# Numerics on a grid

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# Outline

1. Nonparametric galaxy cluster reconstruction
   - The physical problem
   - Reconstruction algorithm
2. Regular Grids
   - Finite differences
   - Solving differential equations
3. Adaptive grids
   - The problem
   - The implementation
4. Summary

# The Task

- Reconstruction of a galaxy cluster out of observed quantitites by using gravitational lensing
- Weak lensing gives constraints on a big field size but with bad resolution
- Strong lensing is limited to the cluster core with well observable effects
- Combine both effects to use all avaiable knowledge and to reduce their individual weaknesses
- To reconstruct the cluster you describe it by a adequate quantity

  $\Longrightarrow$ lensing potential $\psi$

  and you use the thin lens approximation which projects the cluster on a 2D plane

# The reconstruction method

- The reconstruction method is based on a least - $\chi^2$ - minimisation with respect to the lensing potential
- We define a penalty function which contains weak and strong lensing constraints

$$\chi^2(\psi) = \chi^2_w(\psi) + \chi^2_s(\psi)$$

and minimize it with respect to $\psi$

- Finally we have to connect the lensing potential to observable lensing quantities which is done by convergence and shear

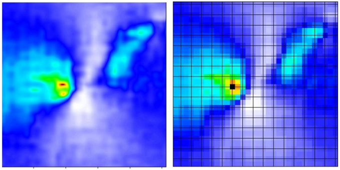
$$\kappa = \frac{1}{2}(\partial_{1,1} + \partial_{2,2})\psi, \quad \gamma_1 = \frac{1}{2}(\partial_{1,1} - \partial_{2,2})\psi, \quad \gamma_2 = \partial_{1,2}\psi$$

$$\hat{\epsilon} = \frac{\gamma}{1-\kappa} \equiv g$$

- $\Longrightarrow \chi^2_w = \frac{|\epsilon - \hat{\epsilon}(\psi)|^2}{\sigma^2_w} = \frac{|\epsilon - \frac{\gamma(\psi)}{1-\kappa(\psi)}|^2}{\sigma^2_w}, \ \chi^2_s = \frac{(det\mathcal{A}(\psi))^2}{\sigma^2_s} = \frac{[(1-\kappa(\psi))^2 - |\gamma(\psi)|^2]^2}{\sigma^2_s}$

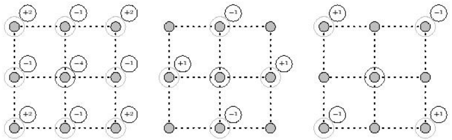
# How the grid comes into play

- We now cover the lens plane by a grid of a given resolution, this resolution is normally limited by the quality of the weak lensing data
- Instead of continous quantities we now calculate only with values on every discrete grid point
- The values of convergence and shear which are connected to the potential by second derivatives are approximated by finite differences



| | $f_{i-2}$ | $f_{i-1}$ | $f_i$ | $f_{i+1}$ | $f_{i+2}$ |
|---|---|---|---|---|---|
| $2(\Delta x)\frac{\partial f}{\partial x}$ | | - 1 | 0 | 1 | |
| $(\Delta x)^2\frac{\partial^2 f}{\partial x^2}$ | | 1 | - 2 | 1 | |
| $2(\Delta x)^3\frac{\partial^3 f}{\partial x^3}$ | -1 | 2 | 0 | -2 | 1 |
| $(\Delta x)^4\frac{\partial^4 f}{\partial x^4}$ | 1 | -4 | 6 | -4 | 1 |

| | $f_{i-3}$ | $f_{i-2}$ | $f_{i-1}$ | $f_i$ | $f_{i+1}$ | $f_{i+2}$ | $f_{i+3}$ |
|---|---|---|---|---|---|---|---|
| $12(\Delta x)\frac{\partial f}{\partial x}$ | | 1 | -8 | 0 | 8 | -1 | |
| $12(\Delta x)^2\frac{\partial^2 f}{\partial x^2}$ | | -1 | 16 | -30 | 16 | -1 | |
| $8(\Delta x)^3\frac{\partial^3 f}{\partial x^3}$ | 1 | -8 | 13 | 0 | -13 | 8 | - 1 |
| $6(\Delta x)^4\frac{\partial^4 f}{\partial x^4}$ | -1 | 12 | -39 | 56 | -39 | 12 | - 1 |



