Lecture Notes in Astrophysical Fluid Dynamics

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Contents

These are the lecture notes for the course in astrophysical fluid dynamics at Heidelberg university. The course is jointly taught by Simon Glover and myself. These notes are work in progress and I will update them as the course proceeds. It is very likely that they contain many mistakes, for which I assume full responsibility. If you find any mistake, please let me know by sending an email at mattia.sormani@uni-heidelberg.de

References

In these notes we make no attempt at originality and we draw from the following sources.

[1] Acheson, D. J., *Elementary Fluid Dynamics*, (Oxford University Press, 1990)
Notes: a very clear text on fluid dynamics. Highly recommended.

[2] Landau, L.D., Lisfshitz, E.M. *Fluid Mechanics*, (Elsevier, 1987)
Notes: part of the celebrated *Course of Theoretical Physics* series. A fantastic book.

[3] Feynman, R.P., *The Feynman Lectures on Physics*
Notes: look at Chapter 40 and 41 in Vol II. The lectures are available for free on the web at the following link: http://www.feynmanlectures.caltech.edu.

[4] Shore, S.N., *Astrophysical Hydrodynamics*, (Wiley, 2007).
Notes: a very comprehensive text with lots of wonderful historical notes.

[5] Balbus, S., *Lecture Notes*,
available here
https://www.studocu.com/fr/document/ecole-normale-superieure-france/m1-hydrodynamics/hydrosun/5137654
and here
https://www.studocu.com/fr/document/ecole-normale-superieure-france/m2-magnetohydrodynamics/mhd-balbus-m2-m2-magnetohydrodynamics-balbus-ens-lra/5137644
Notes: very clear and concise.

[6] Clarke, C.J., Carswell, R.F., *Astrophysical Fluid Dynamics*, (Cambridge University Press, 2007).
Notes: a modern textbook on astrophysical fluid dynamics.

[7] Binney, J., Tremaine, S., *Galactic Dynamics*, (Princeton Series in Astrophysics, 2008)

Notes: comprehensive book that mostly deals with topics that we do not touch in this course, such as dynamics of star systems (which are *collisionless*) and disk dynamics.

[8] Zel'dovich, Ya. B. and Raizer, Yu. P., *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, (Academic Press, New York and London, 1967).
Notes: A classic on shock waves. There is an economic paperback Dover edition, first published in 2002.

[9] Kulsrud, R.M. *Plasma Physics for Astrophysics*, (Princeton, 2004).
Notes: a good book on plasma physics and magnetohydrodynamics.

[10] Pedlosky, J., *Geophysical Fluid Dynamics*, (Springer, 1987)
Notes: an excellent book if you are interested in more "terrestrial" topics.

[11] MIT Fluid Mechanics films. These are highly instructive movies available for free at http://web.mit.edu/hml/ncfmf.html. Description from the website: "In 1961, Ascher Shapiro [...] released a series of 39 videos and accompanying texts which revolutionized the teaching of fluid mechanics. MIT's iFluids program has made a number of the films from this series available on the web." Highly recommended!

[12] Purcell, E.M. and Morin, D.J., *Electricity and Magnetism*, (Cambridge University Press, 2013)
Notes: a nicely updated edition of Purcell's classic. Lots of solved problems.

[13] Jackson, J. D., *Classical electrodynamics*, (Wiley, 1999)
Notes: everything you always wanted to know (and more) about classical electrodynamics.

1 Hydrodynamics

1.1 Introductory remarks

Fluid dynamics is one of the most central branches of astrophysics. It is essential to understand star formation, galactic dynamics (what is the origin of spiral structure?), accretion discs, supernovae explosions, cosmological flows, stellar structure (what is inside the Sun?), planet atmospheres, the interstellar medium, and the list could go on.

Fluids such as water can usually be considered incompressible, because extremely high pressures of the order of thousands of atmospheres are required to achieve appreciable compressions. Air is highly compressible, but it can behave as an incompressible fluid if the flow speed is much smaller than the sound speed. Astrophysical fluids, on the other hand, must usually be treated as compressible fluids. This means that we must account for the possibility of large density changes.

Astrophysical fluids usually consists of gases that are ultimately made of particles. Although they are not *exactly* continuous fluids, most of the time they can be treated as if they are. This approximation is valid if the mean free path of a particle is small compared to the typical length over which macroscopic quantities such as the density vary. If this is the case, one can consider fluid elements that are i) large enough that are much bigger than the mean free path and contain a vast number of atoms ii) small enough that have uniquely defined values for quantities such as density, velocity, pressure, etc. Effects such as viscosity and thermal conduction are a consequence of finite mean free paths, and extra terms can be included in the equations to take them into account in the continuous approximation.

Most astrophysical fluids are magnetised. Although this can sometimes be neglected, there are many instances in which it is necessary to take explicitly into account the magnetised nature of astrophysical fluids. Therefore we shall study magnetohydrodynamics alongside hydrodynamics.

1.2 The state of a fluid

In the simplest case, the state of a fluid at a certain time is fully specified by its density $\rho(\mathbf{x})$ and velocity field $\mathbf{v}(\mathbf{x})$. In some cases it is necessary to know additional quantities, such as the pressure $P(\mathbf{x})$, the temperature $T(\mathbf{x})$, the specific entropy $s(\mathbf{x})$, or in the case of magnetised fluids the magnetic field $B(\mathbf{x})$. Multi-component fluids can have more than one species defined at each point (think for example of a plasma made of positive and negative charged particles which can move at different velocities relative to one another), each with its own density and velocity, but we will not consider this case in this course.

Equations of motion allow us to evolve in time the quantities that define the state of the fluid once we know them at some given time t_0. In other words, they allow us to uniquely determine $\rho(\mathbf{x}, t)$, $\mathbf{v}(\mathbf{x}, t)$, etc at all times once their values $\rho(\mathbf{x}, t = t_0)$, $\mathbf{v}(\mathbf{x}, t = t_0)$, etc are known for all points in space at a particular time t_0.

1.3 The continuity equation

Consider an arbitrary closed volume V that is *fixed* in space and bounded by a surface S (see Fig. 1). The mass of fluid contained in this volume is

$$M(t) = \int_V \rho(\mathbf{x}, t) \mathrm{d}V, \tag{1}$$

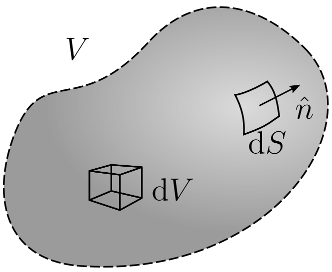

Figure 1: An arbitrary volume V.

and its rate of change with time is (can you show why is it legit to bring the time derivative inside the integral?):

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = \int_V \partial_t \rho(\mathbf{x}, t) \mathrm{d}V. \tag{2}$$

We can equate this with the mass that is instantaneously flowing out through the surface S. The mass flowing out the surface area element $\mathrm{d}\mathbf{S} = \mathrm{d}S\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a vector normal to the surface pointing outwards, is $\rho\mathbf{v} \cdot \mathrm{d}\mathbf{S}$, thus summing contributions over the whole surface we have:

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = -\oint_S \rho\mathbf{v} \cdot \mathrm{d}\mathbf{S}. \tag{3}$$

The **divergence theorem** states that for any vector-valued function $\mathbf{F}(\mathbf{x})$ (can you prove this?):

Divergence theorem

$$\boxed{\int_V \mathrm{d}V\, \boldsymbol{\nabla} \cdot \mathbf{F} = \oint_S \mathrm{d}\mathbf{S} \cdot \mathbf{F}(\mathbf{x})} \tag{4}$$

Applying the divergence theorem with $\mathbf{F} = \rho\mathbf{v}$ to the RHS of Eq. (3) and then equating the result to the RHS of Eq. (2) we obtain:

$$\int_V \partial_t \rho(\mathbf{x}, t) \mathrm{d}V = -\int_V \mathrm{d}V\, \boldsymbol{\nabla} \cdot (\rho\mathbf{v}). \tag{5}$$

Since this equation must hold for any volume V, the arguments inside the integrals must be equal at all points.[1] Hence we find:

$$\boxed{\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0} \tag{6}$$

Continuity equation

This is called **continuity equation** and expresses the conservation of mass: what is lost inside a volume is what has outflown through the surface bounding that volume.

