Perturbation Theory II 9.4.09

I Retrospect and working plan

Goal: Gauge-invariant perturbation variables
- clear physical and geometrical meaning
- no gauge modes (Ellis & Bruni 1989)

Tool: Harmonical analysis, which means a decomposition into eigenfunctions of the Laplacian
- Allows to treat ~~perturbation~~ different kinds of perturbations seperately.
- $V_i = V Q_i^{(s)} + V^{(v)} Q_i^{(v)}$ — depend on geometry
- $H_{ij} = H_L Q^{(s)} \gamma_{ij} + H_T Q_{ij}^{(s)} + H^{(v)} Q_{ij}^{(v)} + H^{(T)} Q_{ij}^{(T)}$

(all coefficients are functions of $\vec{k}$ and t)

How to: ① Look at first-order metric perturbations
$$h_{\mu\nu} dx^\mu dx^\nu = -2A\,dt^2 - 2B_i\,dt\,dx^i + 2H_{ij}\,dx^i dx^j$$
and work out their behaviour under gauge-transformations

Longitudinal gauge and scalar
$$h^{(s)}_{\mu\nu} dx^\mu dx^\nu = -2\Psi\,dt^2 - 2\Phi \gamma_{ij}\,dx^i dx^j$$
gauge-invariant Bardeen potentials.

② Look at first-order energy-momentum tensor perturbations.

③ Find perturbation equations using Einstein's equations

④ Try to solve them.

II Perturbations of the energy-momentum tensor

The perturbed tensor reads:

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \theta^\mu{}_\nu$$

We define energy density ρ and energy flux u:

$$T^\mu{}_\nu u^\nu = -\rho u^\mu \quad ; \quad u^2 = -1$$

and parametrise their perturbations

$$\rho = \bar{\rho}(1+\delta) \quad , \quad u = u^0 \partial_t + u^i \partial_i$$

Stress tensor

Define a projection onto the subspace of tangent space normal to u:

$$P^\mu{}_\nu \equiv u^\mu u_\nu + \delta^\mu{}_\nu$$

and the stress tensor

$$\tau^{\mu\nu} = P^\mu{}_\alpha P^\nu{}_\beta T^{\alpha\beta}$$

This allows us to write:

$$\boxed{T^\mu{}_\nu = \rho u^\mu u_\nu + \tau^\mu{}_\nu}$$

because:

$$\rho u^\mu u_\nu + \tau^\mu{}_\nu = \rho u^\mu u_\nu + P^\mu{}_\alpha P^\beta{}_\nu T^\alpha{}_\beta$$

$$= \rho u^\mu u_\nu + (u^\mu u_\alpha + \delta^\mu{}_\alpha)(u^\beta u_\nu + \delta^\beta{}_\nu) T^\alpha{}_\beta$$

$$= \rho u^\mu u_\nu + u^\mu u_\alpha u^\beta u_\nu T^\alpha{}_\beta + u^\mu u_\alpha \delta^\beta{}_\nu T^\alpha{}_\beta + u^\beta u_\nu \delta^\mu{}_\alpha T^\alpha{}_\beta$$
$$+ \delta^\mu{}_\alpha \delta^\beta{}_\nu T^\alpha{}_\beta$$

$$= \varrho u^\mu u_\nu + \varrho u^\mu u_\alpha u^\alpha u_\nu + u^\mu u_\alpha T^\alpha{}_\nu$$ (2)

$$+ u^\beta u_\nu T^\mu{}_\beta + T^\mu{}_\nu$$

$$= \varrho u^\mu u_\nu + \varrho u^\mu u_\nu - \varrho u^\mu u_\nu - \varrho u^\mu u_\nu + T^\mu{}_\nu$$

In the ~~hom~~ homogeneous case we have:

$$\tau^0{}_\mu = \tau^\mu{}_0 = 0 \quad ; \quad \tau^i{}_j = \bar{p}\,\delta^i{}_j$$

