Scalar fields and vacuum fluctuations II

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Outline

1. What we did so far

2. Primordial curvature perturbations
   - gaussianity
   - spectrum/spectral index
   - beyond slow-roll

2. Gravitational waves
   - spectrum/spectral index
   - gravitational waves vs. adiabatic density perturbations

3. Isocurvature and multicomponent inflation

Some reminders:

slow-roll-inflation (Ewald)

$$H^2 \simeq \frac{V(\phi)}{3M_{pl}^2} \qquad 3H\dot{\phi} \simeq -V'(\phi)$$

$$\epsilon(\phi) \ll 1 \qquad |\eta(\phi)| \ll 1$$

$$\epsilon(\phi) = \frac{M_{pl}^2}{2}\frac{V'^2}{V} \qquad \eta(\phi) = M_{pl}^2 \frac{V''}{V}$$

curvature-perturbations (Claudia/Emanuel)

$$\mathcal{R}_k = -\left[\frac{H}{\dot{\phi}}\delta\phi_k\right]_{t=t_*}$$

$$\delta_k = \frac{4}{9}\left(\frac{k}{aH}\right)^2 \mathcal{R}_k$$

tensor and isocurvature perturbations (Christian)

$$\mathcal{P}_{grav}(k) = \frac{2}{M_{pl}^2}\left(\frac{H}{2\pi}\right)^2\bigg|_{k=aH}$$

scalar fields and vacuum fluctuations (Peter)

$$\mathcal{P}_\phi(k, t_*) = \left(\frac{H}{2\pi}\right)^2\bigg|_{k=aH}$$

Linking inflaton field and structure formation

So far we were discussing kind of separated things:

- inflaton field driving (slow-roll) inflation (comoving Hubble-Length decreases), during this period all cosmologically interesting scales leave horizon including vacuum fluctuations $\delta\phi_k(t_*)$ of the inflaton field
- This vacuum fluctuation is assumed to be gaussian and linked to a curvature perturbation $\mathcal{R}_k(t)$ which has the great advantage that it stays constant until it reenters the horizon again
- $\mathcal{R}_k$ is then taken as the primordial value and cosmological perturbation theory starts...

$$g_k(t) = T_g(t,k)\mathcal{R}_k$$

Primordial curvature perturbations: gaussianity

- inflaton field is in ground/vacuum state
- QFT says: Let's describe a quantum field by an inifinite series of harmonic oscillators
- As long as ϕ is in vacuum state an individual **k**-mode is drawn from a probability-distribution and independent from all others
- Overall fluctuation $\delta\phi_k$ is obtained by summing over an infinite number of **k**-values
- Using central limit theorem makes $\delta\phi_k$ gaussian

(Sideremark: The fact that even every inidividual **k**-mode is gaussian seems to be an accident)

Primordial curvature perturbations: spectrum

Now we are able to put all pieces together:

For the spectrum of the primordial curvature perturbation we obtain

$$\mathcal{P}_{\mathcal{R}}(k) = \left[\left(\frac{H}{\dot{\phi}} \right)^2 \mathcal{P}_{\phi}(k) \right]_{t=t_*}$$

Using the spectrum for the vacuum fluctuations of the inflaton field we get

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\phi}} \right)^2 \left(\frac{H}{2\pi} \right)^2$$

evaluated at horizon exit $k = aH$, now using slow roll conditions:

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{12\pi^2 M_{Pl}^6} \frac{V^3}{V'^2} = \frac{1}{24\pi^2 M_{Pl}^4} \frac{V}{\epsilon}$$

Matching observations

The matter density contrast is given by

$$\delta_{\mathbf{k}}(t) = \frac{2}{5}\left(\frac{k}{aH}\right)^2 T(k)\mathcal{R}_{\mathbf{k}}$$

Which gives us the spectrum

$$\mathcal{P}_\delta(k,t) = \frac{4}{25}\left(\frac{k}{aH}\right)^4 T^2(k)\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{k}{aH}\right)^4 T^2(k)\delta_H^2(k)$$

defining $\delta_H^2(k) \equiv \frac{4}{25}\mathcal{P}_{\mathcal{R}}(k)$ and δ_H as the rms value of δ at horizon entry.

$$\Rightarrow \delta_H^2(k) \simeq \frac{1}{75\pi^2 M_{pl}^6}\frac{V^3}{V'^2} = \frac{1}{150\pi^2 M_{pl}^4}\frac{V^3}{\epsilon}$$

Cobe says: $\delta_H(k_{pivot}) = 1.91x10^{-5}$ normalized on a large scale $k_{pivot} \equiv 7.5a_0H_0$

Matching observations

Assuming only contributions of adiabatic density perturbations

$$\frac{V^{3/2}}{M_{pl}^3 V'} = 5.2x10^{-4}$$

or in other words

$$\frac{V^{1/4}}{\epsilon^{1/4}} = 0.027 M_{Pl} = 6.6x10^{16} GeV$$

This relation now provides us a crucial constraint on every model of inflation with respect to the underlying inflaton potential.

