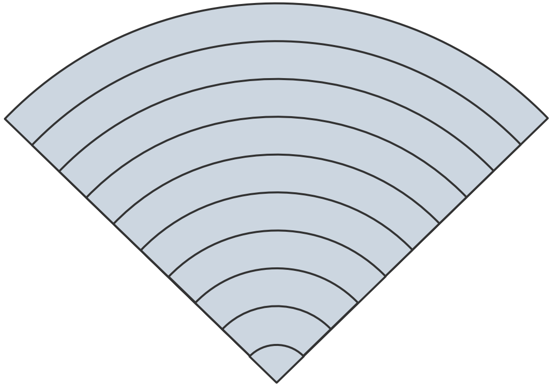
2) Each shell has a local density (same for all thermodynamic parameters)

3) Add up the mass of each shell until reaching M_p

4) Update pressure, temperature etc.

5) Iterate until planet radius converges

$m = M_p; P = P_{atm}$



$m = 0; g = 0$

Mass conservation

The mass $m(r)$ below a sphere of radius r increases with increasing radial coordinate, it is zero in the center of the planet and it is the planet's mass M at the surface:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

If we assume a constant density within a sphere of radius R_p, we obtain:

$$\begin{aligned} m(R_p) &= \int_0^{R_p} 4\pi r^2 \rho \, dr \\ &= 4\pi\rho \int_0^{R_p} r^2 \, dr \\ &= \frac{4}{3}\pi\rho R_p^3 \\ &= \rho V(R_p) \end{aligned}$$

Hydrostatic pressure equation

Hydrostatic equilibrium equation for pressure P:

$$\frac{dP}{dr} = -g\rho$$

The pressure P is zero (or the atmospheric pressure) at the surface of the planet.

Assuming that in the lithosphere density and gravity are constant, then with using zero surface pressure boundary condition

$$P(r) = -\int g\rho \, dr = -g\rho r + c$$

$$0 =: P(R_p) = -g\rho R_p + c$$
$$\leftrightarrow P(r) = g\rho(R_p - r)$$

Atmospheric pressure negligible since $P_0 = 10^5 Pa$ and at depth of $3.3m$ in crust: pressure is $P(3.3m) = 10^5 Pa$ for $g = 10 m/s^2$ and $\rho = 3000 kg/m^3$.

Poisson equation

In the hydrostatic pressure equation the depth-dependent gravity is

$$g(r) = \frac{Gm}{r^2}$$

Instead of solving an additional equation for m, the radial mass dependence is included in the gravity term using gravitational constant G:

$$\begin{aligned}\frac{dg}{dr} &= \frac{d}{dr}\left(Gm(r)r^{-2}\right) \\ &= \frac{G}{r^2}\frac{dm(r)}{dr} + Gm(r)\frac{d(r^{-2})}{dr} \\ &= \frac{G}{r^2}4\pi r^2\rho - \frac{2Gm(r)}{r^3} \\ &= 4\pi G\rho - \frac{2g}{r}\end{aligned}$$

$g(r)$ is zero in the center of the planet.

How to build an interior structure model

Mass conservation:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

Hydrostatic equilibrium equation:

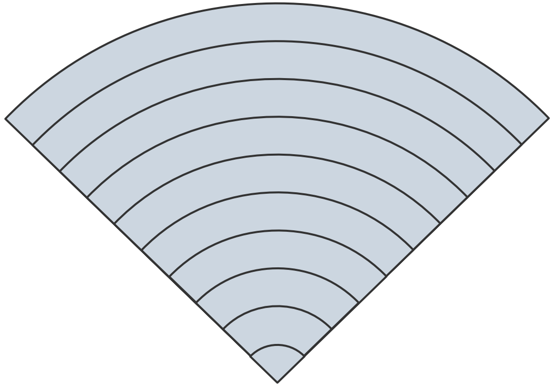
$$\frac{dP}{dr} = -g\rho$$

Poisson equation:

$$\frac{dg}{dr} = 4\pi G\rho - \frac{2g}{r}$$

Equation of state:

$$\rho(P,T); \alpha(T,P); c_P(T,P)$$



Mantle minerals

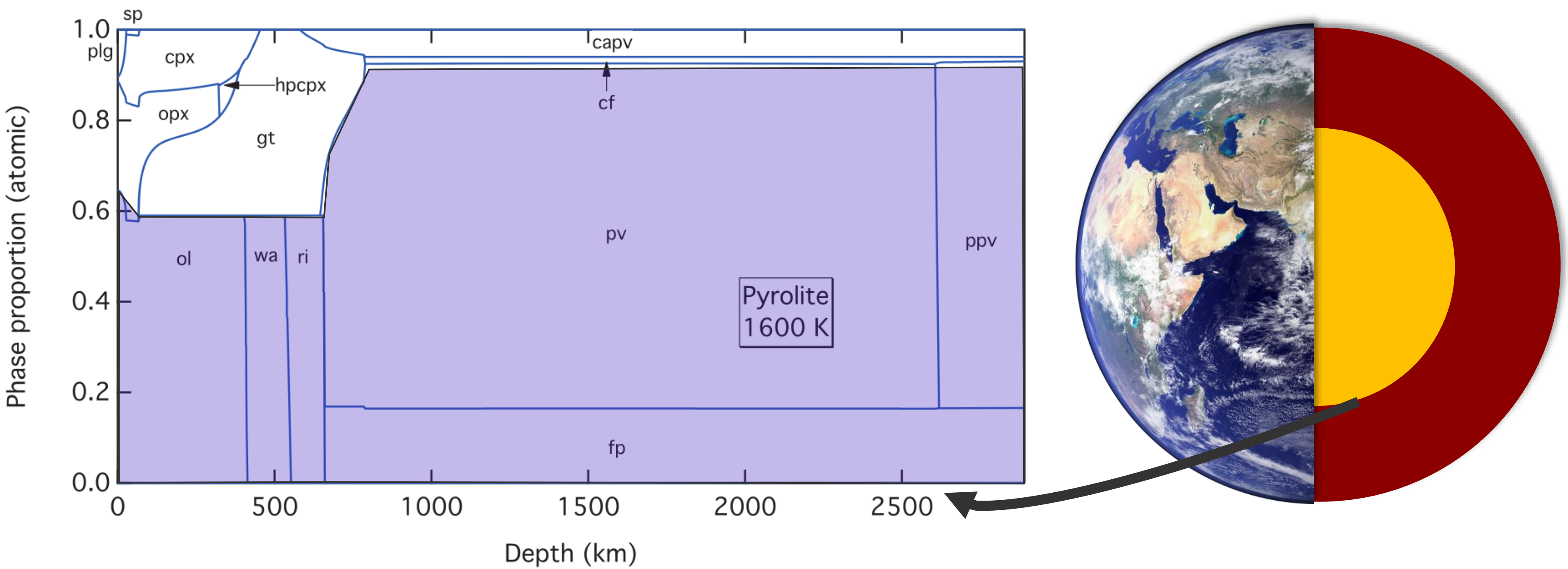


Figure: Phase stability diagram. From Stixrude and Lithgow-Bertelloni, 2011

Mantle minerals - density

- stability of the different minerals depends on temperature and pressure
- phase transitions in the mantle lead to different mineral crystals