1.4 The Euler equation, or $F = ma$

What is the fluid equivalent of Newton's second law $F = ma$? Consider a small fluid element of volume $\mathrm{d}V$ and mass $\mathrm{d}M = \rho \mathrm{d}V$. By Newton's second law,

$$\mathrm{d}M \left(\frac{D\mathbf{v}}{Dt} \right) = \text{sum of the forces acting on the fluid element}, \tag{7}$$

where the quantity $D\mathbf{v}/Dt$ is the acceleration of the fluid element. Note that this is *not* the same as $\partial_t \mathbf{v}(\mathbf{x}, t)$. The difference between the two is as follows: i) $D\mathbf{v}/Dt$ is calculated by comparing the velocities of the *same fluid element* at t and $t + \mathrm{d}t$, which occupies different spatial positions at different times ii) $\partial_t \mathbf{v}(\mathbf{x}, t)$ is calculated by comparing velocities at the *same position in space* at different times.

We can find the relation between the two types of derivatives, which holds for any property $f(\mathbf{x}, t)$ of the fluid and not just for velocities. Consider a fluid element that is initially at $\mathbf{x}(t)$ (see Fig. 2). Its velocity is $\mathbf{v}(\mathbf{x}, t)$ and therefore after a time $\mathrm{d}t$ its new position will be $\mathbf{x}(t+\mathrm{d}t) \simeq \mathbf{x}(t) + \mathbf{v}\mathrm{d}t$. To take the derivative following the fluid element, we must compare f at the new position at time $t + \mathrm{d}t$ with f at the old position at time t:

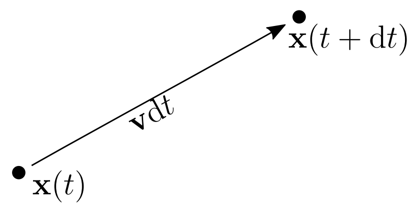

Figure 2: The convective derivative.

$$\begin{aligned}
\frac{Df}{Dt} &\equiv \lim_{\mathrm{d}t \to 0} \frac{f\left(\mathbf{x}(t+\mathrm{d}t), t+\mathrm{d}t\right) - f\left(\mathbf{x}(t), t\right)}{\mathrm{d}t} && (8) \\
&\simeq \frac{f\left(\mathbf{x}(t) + \mathbf{v}\mathrm{d}t, t+\mathrm{d}t\right) - f\left(\mathbf{x}(t), t\right)}{\mathrm{d}t} && (9) \\
&\simeq \frac{\partial_t f \mathrm{d}t + (\boldsymbol{\nabla} f) \cdot (\mathbf{v}\mathrm{d}t)}{\mathrm{d}t} && (10) \\
&= \partial_t f + \mathbf{v} \cdot \boldsymbol{\nabla} f && (11)
\end{aligned}$$

[1] Suppose there is a point in which they are not equal. Then one could just integrate in the neighbourhood of that point, contradicting the result (5). Hence they must be equal.

Thus

Convective derivative

$$\boxed{\frac{D}{Dt} = \partial_t + \mathbf{v} \cdot \boldsymbol{\nabla}} \tag{12}$$

D/Dt is called the **Lagrangian** or **convective derivative**, to distinguish it from the **Eulerian derivative** ∂_t (see also Section 1.9).

Now that we have discussed the LHS of Eq. (7), let us deal with the RHS. What are the forces that act on a fluid element? The most fundamental force acting on a fluid is pressure.

Consider a hypothetical plane slicing through a *static* ($\mathbf{v} = 0$) fluid with an arbitrary orientation. What is the (vector) force that material on one side of that plane exerts on material on the other side? In the most general case the force depends on the orientation of the plane and the force itself can have any direction (different from the orientation of the plane). The thing relating the force to the orientation is in general a second-rank tensor, because it relates a vector to a vector, i.e. it is a 3×3 array of numbers.

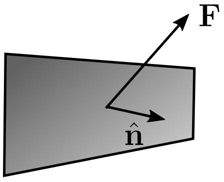

Figure 3: A hypothetical plane slicing through the fluid. $\hat{\mathbf{n}}$ is the normal to the plane. In the general case, the force $\mathbf{F}$ that the fluid on one side exerts on the fluid on the other side can have any direction. In this course, we only consider cases in which $\hat{\mathbf{n}}$ and $\mathbf{F}$ have the same direction, and $\mathbf{F}$ does not depend on the orientation of the plane.

Fortunately, in this course we only consider forces that in such a static situation can always be considered isotropic, i.e. they do not depend on the orientation of $\hat{\mathbf{n}}$ (can you think of an example where this is not true?), and are directed perpendicularly to the surface at each point, i.e. they are directed along $\hat{\mathbf{n}}$ (this is usually not a good approximation in solids. Think of a twisted rubber: it is clearly not true inside it!). The force can then be quantified by a single number called pressure, which is a function of position and time, $P = P(\mathbf{x}, t)$. The pressure force acting on a surface area $\mathrm{d}\mathbf{S}$ is $P\mathrm{d}\mathbf{S}$.

A pressure exerts a net force on a fluid element only if it is not spatially uniform, otherwise the force on opposite sides cancels out. The pressure force acting on a fluid element is

$$-\boldsymbol{\nabla} P \mathrm{d}V. \tag{13}$$

You can show this by considering the forces on the side of a small cube of volume $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

If the fluid is not static, we assume that the pressure force is the same. If two layers of fluid are moving relative to each other, viscous forces can also be present. In contrast to pressure, these are not directed perpendicularly to our hypothetical plane, and will be the subject of Section 1.13.

Now let's put eveything together. First, substitute Eq. (12) into the LHS of (7). If the only force acting on the fluid is pressure, the RHS is simply given by (13). After dividing by $\mathrm{d}M$ and using that $\mathrm{d}M = \rho \mathrm{d}V$ we find:

$$\boxed{\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho}} \tag{14}$$

Euler equation

This is the **Euler equation**. This was derived assuming that pressure is the only force. Other forces can be added as needed. One of obvious importance in astrophysics is gravity. In presence of a gravitational field Φ the force per unit mass acting on a fluid element is $-\boldsymbol{\nabla}\Phi$, therefore the Euler equation becomes in this case

$$\boxed{\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla}\Phi.} \tag{15}$$

Euler equation with gravity.

If the field is externally imposed then Φ is a given function of $\mathbf{x}$ and t; if on the contrary the field is generated by the fluid itself, it must be computed self-consistently.

Other forces of astrophysical interest can arise because of the effects of magnetic fields and viscosity. These will be considered below.

1.5 The choice of the equation of state

Consider the continuity equation (6) and the Euler equation (14). These two equations alone are not enough to evolve the system in time. In other words, if one is given $\rho_0(\mathbf{x}) = \rho(\mathbf{x}, t = t_0)$ and $\mathbf{v}_0(\mathbf{x}) = \mathbf{v}(\mathbf{x}, t = t_0)$ at $t = t_0$, they are not enough to determine uniquely what they are at at $t > t_0$. There are many possible solutions $\rho(\mathbf{x}, t)$, $\mathbf{v}(\mathbf{x}, t = t_0)$ that satisfy them, and one does not know which one to choose. People say that the continuity and Euler equations do not form a **complete system** of differential equations, so we need one more.