Scale dependence/spectral index

Whatever the spectrum will look like wrt the dependence on k we can define an effective spectral index n(k) over an interval of k where n(k) is constant

$$n(k) - 1 \equiv \frac{dln\mathcal{P}_{\mathcal{R}}}{dlnk}$$

Assuming a power law behaviour

$$\mathcal{P}_{\mathcal{R}} \propto k^{n-1}$$

The equations for the spectrum are evaluated at k = aH and the rate of change of H is negligible compared to a

$$\Rightarrow dlnk = Hdt \quad \text{and from slow roll we get} \quad dt = -(3H/V')d\phi$$

$$\Rightarrow \frac{d}{dlnk} = -M_{pl}^2 \frac{V'}{V} \frac{d}{d\phi}$$

Conclusions for spectral index

Now we can take derivatives of the slow roll parameters,

$$\frac{d\epsilon}{dlnk} = 2\epsilon\eta - 4\epsilon^2$$

$$\frac{d\eta}{dlnk} = -2\epsilon\eta + \xi^2$$

$$\frac{d\xi^2}{dlnk} = -2\epsilon\xi^2 + \eta\xi^2 + \sigma^3$$

with

$$\xi^2 \equiv M_{pl}^4 \frac{V'(d^3V/d\phi^3)}{V^2}$$

$$\sigma^3 \equiv M_{pl}^6 \frac{V'^2(d^4V/d\phi^4)}{V^3}$$

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Consequences for spectral index

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{12\pi^2 M_{Pl}^6}\frac{V^3}{V'^2} = \frac{1}{24\pi^2 M_{Pl}^4}\frac{V}{\epsilon}$$
$$\Rightarrow n - 1 = -6\epsilon + 2\eta$$

This leads to the conlcusion
Inflation predicts that the variation of the spectrum is small in an interval $\Delta lnk \sim 1$
we can also caluclate the variation of n to be

$$\frac{dn}{dlnk} = -16\epsilon\eta + 24\epsilon^2 + 2\xi^2$$

This might be observeable by Planck and another test for inflation models

Going beyond slow roll

Simple error estimation

$$\frac{\Delta\mathcal{P}_{\mathcal{R}}}{\mathcal{P}_{\mathcal{R}}} = \mathcal{O}(\epsilon,\eta)$$
$$\Rightarrow n-1 = 2\eta - 6\epsilon + \mathcal{O}(\xi^2)$$
$$\Rightarrow \frac{dn}{dlnk} = -16\epsilon\eta + 24\epsilon^2 + 2\xi^2 + \mathcal{O}(\sigma^3)$$

It is also possible to obtain predictions by simply using linear perturbation theory and some of them are solvebale in a analytic way. The caluclations are quite lengthy and include a lot from Chapter 14, but assuming power-law inflation the result is

$$\mathcal{P}_{\mathcal{R}}^{1/2}(K) = 2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}\left(\nu-\frac{1}{2}\right)^{1/2-\nu}\left.\frac{H^2}{2\pi|\dot{H}|}\right|_{k=aH}, \quad \nu = \frac{3}{2}+\frac{\epsilon_H}{1-\epsilon_H}$$

Additional constraints

Normally impovements to the obtained spectrum are so tiny that they are not measureable with fixed parameters of a given inflation model. Until now (2000) the change can be cancelled by varying model parameters.

$\Rightarrow$ No further constraints on inflation model

Additional constraints are welcome to reduce parameter space.

$\Rightarrow$ Gravitational waves

Gravitational waves: spectrum

As we have seen the spectrum of a gravitational wave amplitudes created by fluctuations of the inflaton field are given by

$$\mathcal{P}_{grav}(k) = \frac{2}{M_{pl}^2}\left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH}$$

again we define a spectral index

$$n_{grav} = \frac{dln\mathcal{P}_{grav}}{dlnk}$$

and by applying the same formalism we obtain

$$n_{grav} = -2\epsilon$$

Gravitational waves: comparison to density perturbations

Looking at the CMB we can see two contributions to the anisotropies and their according spectra

- density perturbations

$$l(l+1)\mathcal{C}_l = \frac{\pi}{2}\left[\frac{\sqrt{\pi}}{2} l(l+1)\frac{\Gamma[(3-n)/2)]\Gamma[l+(n-1)/2]}{\Gamma[(4-n)/2)]\Gamma[l+(5-n)/2]}\right]\delta_H^2(H_0/2)$$

- gravitational waves

$$l(l+1)\mathcal{C}_l = \frac{\pi}{9}\left(1+\frac{48\pi^2}{385}\right)\mathcal{P}_{grav}c_l$$

We can now define the ratio of their contributions

$$r \equiv \frac{\mathcal{C}_l(grav)}{\mathcal{C}_l(ad)} \simeq 12.4\epsilon$$

Gravitational waves: consistency equation

We have seen that the spectra of gravitational waves and density perturbations are related

$$r = -6.2 n_{grav}$$

This equation known as consistency equation holds assuming single-field slow-roll inflation, independent of the underlying potential
The expression consistency simply states that both quantities which are generated at the same period by the same potential $V(\phi)$ must be connected.