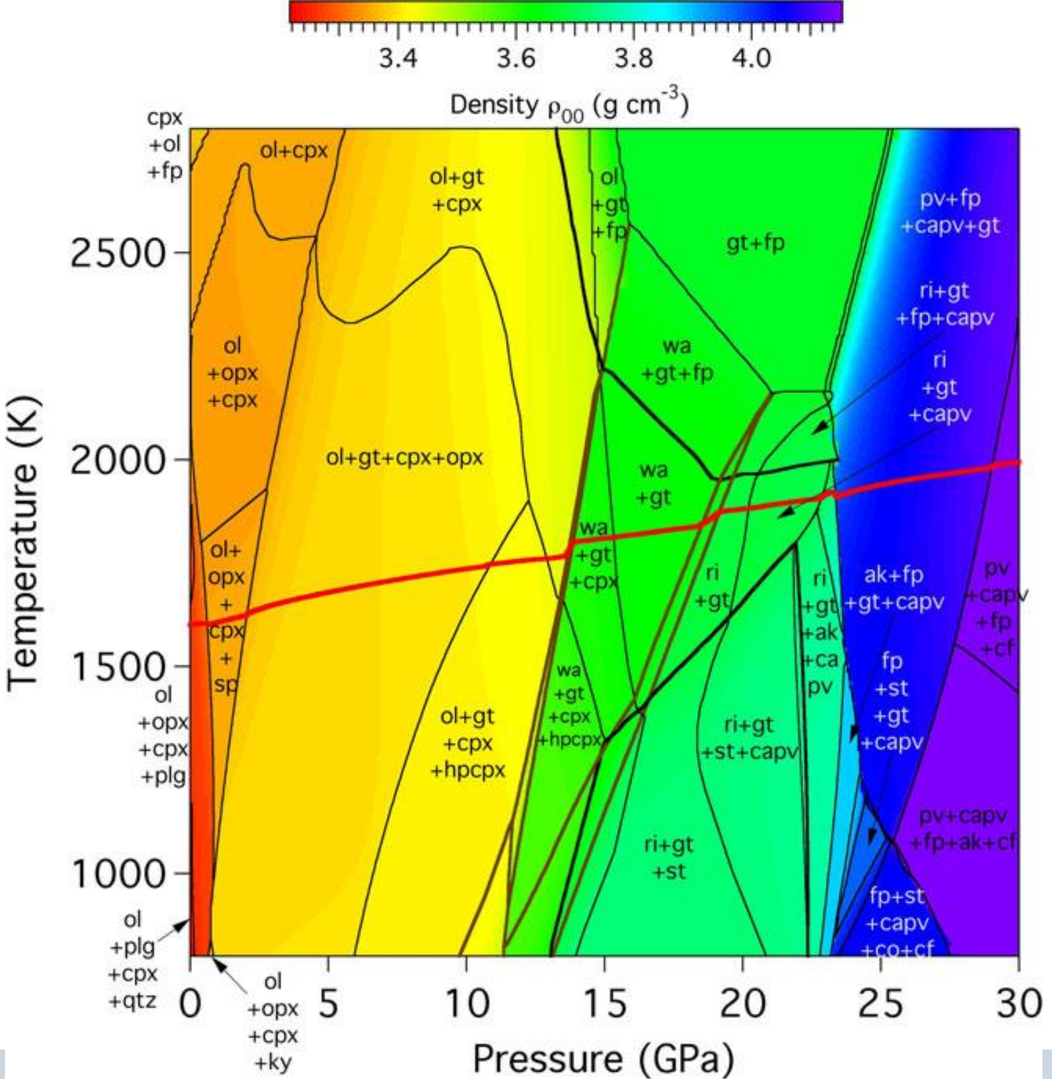
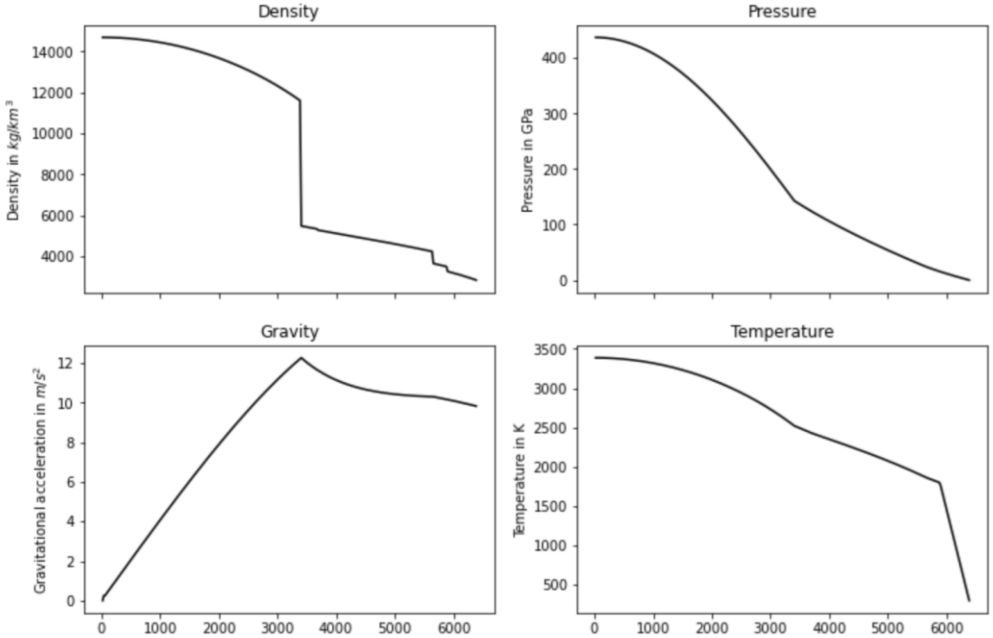


Figure: Density field for different mineral phases for an Earth-like mantle.
From Stixrude and Lithgow-Bertelloni, 2011

Exercise 1



1) Can you reproduce these profiles?
2) What changes when using a different planet mass (`M`), iron content (`x_Fe`) or upper mantle temperature (`Tm`)?