From the mathematical point of view, this can be understood because we have three unknowns functions $(\rho, \mathbf{v}, P)$ and only two equations (6 and 14).[2] From the physical point of view, this can be understood if we consider that to find the time evolution of a fluid element we need to know the forces acting on it, but so far we have said nothing on how to determine P!

It is common to relate pressure and density through an **equation of state**. For most astrophysical applications, it is usually a very good

[2] We have five unknowns and four equations if $\mathbf{v}$ is considered as three scalar functions v_x, v_y, v_z. The Euler equation is a vector equation and counts as three scalar equations.

approximation to consider the equation of state of an ideal gas

Ideal gas

$$P = \frac{\rho k T}{\mu}, \tag{16}$$

where T is the temperature, $k = 1.38 \times 10^{-23}\mathrm{JK}^{-1}$ is the Boltzmann constant, μ is the mass per particle.

The addition of Eq. (16) does not complete our system of equations, because we have introduced a new equation but also a new unknown, the temperature $T(\mathbf{x}, t)$. We have just exchanged one unknown quantity (P) for another (T). The following are common ways of closing the system of equations of astrophysical importance:

- Assume that $T =$ constant in Eq. 16. This is called an **isothermal gas**. In this case P is proportional to ρ. Note that the quantity kT/μ has dimensions of a velocity squared and Eq. (16) can be rewritten as

  Isothermal gas

  $$P = c_{\mathrm{s}}^2 \rho \tag{17}$$

  where

  $$c_{\mathrm{s}}^2 = kT/\mu = \text{constant}. \tag{18}$$

  c_{s} is called the isothermal sound speed, for reasons that will become clear later in the course (see Section 5.2).

- In general, if we follow a fluid element, it changes shape and volume during its motion. It can therefore perform work by expansion or receive work by compression from its surroundings. An **adiabatic** fluid is one in which this work is converted into internal energy of the fluid in a reversible manner and there are no transfers of heat or matter between a fluid element and its surroundings. The temperature of each fluid element is allowed to change, but only as a result of compression and expansion. We will see later how to consider processes able to add or subtract heat from a fluid element, such as exchanges of heat due to thermal conduction between neighbouring fluid elements, viscous dissipation, and extra heating and cooling due to radiative processes.

  We can find the equations governing an adiabatic gas as follows. The internal energy *per unit mass* of an ideal gas is

  Internal energy per unit mass of an ideal gas

  $$\boxed{\mathcal{U} = \frac{P}{\rho(\gamma - 1)}} \tag{19}$$

where $\gamma = 1 + 2/N$ is the adiabatic index and N is the number of degrees of freedom per particle ($N = 3$ for a monoatomic gas, $N = 5$ for a diatomic gas). Note that an isothermal gas corresponds to the limit $N \to \infty$ in which the number of degrees of freedom goes to infinity. In this case, the internal degrees of freedom act like a heat bath that keeps the temperature constant. This is also why $\mathcal{U} \to \infty$ as $\gamma \to 1$.

Consider again a small fluid element of volume $\mathrm{d}V$, mass $\mathrm{d}M = \rho \mathrm{d}V$ and internal energy is $\mathrm{d}M\mathcal{U}$. As it moves through the fluid, this parcel of gas changes volume, doing work by expansion and exchanging heat with the surroundings, just like the ideal gas systems you studied in your first thermodynamics course. The first law of thermodynamics says that

$$D\mathcal{U} = D\mathcal{Q} - D\mathcal{W} \tag{20}$$

where $D\mathcal{U}$ is the change in internal energy, $D\mathcal{Q}$ is the heat added to the system and $D\mathcal{W}$ is the work done *by* the system (all per unit mass). We use the big D to emphasise that these are changes tracked following the fluid. Differentiating (19) yields $D\mathcal{U} = (DP)/[\rho(\gamma-1)] - (D\rho)P/[\rho^2(\gamma-1)]$. The expansion work done by the fluid element due to its volume change is $D\mathcal{W} = PDv = -(D\rho)/\rho^2$, where $v \equiv 1/\rho$ is the specific volume, i.e. the volume per unit mass. In an adiabatic fluid, by definition, $D\mathcal{Q} = 0$. Hence for an adiabatic fluid the first law becomes

$$\frac{1}{\rho(\gamma-1)}DP - \frac{P}{\rho^2(\gamma-1)}D\rho = \frac{P}{\rho^2}D\rho \tag{21}$$

rearranging and dividing by Dt we find

$$\boxed{\frac{D\log\left(P\rho^{-\gamma}\right)}{Dt} = 0} \tag{22}$$

Adiabatic ideal fluid

This last equation together with equations (6) and (14) are the equations of motion of an adiabatic gas. They are a complete system of equations that fully specifies the time evolution given the state of the fluid at $t = 0$. Note that this implies that the **entropy per unit mass** of a fluid element, which up to an unimportant additive constant is given by

$$s = \frac{k}{\mu(\gamma-1)}\log P\rho^{-\gamma} \tag{23}$$

does not change in adiabatic flow, i.e. the following equation is valid:

$$\frac{Ds}{Dt} = 0. \tag{24}$$

Note also that this equation implies that s is a constant for a *particular fluid element*. It does not preclude different fluid elements having different values of s, it just implies that each such element will retain whatever value of s it started with. For example, a medium which is initially isothermal with a non-uniform density, such as an isothermal sphere (see Section 3.2), has s which is initially not uniform in space, hence one must be careful in not confusing $\partial_t \log(P\rho^{-\gamma})$ with $D\log(P\rho^{-\gamma})/Dt$. The particular case in which s is the same for all fluid elements is called **isentropic** fluid. In an isentropic fluid the pressure is related through density by $P = K\rho^\gamma$ where K is a constant which is the same for all fluid elements and therefore is a particular case of barotropic fluid (see below), and in this case $\partial_t s = Ds/Dt = 0$. But in the general case of adiabatic flow, K is not a constant and may be different for different fluid elements.

- An **incompressible** fluid is one in which fluid elements do not change in volume as they move. A liquid like water can usually be considered incompressible, because extremely high density are required to achieve significant compressions. Air is highly compressible, but it can behave as an incompressible fluid if the flow speed is much smaller than the sound speed. For an incompressible fluid the density of a fluid element is constant,

$$\frac{D\rho}{Dt} = 0. \tag{25}$$

Since the continuity equation (6) can be rewritten as

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \tag{26}$$

it implies that for an incompressible fluid

Incompressible fluid

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{v} = 0} \tag{27}$$

Indeed, it can be shown (see problem 2) that the Lagrangian derivative of a volume element is

$$\frac{D(\mathrm{d}V)}{Dt} = (\mathrm{d}V)\boldsymbol{\nabla}\cdot\mathbf{v} \tag{28}$$

This means that in a time dt the volume of a fluid element changes as $\mathrm{d}V \to \mathrm{d}V(1 + (\boldsymbol{\nabla} \cdot \mathbf{v})\mathrm{d}t)$, and $\boldsymbol{\nabla} \cdot \mathbf{v}$ is its rate of change. Hence, $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ means that the volume does not change, i.e. the fluid is incompressible.

Equation (27) together with equations (6) and (14) form a complete system of equations. They are the equations of motion of an incompressible fluid. The density is usually a given constant, so we do not need to solve for it, while the pressure P simply assumes the value needed to sustain the flow, much like the normal reaction of a frictionless surface is as high as it needs to be to sustain an objects moving over it. Note that according to Eq. (25) the density does not *need* to be the same everywhere, i.e. all fluid elements, although this is often the case.