$\kappa \rightarrow \frac{1}{2}\frac{1}{6\Delta}$ $\gamma_1 \rightarrow \frac{1}{2}\frac{1}{\Delta}$ $\gamma_2 \rightarrow \frac{1}{4\Delta}$

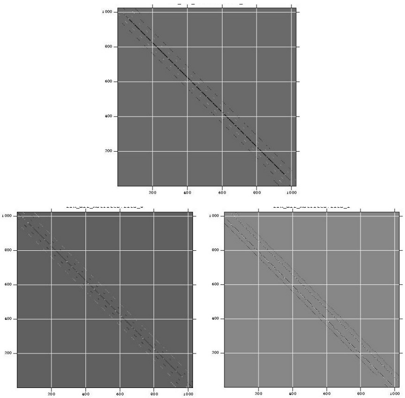
# Implementation of finite differencing

- Values of a certain quantity on the grid are now written as a 1D array $\psi_i$,$\epsilon_i$,$\kappa_i$,$\gamma_i$
- This allows us to write finite differences, means derivatives, as matrix multiplications:

$$\kappa_i = \mathcal{K}_{ij}\psi_j$$
$$\gamma_i^1 = \mathcal{G}_{ij}^1\psi_j$$
$$\gamma_i^2 = \mathcal{G}_{ij}^2\psi_j$$

- As $\chi^2$-function we now define the sum over all grid points

$$\chi^2 = \sum_{i=1}^{n} \frac{|\epsilon_i - \hat{\epsilon}_i(\psi_j)|^2}{\sigma_{wi}^2} + \sum_{i=1}^{n^*} \frac{(det\mathcal{A})_i^2}{\sigma_{si}^2}$$



# Finite Differences Matrix

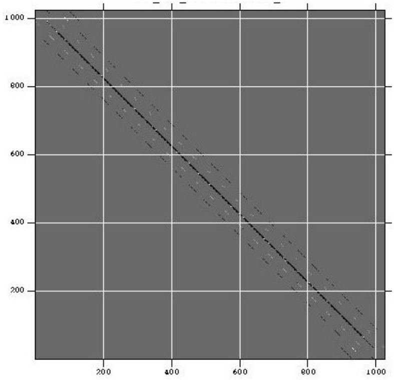



Figure: Finite differences Matrix for convergence

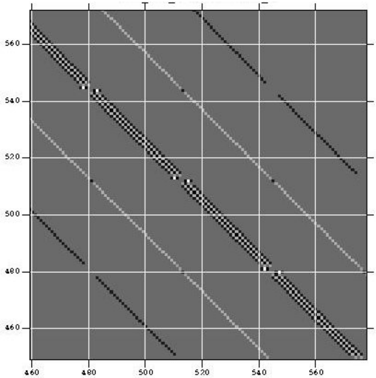



Figure: A more detailed look at the matrix

# Solving a differential equation on the grid

- What we now want to solve is: $\frac{\partial \chi^2(\psi_k)}{\partial \psi_j} = 0$
- This can now be linearized: $\frac{\partial \chi_w^2(\psi_k)}{\partial \psi_j} = 0$

$$-2 \sum_{i=1}^{n} \frac{1}{\sigma_{wi}^2} \left[ (\epsilon_i^1 - \gamma_i^1) \frac{\partial \gamma_i^1}{\partial \psi_j} + (\epsilon_i^2 - \gamma_i^2) \frac{\partial \gamma_i^2}{\partial \psi_j} \right] = 0$$

$$-2 \sum_{i=1}^{n} \frac{1}{\sigma_{wi}^2} \left[ (\epsilon_i^1 - \gamma_i^1) \frac{\partial (\mathcal{G}_{ik}^1 \psi_k)}{\partial \psi_j} + (\epsilon_i^2 - \gamma_i^2) \frac{\partial (\mathcal{G}_{ik}^2 \psi_k)}{\partial \psi_j} \right] = 0$$

$$\sum_{i=1}^{n} \frac{2}{\sigma_{wi}^2} \left[ (\mathcal{G}_{ij}^1 \mathcal{G}_{ik}^1 + \mathcal{G}_{ij}^2 \mathcal{G}_{ik}^2) \psi_k - (\epsilon_i^1 \mathcal{G}_{ij}^1 + \epsilon_i^2 \mathcal{G}_{ij}^2) \right] = 0$$