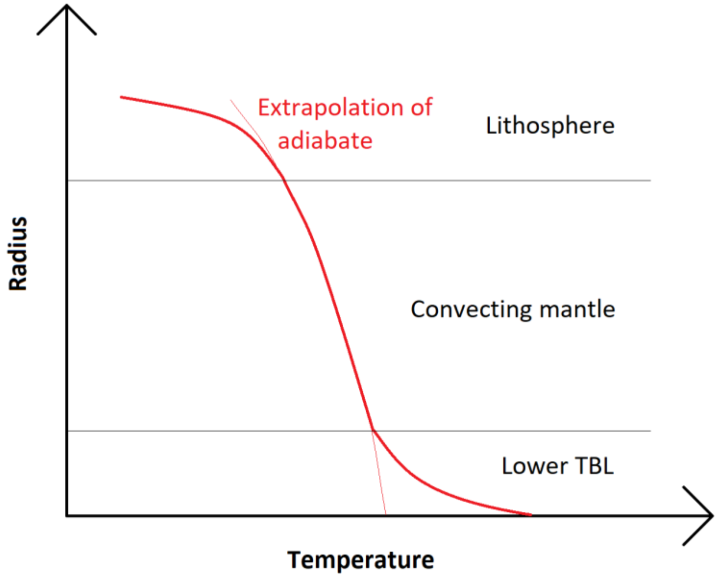
Here, we provide three python files (EOS.py, InteriorIterativeModel.py and InteriorStructure.py)

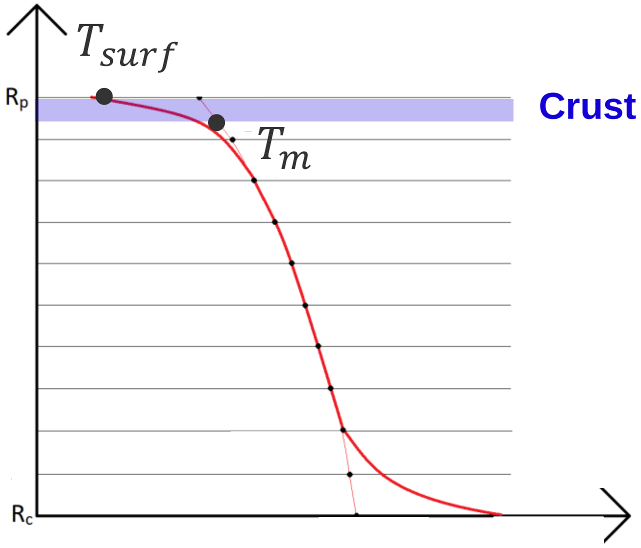
In InteriorIterativeModel we obtain properties per shell from an EOS:

```
[density,alpha,cp] = EOS_all(P[GPa],T[K],mat)
```

where `mat` is the material number of the layer (1-crust, 2-upper mantle, 3-lower mantle, 4-high pressure mantle, 5-iron core)

Interior temperature profile



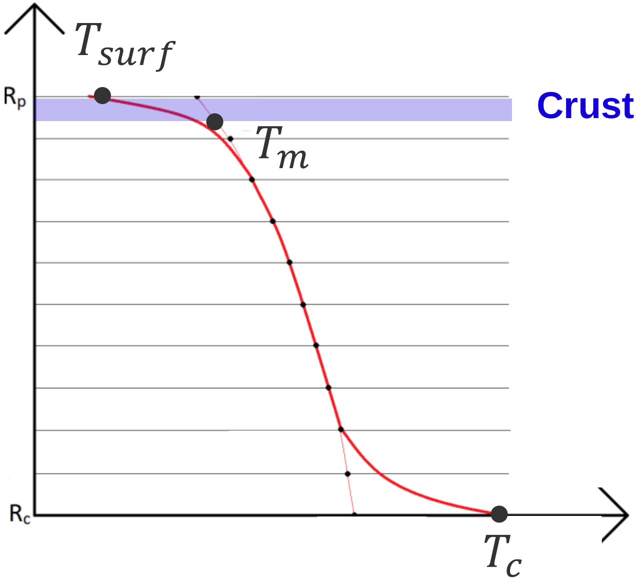


$$r_i = \{r_0, r_2, \ldots, r_n\}$$

$$\frac{\partial T}{\partial r} = -\frac{\alpha(T,P)g(P)}{c_P(T,P)}T$$

where the first n_c shells are set within the core, and n_m shells within the mantle (hence shell thickness $dr = r_{i+1} - r_i$ varies between core and mantle)