- An important case in astrophysics is that of a **barotropic** equation of state, in which the pressure is a function of density only:

$$P = P(\rho). \tag{29}$$

Barotropic fluid

A subclass of barotropics model often used in astrophysics is **polytropic** models, in which $P = K\rho^{(n+1)/n}$, where K is a constant and the constant n is the polytropic index. Examples of polytropic fluids are an isothermal gas ($n = \infty$), an isentropic gas ($\gamma = (n+1)/n$), and a degenerate gas of electrons ($n = 3/2$ for the non relativistic case, $n = 3$ for the relativistic case).

1.6 Manipulating the fluid equations

When dealing with the fluid equations there are several tricks and techniques that can make your calculation faster and your life easier.

1.6.1 Writing the equations in different coordinate systems

Suppose that you want to write down the continuity and Euler equations in a cylindrical or spherical coordinate system. There are several possible ways to approach this problem. You could calculate them by brute force by writing down the new variables as a function of the old ones, calculating the old derivatives as a function of the new ones using the chain rule and then substitute them in the Cartesian equations, but this is the lengthy route. A least action route is to look up on Wikipedia the expressions of the various differential operators in your coordinate system.[3]

The most instructive route, which allows you to derive most expression without the need to look them up, is probably as follows:

1. Start from expressions in vector form, which are valid in all coordinate systems.

2. Manipulate these expressions by expanding vectors using unit vectors in the coordinate system of interest. This requires knowledge of the unit vectors derivatives and how to write down the gradient operator, but these are generally easy to derive or remember.

For example, let us find $\boldsymbol{\nabla} \cdot \mathbf{v}$ in cylindrical coordinates (R, ϕ, z). We write

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_R \frac{\partial}{\partial R} + \hat{\mathbf{e}}_\phi \frac{1}{R} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{30}$$

and

$$\mathbf{v} = \hat{\mathbf{e}}_R v_R + \hat{\mathbf{e}}_\phi v_\phi + \hat{\mathbf{e}}_z v_z \tag{31}$$

where v_R is the R component of velocity, and so on. Putting these together we write the divergence as:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \left(\hat{\mathbf{e}}_R \frac{\partial}{\partial R} + \hat{\mathbf{e}}_\phi \frac{1}{R} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \cdot \left(\hat{\mathbf{e}}_R v_R + \hat{\mathbf{e}}_\phi v_\phi + \hat{\mathbf{e}}_z v_z \right) \tag{32}$$

Note that *the derivatives must operate on the unit vectors*! In cartesian coordinates the unit vectors are constant so this does not matter, but in other coordinate systems the unit vectors generally change with position. Using the relations in Section 1.6.3 we find:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{v} &= \left(\hat{\mathbf{e}}_R \frac{\partial}{\partial R} + \hat{\mathbf{e}}_\phi \frac{1}{R} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \right) \cdot \left(\hat{\mathbf{e}}_R v_R + \hat{\mathbf{e}}_\phi v_\phi + \hat{\mathbf{e}}_z v_z \right) && (33) \\ &= \frac{\partial v_R}{\partial R} + \frac{1}{R} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} + \frac{v_R}{R} && (34) \end{aligned}$$

where the last term originates because of the nonvanishing derivatives of the unit vectors.

As another example, let us find $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ in cylindrical coordinates. The radial component of this vector is *not* $(\mathbf{v} \cdot \boldsymbol{\nabla}) v_R$, because again we

[3] https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

have to take the derivatives of the unit vectors. Instead we should write:

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \left[(\hat{\mathbf{e}}_R v_R + \hat{\mathbf{e}}_\phi v_\phi + \hat{\mathbf{e}}_z v_z)\cdot\left(\hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \hat{\mathbf{e}}_\phi\frac{1}{R}\frac{\partial}{\partial\phi} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}\right)\right] \tag{35}$$

$$(\hat{\mathbf{e}}_R v_R + \hat{\mathbf{e}}_\phi v_\phi + \hat{\mathbf{e}}_z v_z) \tag{36}$$

and take all the derivatives and the scalar products according to relations in Section 1.6.3. The result should be (exercise)

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \left(v_R\frac{\partial v_R}{\partial\rho} + \frac{v_\phi}{\rho}\frac{\partial v_R}{\partial\phi} + v_z\frac{\partial v_R}{\partial z} - \frac{v_\phi v_\phi}{\rho}\right)\hat{\mathbf{e}}_R \tag{37}$$

$$+\left(v_R\frac{\partial v_\phi}{\partial\rho} + \frac{v_\phi}{\rho}\frac{\partial v_\phi}{\partial\phi} + v_z\frac{\partial v_\phi}{\partial z} + \frac{v_\phi v_R}{\rho}\right)\hat{\mathbf{e}}_\phi \tag{38}$$

$$+\left(v_R\frac{\partial v_z}{\partial\rho} + \frac{v_\phi}{\rho}\frac{\partial v_z}{\partial\phi} + v_z\frac{\partial v_z}{\partial z}\right)\hat{\mathbf{e}}_z \tag{39}$$

1.6.2 Indecent indices

Vector notation is nice because it is coordinate independent, thus the same expression is valid in Cartesian, Spherical or Cylindrical coordinates. However it is often useful to switch to index notation in calculations, for example in proving vector identities. The index i,j and k here will represent any of the Cartesian coordinates x, y and z. For example, v_i is the ith component of $\mathbf{v}$, which may be any of the three (x, y or z). The gradient operator $\boldsymbol{\nabla}$ is written ∂_i.

We also use the Einstein **summation convention**: when an index variable appears twice in a single term it implies summation of that term over all the values of the index (unless otherwise specified).

For example, the dot product between two vectors $\mathbf{A}$ and $\mathbf{B}$ is written

$$\mathbf{A}\cdot\mathbf{B} = A_i B_i. \tag{40}$$

The term "v dot grad v" is written

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = (v_i\partial_i)v_j, \tag{41}$$

where the term on the right represents the component in the direction $j = x, y$ or z. The continuity equation (6) is rewritten in index notation as follows:

$$\partial_t\rho + \partial_i(\rho v_i) = 0, \tag{42}$$

and the Euler equation (14):

$$\partial_t v_j + (v_i\partial_i)v_j = -\frac{\partial_j P}{\rho}. \tag{43}$$

Note that these are three scalar equations, one for each possible value of $j = x, y$ or z.

The cross product and the curl can be written using the Levi-Civita symbol ε_{ijk}. This is defined as

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),(2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),(1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if any two indices are equal to one another} \end{cases} \tag{44}$$

Then we can write for example

$$\boldsymbol{\nabla} \times \mathbf{A} = \varepsilon_{ijk} \partial_i A_j, \tag{45}$$

and

$$\mathbf{A} \times \mathbf{B} = \varepsilon_{ijk} A_i B_j. \tag{46}$$

In expressions involving two cross products we have two ε_{ijk} symbols. In those cases, the following identity is very useful:

$$\varepsilon_{ijk} \varepsilon^{imn} = \delta_j{}^m \delta_k{}^n - \delta_j{}^n \delta_k{}^m \tag{47}$$

Here the position of the index (subscript or superscript) has no meaning and different arrangements are only used for clarity. Such distinction is instead important general relativity, but we do not discuss that here.