$$\mathcal{B}_{jk} \psi_k = \mathcal{V}_j$$

# More true is:

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$$\sum_{i=1}^{n} \frac{1}{(1-\kappa_i)^2\sigma_{wi}^2}[\mathcal{G}_{ij}^1\mathcal{G}_{ik}^1 + \mathcal{G}_{ij}^2\mathcal{G}_{ik}^2 + \epsilon_i^1(\mathcal{G}_{ij}^1\mathcal{K}_{ik} + \mathcal{G}_{ij}^2\mathcal{K}_{ik})$$

$$+\epsilon_i^2(\mathcal{G}_{ij}^1\mathcal{K}_{ik} + \mathcal{G}_{ij}^2\mathcal{K}_{ik}) + |\epsilon_i|^2\mathcal{K}_{ij}\mathcal{K}_{ik} + \eta\mathcal{K}_{ij}\mathcal{K}_{ik}]\psi_k$$

# More true is:

$$\sum_{i=1}^{n} \frac{1}{(1-\kappa_i)^2\sigma_{wi}^2}[\mathcal{G}_{ij}^1\mathcal{G}_{ik}^1 + \mathcal{G}_{ij}^2\mathcal{G}_{ik}^2 + \epsilon_i^1(\mathcal{G}_{ij}^1\mathcal{K}_{ik} + \mathcal{G}_{ij}^2\mathcal{K}_{ik})$$

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$$=$$

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$$=$$

$$\sum_{i=1}^{n} \frac{1}{(1-\kappa_i)^2\sigma_{wi}^2}(\epsilon_i^1\mathcal{G}_{ij}^1 + \epsilon_i^2\mathcal{G}_{ij}^2 + |\epsilon_i|^2\mathcal{K}_{ij} + \eta\kappa_i^{(n_2-1)}\mathcal{K}_{ij})$$

# More true is:

$$\sum_{i=1}^{n} \frac{1}{(1-\kappa_i)^2\sigma_{wi}^2}[\mathcal{G}_{ij}^1\mathcal{G}_{ik}^1 + \mathcal{G}_{ij}^2\mathcal{G}_{ik}^2 + \epsilon_i^1(\mathcal{G}_{ij}^1\mathcal{K}_{ik} + \mathcal{G}_{ij}^2\mathcal{K}_{ik})$$

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$$=$$

$$\sum_{i=1}^{n} \frac{1}{(1-\kappa_i)^2\sigma_{wi}^2}(\epsilon_i^1\mathcal{G}_{ij}^1 + \epsilon_i^2\mathcal{G}_{ij}^2 + |\epsilon_i|^2\mathcal{K}_{ij} + \eta\kappa_i^{(n_2-1)}\mathcal{K}_{ij})$$

$$-\sum_{i=1}^{n^*} \frac{-4det\mathcal{A}_i}{\sigma_{si}^2}((1-\kappa_i)\mathcal{K}_{ij} - \gamma_i^1\mathcal{G}_{ij}^1 - \gamma_i^2\mathcal{G}_{ij}^1)$$

# More true is:

$$\sum_{i=1}^{n} \frac{1}{(1-\kappa_i)^2\sigma_{wi}^2}[\mathcal{G}_{ij}^1\mathcal{G}_{ik}^1 + \mathcal{G}_{ij}^2\mathcal{G}_{ik}^2 + \epsilon_i^1(\mathcal{G}_{ij}^1\mathcal{K}_{ik} + \mathcal{G}_{ij}^2\mathcal{K}_{ik})$$

$$+\epsilon_i^2(\mathcal{G}_{ij}^1\mathcal{K}_{ik} + \mathcal{G}_{ij}^2\mathcal{K}_{ik}) + |\epsilon_i|^2\mathcal{K}_{ij}\mathcal{K}_{ik} + \eta\mathcal{K}_{ij}\mathcal{K}_{ik}]\psi_k$$

$$=$$

$$\sum_{i=1}^{n} \frac{1}{(1-\kappa_i)^2\sigma_{wi}^2}(\epsilon_i^1\mathcal{G}_{ij}^1 + \epsilon_i^2\mathcal{G}_{ij}^2 + |\epsilon_i|^2\mathcal{K}_{ij} + \eta\kappa_i^{(n_2-1)}\mathcal{K}_{ij})$$

$$-\sum_{i=1}^{n^*} \frac{-4det\mathcal{A}_i}{\sigma_{si}^2}((1-\kappa_i)\mathcal{K}_{ij} - \gamma_i^1\mathcal{G}_{ij}^1 - \gamma_i^2\mathcal{G}_{ij}^1)$$

The problem now is that this system is not linear in $\psi$ anymore. This is solved by an iterative process.