Problems:

1. A solid signal for r will be very hard to detect (even for Planck)
2. With respect to the detection efficiency every suggested inflation model delivers a negligible contribution for gravitational waves

Isocurvature perturbations

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- So what...??
- The only physically motivated source of such an isocurvature perturbation is the scalar axion field
- Interesting for particle physics
- possible candidate for CDM
- Even the requirement that its isocurvature contribution is not bigger than the total density perturbations gives significant constraint on the axion properties

Isocurvature: the general case

- Assume a non-inflaton-scalar-field χ with a simple potential

$$V(\chi) = \frac{1}{2} m_\chi^2 \chi^2$$

- We want the field to become classical after horizon exit

$$m_\chi \lesssim H$$

- We focus also on a field which mass is small that it has negligible motion during some e-folds of inflation

$$m_\chi^2 \ll V''(\ll H^2)$$

- At horizon exit we obtain the spectrum

$$\mathcal{P}_\chi(k) = \left(\frac{H}{2\pi}\right)^2$$

Isocurvature: dark matter contribution

- If χ survived inflation it can create isocurvature perturbations by contributing with a fraction f_χ to the nonbaryonic dark matter (only case which is taken into account)
- mechanism is called misalignment: χ starts to oscillate around the minimum of potential, which happens when H falls below m_χ
- This leads to an initial isocurvature perturbation

$$S = \frac{\delta\rho_c}{\rho_c} = f_\chi \frac{\delta\rho_\chi}{\rho_\chi}$$

- With $\rho_\chi \propto (\chi + \delta\chi)^2$ and assuming $|\delta\chi| \ll |\chi|$ we get the spectrum

$$\mathcal{P}_S = \frac{4f_\chi}{\chi}\left(\frac{H}{2\pi}\right)^2 \quad \overset{obs}{\Rightarrow} \quad f_\chi \sim 1$$

- For $|\delta\chi| \gg |\chi|$ we find

$$f_\chi \lesssim 10^{-5}$$

Isocurvature: the axion

- Pseudo-Goldstone-Boson of the spontaneously broken Peccei-Quinn-symmetry
- predicted by some extensions of the standard model
- in the (very) early universe this symmetry is taken to be exact, what makes the axion massless
  between $T \sim 1GeV - 100MeV$ mass increases to its real value
- Astro .-and colliderphysics predict $m \lesssim 10^{-2}eV$
- If symmetrie is broken spontaneously at all epochs after horizon exit, the axion give an isocurvature distribution by the misalignment mechanism and axion number will be conserved
  $\Rightarrow$ Calculate Ω_{a0} with standard $(T > 1GeV)$-cosmology
- $$\Omega_{a0} \lesssim 1 \quad \Rightarrow \quad m \gtrsim 10^{-3} - 10^{-4}eV$$

Multicomponent inflation

- One-component solutions in scalar field space refer to straight line-trajectories
- Multi-component slow-roll models represent a family of curved trajectories in the space of two or more fields
- We refer to the coordinates of the inflation-trajectory in field space as components of the inflaton
- Let's assume that all components fulfil the slow roll condition

$$3H\dot{\phi}_a = -\frac{\partial V}{\partial \phi_a}$$

$$\Rightarrow \quad \epsilon_a \equiv M_{pl}^2 \left(\frac{\partial V / \partial \phi_a}{V} \right) \ll 1 \qquad |\eta_{ab}| \equiv M_{pl}^2 \frac{\partial^2 V / \partial \phi_a \partial \phi_b}{V} \ll 1$$

- Now we have to check how the primordial curvature perturbations arise

Consequences of multicomponent inflation

A formalism that I don't understand at all says
(using $\mathcal{R} = H\delta t$ proved in Chapter 14)

$$\mathcal{R} = \delta N$$
$$\Rightarrow \mathcal{R} = \frac{\partial N}{\partial \phi_a}\delta\phi_a$$

where N is the number of Hubble times measured by a comoving observer between an initial slice defining $\delta\phi$ and the final slice defining $\mathcal{R}$

Consequences

- $\delta_H^2 = \frac{V}{75\pi^2 M_{pl}^2}\frac{\partial N}{\partial\phi_a}\frac{\partial N}{\partial\phi_a}$
- $n - 1 = -\frac{M_{pl}^2 V_{,a} V_{,a}}{V^2} - \frac{2}{M_{pl} N_{,a} N_{,a}} + 2\frac{M_{pl}^2 N_{,a} N_{,b} V_{,ab}}{V N_{,d} N_{,d}}$
- The classical trajectory is not specified by the potential, but has to be given seperately

Conclusions

- The origin of primordial perturbations from vacuum fluctuations of the inflaton field justifies gaussianity and gives theoretical values for the exspected spectra
- By comparing with observations one can reduce parameter-space for the inflaton-model
- Observation of gravitational waves would reduce it even more

- By looking at a hypothetical isocurvature creation by an axion field, one can obtain a relativley small window for the axion mass
- Assuming multicomponent inflation makes things complicated