As an exercise, you can use (47) to prove the following identities:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\, \mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\, \mathbf{C} \tag{48}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{49}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{50}$$

1.6.3 Tables of unit vectors and their derivatives

Cylindrical unit vectors:

$$\hat{\mathbf{e}}_R = (\cos\phi, \sin\phi, 0) \tag{51}$$

$$\hat{\mathbf{e}}_\phi = (-\sin\phi, \cos\phi, 0) \tag{52}$$

$$\hat{\mathbf{e}}_z = (0, 0, 1) \tag{53}$$

Spherical unit vectors:

$$\hat{\mathbf{e}}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \sin\theta \hat{\mathbf{e}}_R + \cos\theta \hat{\mathbf{e}}_z \tag{54}$$

$$\hat{\mathbf{e}}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta) = \cos\theta \hat{\mathbf{e}}_R - \sin\theta \hat{\mathbf{e}}_z \tag{55}$$

$$\hat{\mathbf{e}}_\phi = (\sin\phi, \cos\phi, 0) \tag{56}$$

Nonvanishing derivatives of cylindrical unit vectors

$$\partial_\phi(\hat{\mathbf{e}}_R) = \hat{\mathbf{e}}_\phi \tag{57}$$
$$\partial_\phi(\hat{\mathbf{e}}_\phi) = -\hat{\mathbf{e}}_R \tag{58}$$

Nonvanishing derivatives of spherical unit vectors

$$\partial_\theta(\hat{\mathbf{e}}_r) = \hat{\mathbf{e}}_\theta \tag{59}$$
$$\partial_\phi(\hat{\mathbf{e}}_r) = \sin\theta\hat{\mathbf{e}}_\phi \tag{60}$$
$$\partial_\theta(\hat{\mathbf{e}}_\theta) = -\hat{\mathbf{e}}_r \tag{61}$$
$$\partial_\phi(\hat{\mathbf{e}}_\theta) = \cos\theta\hat{\mathbf{e}}_\phi \tag{62}$$
$$\partial_\phi(\hat{\mathbf{e}}_\phi) = -(\sin\theta\hat{\mathbf{e}}_r + \cos\theta\hat{\mathbf{e}}_\theta) = -\hat{\mathbf{e}}_R \tag{63}$$

More comprehensive list of properties can be found for example at:
http://mathworld.wolfram.com/CylindricalCoordinates.html
http://mathworld.wolfram.com/SphericalCoordinates.html

1.7 Conservation of energy

A useful equation related to the conservation of kinetic energy can be obtained by taking the dot product of $\mathbf{v}$ with the Euler equation (14). The LHS becomes

$$\mathbf{v}\cdot[\partial_t\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}] = \frac{1}{2}\left[\partial_t\mathbf{v}^2 + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}^2\right] \tag{64}$$
$$= \frac{1}{2}\frac{D\mathbf{v}^2}{Dt} \tag{65}$$

The first step can be proved easily by switching to index notation (see Section 1.6.2). Putting this back together with the RHS we find

$$\frac{1}{2}\frac{D\mathbf{v}^2}{Dt} = \mathbf{v}\cdot\left[-\frac{\boldsymbol{\nabla}P}{\rho}\right] \tag{66}$$

The interpretation of this equation is simple. Consider a small fluid element of mass $\mathrm{d}M$. Its kinetic energy is $\mathrm{d}M\mathbf{v}^2/2$. Then this equation simply states that the change in kinetic energy of a fluid element is the dot product between the force, $-\mathrm{d}M\boldsymbol{\nabla}P/\rho$, and the velocity, i.e. it is the fluid mechanics equivalent of the familiar Newtonian mechanics statement that $\mathrm{d}(m\mathbf{v}^2/2)/\mathrm{d}t = \mathbf{F}\cdot\mathbf{v}$.

We can rewrite (66) in another useful way. After multiplying both sides by ρ, the LHS can be rewritten

$$\frac{1}{2}\rho\frac{D\mathbf{v}^2}{Dt} = \frac{1}{2}\rho\left(\partial_t\mathbf{v}^2 + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}^2\right) \tag{67}$$

$$= \frac{1}{2}\rho\left(\partial_t\mathbf{v}^2 + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}^2\right) + \frac{1}{2}\mathbf{v}^2\left[\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\right] \tag{68}$$

$$= \partial_t\left(\frac{\rho\mathbf{v}^2}{2}\right) + \boldsymbol{\nabla}\left(\frac{\rho\mathbf{v}^2}{2}\mathbf{v}\right) \tag{69}$$

where in the second step we have used the continuity equation (6). The RHS can be rewritten as

$$-\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)P = -\boldsymbol{\nabla}\cdot(P\mathbf{v}) + P(\boldsymbol{\nabla}\cdot\mathbf{v}) \tag{70}$$

Putting all together and rearranging we find

$$\partial_t\left[\frac{\rho\mathbf{v}^2}{2}\right] + \boldsymbol{\nabla}\cdot\left[\left(\frac{\rho\mathbf{v}^2}{2} + P\right)\mathbf{v}\right] = P(\boldsymbol{\nabla}\cdot\mathbf{v}) \tag{71}$$

We have only used the continuity and Euler equation to derive this result, so it applies both to incompressible fluids and to compressible gases.

For an incompressible fluid $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ so the RHS side vanishes, and this constitutes a statement of the conservation of kinetic energy,[4] i.e. the total kinetic energy is a conserved quantity in an incompressible inviscid fluid.

For an adiabatic gas, the term $P(\boldsymbol{\nabla}\cdot\mathbf{v})$ represents the expansion work done by the gas, and kinetic energy is not a conserved quantity. To discuss the conservation of energy in this case one must consider the internal energy of the gas. The continuity equation can be rewritten (see Eq. 26) as

$$\boldsymbol{\nabla}\cdot\mathbf{v} = -\frac{D\log\rho}{Dt} \tag{73}$$

[4] Any statement of the type

$$\partial_t Q + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = 0 \tag{72}$$

is a conservation law where Q is the conserved quantity and $\boldsymbol{\mathcal{F}}$ is the associated flux. To see this, integrate over a finite volume and use the divergence theorem (4). This shows that the change in the amount of $\int_V Q\mathrm{d}t$ inside the volume is related to the outflux quantified by $\boldsymbol{\mathcal{F}}$. Integrating over the whole space and assuming that $\boldsymbol{\mathcal{F}} = 0$ at infinity, which is usually the case, you get that $\partial_t\left(\int Q\mathrm{d}V\right) = 0$, hence the quantity between parentheses is constant in time, i.e. it is globally conserved. What gets out of one volume just enters into an adjacent one.

From equation (22) for an adiabatic gas,

$$\frac{D\log\rho}{Dt} = \frac{1}{\gamma}\frac{D\log P}{Dt} \tag{74}$$

Eliminating $D\log\rho/Dt$ from these last two equations we find

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= -\frac{1}{\gamma}\frac{D\log P}{Dt} && (75)\\
&= -\frac{1}{P\gamma}\left[\partial_t P + (\mathbf{v}\cdot\boldsymbol{\nabla})\, P\right] && (76)\\
&= -\frac{1}{P\gamma}\left[\partial_t P + \boldsymbol{\nabla}\cdot(P\mathbf{v}) - P(\boldsymbol{\nabla}\cdot\mathbf{v})\right] && (77)
\end{aligned}$$

where in the last step we have used (70). Isolating $\boldsymbol{\nabla}\cdot\mathbf{v}$ from this last equation we find

$$\boldsymbol{\nabla}\cdot\mathbf{v} = -\frac{1}{P(\gamma-1)}\left[\partial_t P + \boldsymbol{\nabla}\cdot(P\mathbf{v})\right] \tag{78}$$

Substituting this into (71) and rearranging we finally find:

$$\partial_t\left[\frac{\rho\mathbf{v}^2}{2} + \frac{P}{\gamma-1}\right] + \boldsymbol{\nabla}\cdot\left[\left(\frac{\rho\mathbf{v}^2}{2} + P + \frac{P}{\gamma-1}\right)\mathbf{v}\right] = 0 \tag{79}$$

Conservation of energy for an adiabatic fluid.

which is a statement of the conservation of energy for an adiabatic gas. The terms between the first bracket parentheses are respectively the kinetic energy per unit volume and the internal energy per unit volume. Neither mechanical nor internal energy is separately conserved, but their sum integrated over all space is conserved. The three terms between the second square brackets are the fluxes of energy due to advection of kinetic energy, to pressure forces exchanged between adjacent fluid elements, and to advection of internal energy respectively. Since we are considering an adiabatic gas, there are no other fluxes of energy related to, for example, viscous forces or thermal conduction. The sum of kinetic plus internal energy must be conserved even if the gas is viscous, since dissipation does not constitute an external heat source (see Section 1.13).