Exercise 2



$$T_c = 4370\ K \left(\frac{P[GPa]}{140}\right)^{0.48}$$

So far temperature increases adiabatically from mantle to core

More realistic: core cools slower than mantle (as seen in Earth)

→ Exercise: add temperature jump at CMB → first temperature point in core shall be set to T_c.

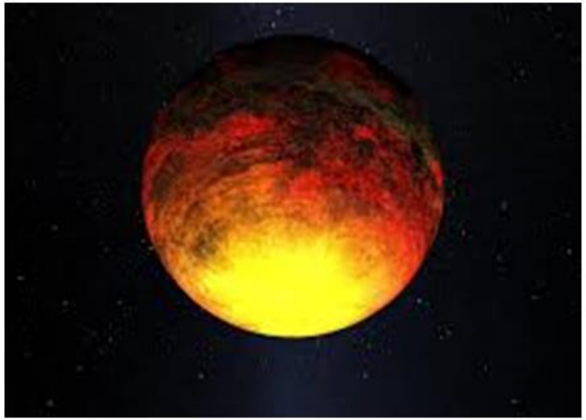
Attention: equation only valid from ~0.5 Earth mass on, need to set condition that Tc cannot be lower than last mantle value

Exercise 3

Mass and radius of several planets and moons of the solar system and beyond:

Corot-7b

Kepler 10b



Exercise 3

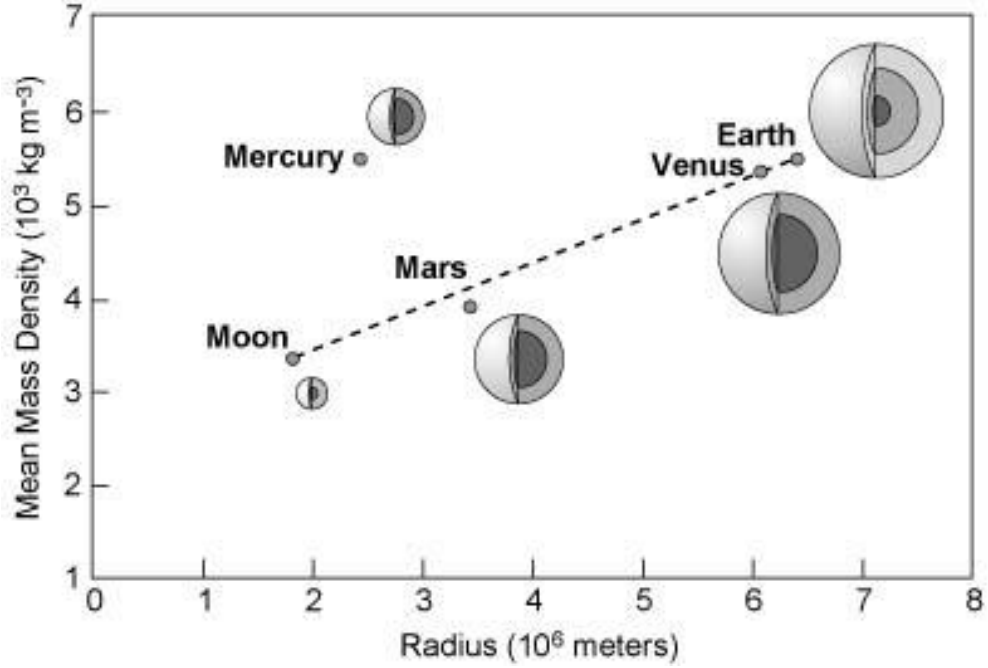
Mass and radius of several planets and moons of the solar system and beyond:

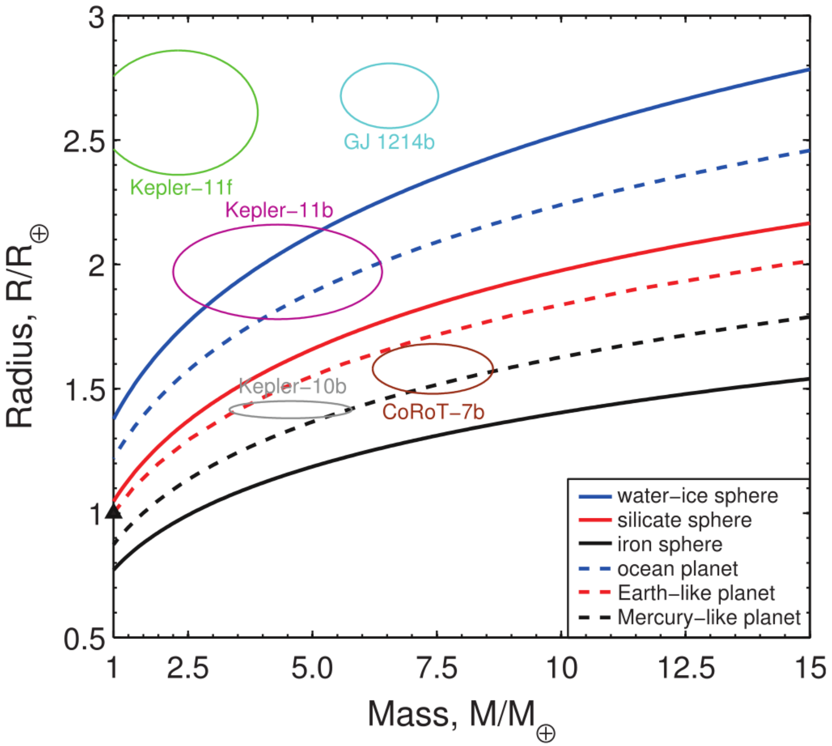
Body	**Mass [M_{Earth}]**	**Radius [R_{Earth}]**	**x_{Fe}**
Mercury	0.0553	0.383	60, 63
Earth	1	1	35
Moon	0.0123	0.273	5,7.5,10
Mars	0.1075	0.532	25,30
Callisto	0.018	0.377	2+40 / 9+45 / 20+50
CoRoT-7b	7.42	1.58	55,56,59,60
Kepler-10b	4.56	1.416	54,55,56

→ Exercise: Determine iron content that yields observed radius

Exercise 4

Mass-radius / density-radius relationship





→ Exercise: Can you generate a similar plot?

From Wagner et al. (2011)

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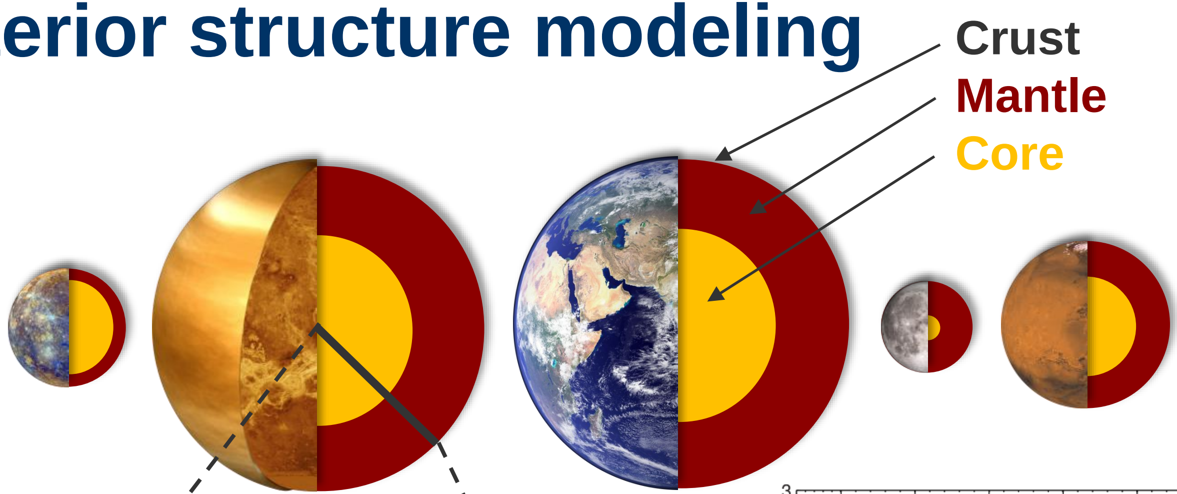

Thanks!!!