If we define velocity components:

$$u^i = \frac{1}{a} v^i = \frac{1}{a}\left(v\, Q^{(s)i} + v^{(v)} Q^{(v)i}\right)$$

we find with the metric perturbations:

$$\tau^0{}_0 = 0 \quad ; \quad \tau^j{}_0 = -\bar{p}\, v^j ; \quad \tau^0{}_j = \bar{p}\,(v_j - h_j)$$

For $\tau^i{}_j$ we have to introduce new perturbations:

$$\tau^i{}_j = \bar{p}\left[(1+\pi_L)\,\delta^i{}_j + \underbrace{\Pi^i{}_j\,] \; ; \quad \Pi^i{}_i = 0}_{\text{anisotropic stress}}$$

$$\Pi^i{}_j = \Pi\, Q^{(s)i}_j + \Pi^{(v)} Q^{(v)i}_j + \Pi^{(T)} Q^{(T)i}_j$$

Then we find:

$$T^\mu{}_\nu = \begin{pmatrix} -\bar{\varrho}(1+\delta) & (\bar{\varrho}+\bar{p})(v_j - h_j) \\ -(\bar{\varrho}+\bar{p})\,v^j & \bar{p}\left[(1+\pi_L)\,\delta^i{}_j + \Pi^i{}_j\right] \end{pmatrix}$$

Gauge trafos

while using $S^{(1)} \to S^{(1)} + L_X \bar{S}$

for $S = \bar{S} + \varepsilon S^{(1)}$ for arbitrary quantities

we find:

$$\delta \to \delta - 3(1+w)\mathcal{H} T$$
$$\pi_L \to \pi_L - 3\frac{c_s^2}{w}(1+w)\mathcal{H} T$$
$$v \to v - \dot{L}$$
$$\Pi \to \Pi$$
$$v^{(v)} \to v^{(v)} - \dot{L}^{(v)}$$
$$\Pi^{(v)} \to \Pi^{(v)}$$
$$\Pi^{(T)} \to \Pi^{(T)}$$

Out of this you can find only one gauge-invariant perturbation variable:

$$\Gamma = \pi_L - \frac{c_s^2}{w}\delta$$

To find others, we need the metric perturbations.

$$V \equiv v - \frac{1}{k}\dot{H}_T = v_{long}$$
$$D_s \equiv \delta + 3(1+w)\mathcal{H}(k^{-2}\dot{H}_T - k^{-1}B) \equiv \delta_{long}$$
$$D \equiv \delta_{long} + 3(1+w)\frac{\mathcal{H}}{k}V = \delta + 3(1+w)\frac{\mathcal{H}}{k}(v - B)$$
$$= D_s + 3(1+w)\frac{\mathcal{H}}{k}V$$
$$D_g \equiv \delta + 3(1+w)\left(H_L + \frac{1}{3}H_T\right) = \delta_{long} - 3(1+w)\phi$$
$$= D_s - 3(1+w)\phi$$

$$V^{(v)} \equiv v^{(v)} - \frac{1}{k}\dot{H}^{(v)} = v_{vec}$$
$$\Omega \equiv v^{(v)} - B^{(v)} = v_{vec} - B^{(v)}$$
$$\Omega - V^{(v)} = \sigma^{(v)}$$ (metric perturbation in vector gauge

III The perturbation equations

We identified all gauge-invariant variable and have to derive their EOM's.