What about energy conservation in an isothermal fluid? Well, energy is not conserved in an isothermal fluid.[5] One must imagine that any excess/lack of heat is removed/provided as needed by an external heat

[5] If one tries to see an isothermal gas as the $\gamma \to 1$ limit as discussed in Sect. 1.5, Eq. (79) diverges so it cannot be applied! Note however that *it is* possible to derive an "energy conservation" theorem similar to Eq. (79) for an isothermal fluid. This

bath to each fluid element, in such a way that the temperature is constant during its motion. In an astrophysical context, it is sometimes a useful approximation to treat the interstellar medium as an isothermal fluid, in which the thermal balance is maintained for example by external heating sources and collisional cooling.

1.8 Conservation of momentum

Conservation of momentum can be proved starting from the continuity (6) and Euler (14) equations. Multiplying (14) by ρ and considering the ith component we can rewrite the LHS as

$$\begin{aligned}\rho\frac{Dv_i}{Dt} &= \rho\left[\partial_t v_i + (\mathbf{v}\cdot\boldsymbol{\nabla})\, v_i\right] &(81)\\ &= \rho\left[\partial_t v_i + (\mathbf{v}\cdot\boldsymbol{\nabla})\, v_i\right] + v_i\left[\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\right] &(82)\\ &= \partial_t(\rho v_i) + \boldsymbol{\nabla}\cdot\left[(\rho v_i)\mathbf{v}\right] &(83)\\ &= \partial_t(\rho v_i) + \partial_j\left[(\rho v_i)v_j\right] &(84)\end{aligned}$$

where in the second step we have added a term which vanishes thanks to the continuity equation. The RHS can be rewritten as

$$-\partial_i P = -\delta_{ij}\partial_j P \tag{85}$$

Putting all together and rearranging we find

Conservation of momentum.

$$\boxed{\partial_t(\rho v_i) + \partial_j\left[(\rho v_i)v_j + \delta_{ij}P\right] = 0} \tag{86}$$

This equation is valid both for an incompressible and compressible fluid and is a statement of the conservation of momentum. To see this, just integrate over all space. If all the fluid quantities v_i, P and ρ vanish at infinity, then the integral of the second term in Eq. (86) disappears and we are left with

$$\partial_t\left(\int_V \mathrm{d}V\,\rho v_i\right) = 0 \tag{87}$$

which means that the quantity between parentheses, i.e. the total momentum in the ith direction, is constant in time.

---

reads:

$$\partial_t\left[\frac{\rho\mathbf{v}^2}{2} + \rho H - P\right] + \boldsymbol{\nabla}\cdot\left[\left(\frac{\rho\mathbf{v}^2}{2} + \rho H\right)\mathbf{v}\right] = 0 \tag{80}$$

where $H = c_\mathrm{s}^2\log(\rho/\rho_0)$ and ρ_0 is an arbitrary constant. The steps to derive this equation are similar to those to derive Eq. (79). The conserved quantity (i.e. the integral over all space of the terms within the first bracket parentheses on the LHS) is similar but not exactly the same as what you would normally call the energy.

1.9 Lagrangian vs Eulerian view

In astrophysics you often hear talking about "Lagrangian view" as opposed to "Eulerian view". What is meant exactly may depend on who you are speaking to. In practice, most of the time it is referred to equivalent ways of solving the equations of hydrodynamic. Since they are equivalent, one should choose the most convenient for the problem at hand.

- In a **Eulerian** description, quantities are written as a function of *fixed* spatial coordinates. For example, $\rho = \rho(\mathbf{x}, t)$ is the density at a fixed location in space $\mathbf{x}$. If we write the continuity equation as $\partial_t \rho(\mathbf{x}, t) + \boldsymbol{\nabla} \cdot (\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t))$, it is meant that ∂_t takes the time derivatives by comparing quantities at different times but at the same location in space, and $\boldsymbol{\nabla}$ are derivatives obtained comparing quantities at neighbouring locations in space at the same time.

- In a **Lagrangian** description, quantities are written as a function of coordinates that move with the flow. For example, suppose that we label all fluid elements by the position $\mathbf{x}_0$ they had ad time $t = t_0$. Then we can write $\rho = \rho(\mathbf{x}_0, t)$ meaning "what is the density at time $t > t_0$ of the fluid element that was at $\mathbf{x} = \mathbf{x}_0$ at time $t = t_0$"? In general the position of such a fluid element changes with time, so $\rho(\mathbf{x}_0, t)$ is not the density of a fixed point in space. It is the density that we would see by following a given fluid element in time. Since the convective derivative (12) follows the flow, it is also called the Lagrangian derivative. Equations of motion written in terms of D/Dt are sometimes said to be in "Lagrangian form".

1.10 Vorticity

In fluid dynamics it often useful to consider the **vorticity**

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}. \tag{88}$$

Definition of vorticity.

The physical meaning of this quantity can be obtained considering two short fluid line elements which are perpendicular at a certain instant and move with the fluid (see Fig. 4). The vorticity is twice the average angular velocity of two such short fluid elements. *In this sense*, the vorticity is a measure of the local degree of spin, or rotation, of the fluid. Note that this may not always be the same of our intuitive notion of rotation; if the two lines rotate in opposite directions with equal angular

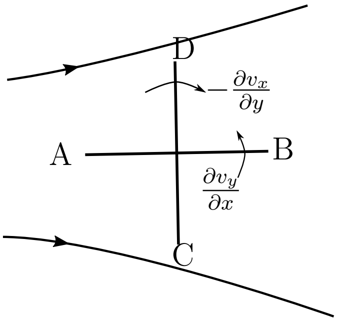

Figure 4: Physical meaning of vorticity. $\boldsymbol{\omega}$ is taken to point out of the page. The angular velocity of the two perpendicular fluid element lines AB and CD is $\partial_x v_y$ and $-\partial_y v_x$ respectively. Hence their average angular velocity is $(\partial_x v_y - \partial_y v_x)/2 = \omega/2$.

velocity the vorticity is zero. Note also that $\boldsymbol{\omega}$ is not directly related to global rotation. Velocity configurations are possible for which there is no global rotation but $\boldsymbol{\omega} \neq 0$. As an exercise, consider for example the vorticity of the flow fields defined by $\mathbf{v} = \beta y \hat{\mathbf{e}}_x$, where β is a constant, and $\mathbf{v} = (k/R)\hat{\mathbf{e}}_\phi$.