# Solving the linear system

- The last task is solving the linear system to obtain the solution for $\psi$
- In my code this is done by standard GSL routines (LU decomposition)
- In the end it is a little bit more complicated: iterative process, increasing grid resolution
- Especially you have to think about fast routines to build up the coefficient matrix

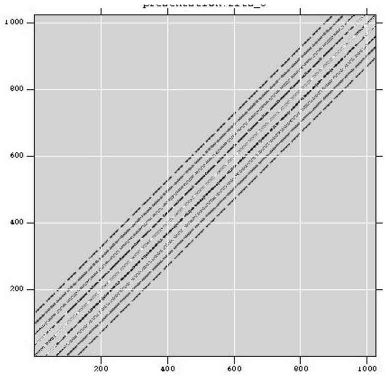
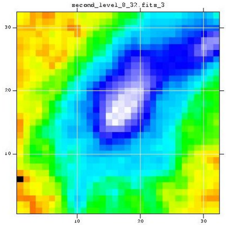
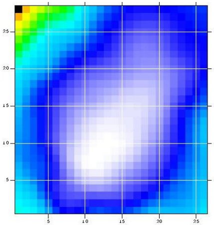


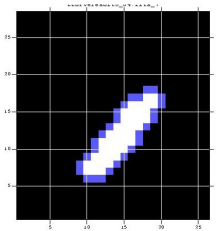

Figure: The complete coefficient matrix

# Adaptive grids: The problem

- During the cluster reconstruction you want to focus on the innermost cluster center and cut out the core
- If you do so you are not allowed to work on regular rectangular grids anymore, they become arbitrarily shaped now and things become more complicated
- Again all points of the grid are stored in a 1D array
- Problems: Positions of points for finite differencing, you have to be more careful with the borders



# Adaptive grids: The implementation

- First, a proper border is constructed to avoid the finite differencing from taking points which don't exist on that grid
- Every point of the grid needs properties on what finite differencing scheme could be used
- To take into account the more complex shape of the grid, additional information for every point needs to be stored
- These informations are: Position in original grid, type of the point, position of points up and down in the grid

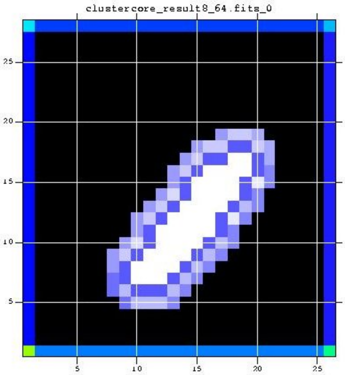



Figure: Typical type map of an adaptive grid

# Again some matrices

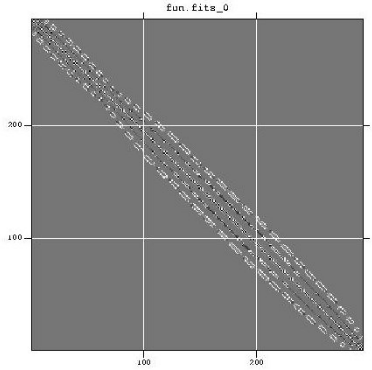



Figure: Adaptive finite differences matrix

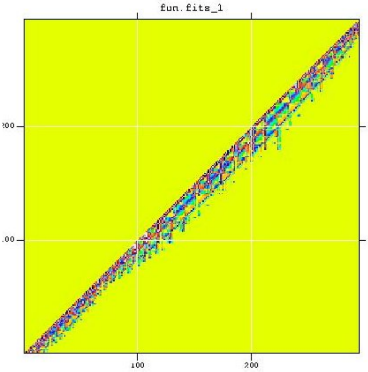



Figure: LU decomposition

# Summary

- Use of a grid with finite resolution to describe a lensing plane
- Use of finite differencing schemes to approximate derivatives
- Linearising of differential equations on our grid
- Use of adaptive grids to focus on more complicated structures