Two "standard" ways:

1.) Go through the Christoffel in a certain gauge (e.g. Dodelson does it similary)

2.) Use 3+1 formulation of GR and Cartan's formalism (Durrer and Straumann 1988)

We find:

$$\underline{G_{0\mu} = 8\pi G T_{0\mu}} \quad \text{(constraints)}$$

$$4\pi G a^2 \rho D = -(k^2 - 3K)\phi \qquad \underline{\text{scalar}}$$

$$4\pi G a^2 (\rho + P) V = k(\mathcal{H}\psi + \dot{\phi}) \qquad \underline{\text{scalar}}$$

$$8\pi G a^2 (\rho + P)\Omega = \frac{1}{2}(2K - k^2)\sigma^{(v)} \qquad \underline{\text{vector}}$$

$$G_{ij} = 8\pi G T_{ij} \quad \text{(dynamical)}$$

$$k^2(\phi - \psi) = 8\pi G a^2 P \Pi^{(s)} \qquad \underline{\text{scalar}}$$

$$\ddot{\phi} + 2\mathcal{H}\dot{\phi} + \mathcal{H}\dot{\psi} + \left[2\dot{\mathcal{H}} + \mathcal{H}^2 - \frac{k^2}{3}\right]\psi \qquad \underline{\text{scalar}}$$

$$= 4\pi G a^2 \rho\left[\frac{1}{3}D + c_s^2 D_s + w\Gamma\right]$$

$$k(\dot{\sigma}^{(v)} + 2\mathcal{H}\sigma^{(v)}) = 8\pi G a^2 P \Pi^{(v)} \qquad \underline{\text{vector}}$$

$$\ddot{H}^{(T)} + 2\mathcal{H}\dot{H}^{(T)} + (2K + k^2)H^{(T)} = 8\pi G a^2 P \Pi^{(T)} \qquad \underline{\text{tensor}}$$

$$w = \frac{P}{\rho} \; ; \quad c_s = \frac{P'}{\rho'} = \frac{\dot{P}}{\dot{\rho}}$$

Some remarks:

- For a perfect fluid we have $\Pi^i_j \equiv 0$
  $\Rightarrow \phi = \psi$

- Adiabatic perturbations are characterised by
  $\Gamma = 0$

- Scalar perturbations with $\Pi = \Gamma = 0$ are described by a damped wave equation with the propagation speed c_s^2.

- $H^{(T)}$ in perfect fluids is also given by a damped wave equation, where the damping can be neglected on small scales and time periods.
  $\Rightarrow$ Gravitational wave propagation with the speed of light.

- Vector perturbations simply decay.

Energy - momentum conservation

$$T^{\mu\nu}{}_{;\nu} = 0$$

$$\dot{D}_g + 3(c_s^2 - w)\mathcal{H} D_g + (1+w) k V + 3w\mathcal{H}\Gamma = 0 \qquad \underline{\text{scalar}}$$

$$\dot{V} + \mathcal{H}(1-3c_s^2)V = k(\psi + 3c_s^2\phi) + \frac{c_s^2 k}{k^2} D_g + \frac{w k}{1+w}\left[\Gamma - \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\Pi\right] \qquad \underline{\text{scalar}}$$

$$\dot{\Omega} + (1-3c_s^2)\mathcal{H}\Omega = -\frac{w}{2(1+w)}\left(k - \frac{2K}{k}\right)\Pi^{(v)} \qquad \underline{\text{vector}}$$

Some more remarks

- Scalar perturbations: 4 independent equations
  6 perturbation variables

- Vector perturbations: 2 equations
  3 variables

- Tensor perturbations: 1 equation
  2 variables

- One way to close the system are adiabatic perturbations of the perfect fluid:
  $$\Gamma = \Pi_{ij} = 0$$

- ~~The conservat~~
- Different fluid components:
  - The conservation equations hold for each component separately
  - In the Einstein equation one has to be careful, since the metric perturbations are induced by the full perturbation:

$$\rho D_g = \rho_\alpha D_{g\alpha}$$
$$(\rho + p)v = (\rho_\alpha + p_\alpha)v_\alpha$$
$$p\Pi = p_\alpha \Pi_\alpha$$
$$p\Gamma = p_\alpha \Gamma_\alpha + \underbrace{p\Gamma_{rel}}_{\text{mixing term}}$$

The Bardeen equation

For later use we want a scalar perturbation equation for the ~~potentia~~ Bardeen potentials with Γ and Π as sources:

Recipe:

1. Take 2nd scalar Einstein equation
2. Replace D_s by D via $D = D_s + 3(1+w)\frac{\mathcal{H}}{k}V$
3. Replace D via "Poisson equation".
4. Take 1st scalar equation and differentiate
5. Replace $\dot{\Psi}$ and Ψ
6. Use $\dot{\rho} = -3(\rho + P)\mathcal{H}$, $\mathcal{H} = \frac{\dot{a}}{a}$, $\dot{\mathcal{H}} = -\frac{1+3w}{2}(\mathcal{H}^2 + K)$

This results in:

$$\ddot{\Phi} + 3\mathcal{H}(1+c_s^2)\dot{\Phi} + [3(c_s^2 - w)\mathcal{H}^2 - (2 + 3w + 3c_s^2)K + c_s^2 k^2]\Phi$$

$$= \frac{8\pi G a^2 P}{k^2}\left[\mathcal{H}\dot{\Pi} + [2\dot{\mathcal{H}} + 3\mathcal{H}^2(1 - \frac{c_s^2}{w})\Pi - \frac{1}{3}k^2\Pi + \frac{k^2}{2}\Gamma\right]$$

Applications of the Bardeen equation

We discuss adiabatic, scalar perturbations in a perfect fluid.

$$\Pi = \Gamma = 0$$

$$\Rightarrow \quad k^2(\phi - \Psi) = 8\pi G a^2 p\, \Pi^{(s)}$$

$$\Rightarrow \quad \phi = \Psi$$

with Bardeen equation:

$$\cancel{\ddot{\Psi} + 3\mathcal{H}(1+c_s^2)\dot{\Psi} + [(1+3c_s^2)(\mathcal{H}^2-K)}$$

$$\ddot{\Psi} + 3\mathcal{H}(1+c_s^2)\dot{\Psi}$$

$$+ \left[(1+3c_s^2)(\mathcal{H}^2-K) - (1+3w)(\mathcal{H}^2+K) + c_s^2 k^2\right]\Psi = 0$$

This is a damped wave equation.

Furthermore we neglect curvature and assume $w = \text{const.}$

$$\Rightarrow K = 0 \;; \quad w = c_s^2$$

$$\Rightarrow \ddot{\Psi} + 3\mathcal{H}(1+w^2)\dot{\Psi} + w k^2 \Psi = 0$$

Now we use:

$$a \propto t^{\frac{2}{1+3w}} = t^q \qquad \Rightarrow \ddot{\Psi} + 6\frac{1+w}{(1+3w)t}\dot{\Psi} + w k^2 \Psi = 0$$

$$\Rightarrow \mathcal{H} = \frac{\dot{a}}{a} = \frac{2}{1+3w}\frac{1}{t}$$

$$q = \frac{2}{1+3w}$$

This has an analytic solution:

$$\Psi = \frac{1}{a}\left(A\, j_q(\sqrt{w}\, k t) + B\, y_q(\sqrt{w}\, k t)\right)$$

with the spherical Bessel functions of order q.

We distinguish between sub and super-horizon scales:

$$\Psi = \begin{cases} \text{constant} & \text{for } \sqrt{w}\, k t \ll 1 \\ \dfrac{A}{a\sqrt{w}\, k t} \sin(\sqrt{w}\, k t - \frac{q}{2}\pi) & \text{for } \sqrt{w}\, k t \gg 1 \end{cases}$$

Assuming:
- radiation domination in the beginning
- matter domination at t_{eq}
- scale invariant initial spectrum

$\Rightarrow$ Power spectrum in later (matter-dominated) times.

$$\langle |\Psi|^2 \rangle k^3 = A_s \left(\frac{k}{H_0}\right)^{n-1} \begin{cases} 1 & \text{for } k t_{eq} < 1 \\ (k t_{eq})^{-4} \cos^2(k t_{eq}) & \text{for } k t_{eq} > 1 \end{cases}$$