1.10.1 The vorticity equation

To obtain an equation for the evolution of vorticity, consider the following identity:

$$\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 - (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v} \tag{89}$$

which can be proved using (70). Using this we can rewrite the Euler equation (14) as

$$\partial_t \mathbf{v} - \mathbf{v} \times \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 = -\frac{\boldsymbol{\nabla} P}{\rho} \tag{90}$$

Taking the curl of this equation and using that the curl of a gradient vanishes, we obtain

$$\partial_t \boldsymbol{\omega} - \boldsymbol{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}) = \frac{1}{\rho^2}\left(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P\right). \tag{91}$$

Using identity (50) with $\mathbf{A} = \mathbf{v}$ and $\mathbf{B} = \boldsymbol{\omega}$, and remembering that the divergence of the curl vanishes we can rewrite it as:

$$\partial_t \boldsymbol{\omega} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\,\mathbf{v} - \boldsymbol{\omega}\,(\boldsymbol{\nabla} \cdot \mathbf{v}) + \frac{1}{\rho^2}\left(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P\right). \tag{92}$$

Now substitute $\boldsymbol{\nabla} \cdot \mathbf{v}$ from the continuity equation in the form (73):

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\,\mathbf{v} + \boldsymbol{\omega}\frac{D\log\rho}{Dt} + \frac{1}{\rho^2}\left(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P\right). \tag{93}$$

which can be rewritten as

Vorticity equation.

$$\boxed{\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left[\left(\frac{\boldsymbol{\omega}}{\rho}\right) \cdot \boldsymbol{\nabla}\right]\mathbf{v} + \frac{1}{\rho^3}\left(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P\right)} \tag{94}$$

This is the **vorticity equation**, which describes the evolution of vorticity. The quantity ω/ρ is called **potential vorticity**. Note that in the presence of a gravitational field the vorticity equation would be exactly the same since the extra term vanishes in the step in which we take the curl of the Euler equation (the curl of a gradient is zero, and the gravitational force is the gradient of Φ).

Now assume that

$$\nabla \rho \times \nabla P = 0. \tag{95}$$

Barotropic condition.

This is called **barotropic condition** and it means that surfaces of constant density are the same as surfaces of constant pressure. It is satisfied for barotropic fluids (29), which include isothermal and isentropic fluids as particular cases, and for incompressible fluids (27), for which the density is constant. Under this assumption the vorticity equation becomes

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left[\left(\frac{\boldsymbol{\omega}}{\rho}\right) \cdot \nabla\right] \mathbf{v}. \tag{96}$$

If the flow is *two-dimensional*, so that $\boldsymbol{\omega}$ is always perpendicular to $\mathbf{v}$, the term on the right hand side vanishes which implies that the potential vorticity is conserved, i.e. it is constant for a given fluid element. In the case of an incompressible flow, we can simplify ρ and the vorticity itself is conserved. These are powerful constraints: suppose for example that we know that the flow is steady, so that it does not change with time, and consider a streamline. If $\boldsymbol{\omega}/\rho$ or $\boldsymbol{\omega}$ is zero at some point along the streamline, it will be zero at all points along the streamline. Often we know that the potential vorticity (or the vorticity) vanishes at special points, for example at infinity, and we can use this condition to conclude that it is zero at all points reached by streamlines, which could be everywhere.

In the presence of viscosity (Section 1.13) the vorticity is not conserved anymore, even for an incompressible fluid. In this case it is possible to study how the vorticity is diffused, but we will not do it in this course (if you are interested, see for example [1]).

1.10.2 Kelvin circulation theorem

Consider a closed curve that moves with the fluid, see Fig. 5. The integral of the component of the velocity parallel to the curve around the closed curve

$$C = \oint_\gamma \mathbf{v} \cdot \mathrm{d}\mathbf{l} \tag{97}$$

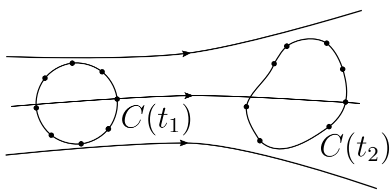

Figure 5: Kelvin circulation theorem.

is called the **circulation**. Kelvin circulation theorem states that, in an inviscid fluid in which the barotropic condition (95) is valid, this integral is constant in time if we follow the fluid, i.e. $C(t_1) = C(t_2)$.

To prove Kelvin's theorem, let us consider a slightly more general situation, which will be useful later when we consider flux freezing in magnetohydrodynamics. Suppose you have an equation of the following form

$$\partial_t \mathbf{A} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \boldsymbol{\nabla} f \tag{98}$$

where f is a generic function. We will now prove that

$$\Gamma(t) = \oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{l}, \tag{99}$$

calculated following the fluid, is constant in time, i.e. $\Gamma(t_1) = \Gamma(t_2)$. First, take the Lagrangian derivative of $\Gamma(t)$:[6]

$$\frac{D\Gamma(t)}{Dt} = \oint_\gamma \left(\frac{D}{Dt}\mathbf{A}\right) \cdot \mathrm{d}\mathbf{l} + \oint_\gamma \mathbf{A} \cdot \left(\frac{D}{Dt}\mathrm{d}\mathbf{l}\right) \tag{100}$$

We need to show that this is zero. (98) can be rewritten with the help of identity (47) as:

$$\frac{DA_i}{Dt} = v_j \partial_i A_j + \partial_i f \tag{101}$$

Using this equation the first term on the RHS of (100) can be rewritten as:

$$\oint_\gamma \left(\frac{D}{Dt}\mathbf{A}\right) \cdot \mathrm{d}\mathbf{l} = \oint_\gamma \left(v_j \partial_i A_j + \partial_i f\right) \mathrm{d}l_i \tag{102}$$

Next, use the result of problem 2, equation (**??**)

$$\frac{D}{Dt}\mathrm{d}\mathbf{l} = (\mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla})\, \mathbf{v} \tag{103}$$

to rewrite the second term on the RHS of (100) as:

$$\oint_\gamma \mathbf{A} \cdot \left(\frac{D}{Dt}\mathrm{d}\mathbf{l}\right) = \oint_\gamma A_j \mathrm{d}l_i \partial_i v_j \tag{104}$$

Adding (102) and (104) gives

$$\oint_\gamma \frac{D}{Dt}(\mathbf{A} \cdot \mathrm{d}\mathbf{l}) = \oint_\gamma \mathrm{d}l_i \partial_i \left(A_j v_j + f\right) = \oint_\gamma \mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla}\left(\mathbf{A} \cdot \mathbf{v} + f\right) = 0 \tag{105}$$

which vanishes because the integral of a gradient over a closed loop is zero, which proves the theorem.

[6] as an exercise, you can show that it is allowed to take the derivative inside the integral.

Note that using **Stokes theorem**, which states that

$$\oint_{\gamma} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \oint_{S} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} \tag{106}$$

Stokes theorem.

where S is an open surface[7] bounded by the curve γ, we can rewrite (99) as

$$\Gamma(t) = \oint_{S} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S}. \tag{107}$$

This can be interpreted as if the vector $\boldsymbol{\nabla} \times \mathbf{A}$ is "frozen" into the fluid. A small patch of fluid will retain during its motion the amount of flux of this vector that was given initially.

To see how this theorem applies with $\mathbf{A} = \mathbf{v}$ and recover Kelvin theorem, consider the Euler equation in its form (90):

$$\partial_t \mathbf{v} - \mathbf{v} \times \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\nabla} \mathbf{v}^2 = -\frac{\boldsymbol{\nabla} P}{\rho} \tag{108}$$

To bring this equation in the form (98), note that if the barotropic condition (95) holds, then our fluid is either barotropic or incompressible. In both cases we can define a function H such that

$$\frac{\boldsymbol{\nabla} P}{\rho} = \boldsymbol{\nabla} H \tag{109}$$

For a barotropic fluid this is

$$H(\rho) = \int \mathrm{d}\rho \frac{P(\rho)}{\rho} \tag{110}$$

while for an incompressible fluid it is

$$H(\rho) = \frac{P}{\rho} \tag{111}$$

hence, (108) is of the form (98) with $f = -H - \mathbf{v}^2/2$ and $\mathbf{A} = \mathbf{v}$. The vorticity $\boldsymbol{\omega}$ is frozen into the fluid.

[7] Note that this need not to be flat, it can be any of the infinite possible surfaces that have γ as boundary. This implies that the integral of a curl over a closed surface is zero.

1.11 Steady flow: the Bernoulli's equation

A flow is said to be steady when all quantities do not depend on time ($\partial_t = 0$). Of course, by this we do not mean that the fluid does not move, but only that the velocity and all other quantities do not change with time at each fixed point in space. A useful theorem for this type of flow can be derived as follows. Consider Euler equation in the presence of a gravitational field (15):

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla}\Phi \tag{112}$$

Then assume that the flow is steady so that $\partial_t \mathbf{v} = 0$ and then use the identity (89) to rewrite this equation as:

$$\frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = -\frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla}\Phi \tag{113}$$

Now assume that the flow is either barotropic, $P = P(\rho)$, or incompressible, so that the function H given by (110) or (111) can be defined. Then the Euler equation becomes

$$\frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = -\boldsymbol{\nabla} H - \boldsymbol{\nabla}\Phi \tag{114}$$

The term $\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})$ is perpendicular to $\mathbf{v}$ at each point in the flow. Hence, if we take the scalar product of with $\mathbf{v}$, we find that

$$\mathbf{v} \cdot \boldsymbol{\nabla} \left(\frac{1}{2}\mathbf{v}^2 + H + \Phi \right) = 0 \tag{115}$$

Since the direction of $\mathbf{v}$ is the same as the tangent to streamlines, this means that the quantity

Bernoulli's theorem.

$$\boxed{\frac{1}{2}\mathbf{v}^2 + H + \Phi = \text{constant}} \tag{116}$$

is constant along each streamline. This is known as **Bernoulli's theorem**. It is valid if the flow is steady, barotropic or incompressible, and nonviscous. For an incompressible fluid, $H = P/\rho$, and the theorem says that where velocity goes up, pressure goes down.

Note that this theorem says nothing about the constant being the same on different streamlines, only that it remains constant along each one. The constant *is* the same if we make one further assumption, namely that the flow is **irrotational**, which means that

$$\boldsymbol{\nabla} \times \mathbf{v} = 0. \tag{117}$$

In this case, we do not need to take the scalar product with $\mathbf{v}$ in the derivation above, and the constant is the same throughout the whole fluid.

1.12 Rotating frames

In astrophysics it is often useful to work in a frame rotating with constant angular speed $\boldsymbol{\Omega}$. This may be the frame in which a binary system is stationary, a rotating fluid is at rest, or a spiral pattern is stationary. In a rotating frame, we simply add the Coriolis and Centrifugal forces to the RHS of the Euler equation (14), which gives the forces acting on fluid elements, just as we would do for Newtonian point-particle mechanics. Thus the Euler equation in a rotating frame is:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} - 2\boldsymbol{\Omega} \times \mathbf{v} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) \tag{118}$$

Euler equation in a rotating frame.

where $-2\boldsymbol{\Omega} \times \mathbf{v}$ is the Coriolis force, $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})$ is the centrifugal force and velocities are measured in the rotating frame and are related to those in the inertial frame by

$$\mathbf{v}_{\text{inertial}} = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{x}. \tag{119}$$

Note that if $\boldsymbol{\Omega} = \Omega \hat{\mathbf{e}}_z$ we can rewrite the centrifugal force as $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = R\Omega^2 \hat{\mathbf{e}}_R$.

1.13 Viscosity and thermal conduction

The fact that the mean free path is small but finite has the consequence that particles can be exchanged between adjacent fluid elements, which creates transfers of momentum and energy in addition to those that we have studied in the previous sections. This is the origin of dissipative processes such as **viscosity** and **thermal conduction** (see Fig. 6).

Viscosity creates forces that tend to prevent velocity gradients, i.e. it is a force that opposes layers moving relative to each other. These are not contained in the equations considered so far. To obtain the equation of motion of a viscous fluid we need to modify our equations.

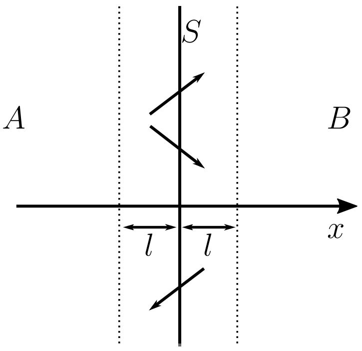

Figure 6: Particles exchanged between adjacent fluid elements A and B are the origin of viscosity. Molecules are exchanged between layers whose thickness is of the order of the mean free path l. Fluid elements are much bigger than l. If A and B are moving in the y direction with different velocities, particles escaping from the faster fluid element will accelerate the slower one and viceversa.

The continuity equation (6) is unchanged. We need to add other terms to the Euler equation (14). To do this, it is convenient to start from the version written as conservation of momentum of an inviscid fluid, equation (86):

$$\partial_t(\rho v_i) + \partial_j \left[\Pi_{ij}\right] = 0 \tag{120}$$

In this equation, the term

$$\Pi_{ij} = \rho v_i v_j + \delta_{ij} P \tag{121}$$

represents the flux of the ith component of momentum in the j direction. The equations of motion for a viscous flow are obtained adding to Π_{ij} an additional term σ_{ij} that accounts for the transfer of momentum due to viscous processes:

$$\Pi_{ij} = \rho v_i v_j + \delta_{ij} P + \sigma_{ij}. \tag{122}$$

The form of σ_{ij} is derived "heuristically", i.e. it is not a derivation from first principles, much like the description of friction in first year mechanics. We do not go through the whole derivation here, and the reader is referred for example to [2] or [1]. The derivation proceeds along the following lines. σ_{ij} is required to satisfy the following properties:

1. $\sigma_{ij} = 0$ when there are no relative motions between different parts of the fluid. This means that σ_{ij} must depend only on the space derivatives of the velocity.

2. We assume that σ_{ij} depends *only on the first derivatives* of the velocity and that it is a *linear* function of these first derivatives. This is a heuristic reasonable approximation. It is true for example if an expression for the momentum transfer is obtained from the simple picture sketched in Fig. 6.

3. We require that σ_{ij} does not look special in any inertial frame of reference. If this weren't the case, that frame would be different from the other frames and Galilean invariance would not be satisfied. Thus we require that when we change frame of reference (i.e. we perform a translation, rotation or add a constant relative motion) σ_{ij} transforms as a second-rank tensor.[8]

[8] For example, the expression $\sigma_{ij} = v_1$ cannot be true in all inertial frames. Hence the frames in which it *is* true would be special. The expression $\sigma_{ij} = \partial_{(i+1)} v_j$ is also not allowed, while the expression $\sigma_{ij} = \partial_i v_j$ is allowed because it has the same form in all inertial frames.

4. We require σ_{ij} to vanish when the whole fluid is in rigid rotation since in such a motion no internal friction occurs, i.e. $\sigma_{ij} = 0$ if $\mathbf{v} = \mathbf{\Omega} \times \mathbf{x}$ where $\mathbf{\Omega}$ is a constant vector.

5. We require σ_{ij} to vanish for uniform expansion. This is not a necessary requirement and there is a type of friction called **second viscosity**, which we do not consider in this course, which arises from such a term. This is important for example when internal degrees of freedom are slow in being excited, so that a fast expansion can result in instantaneous thermodynamic equilibrium not being satisfied (see for example [2] and [8] for more information), but will not be important in this course.

The most general second rank tensor satisfying the above conditions is

$$\sigma_{ij} = \eta \left(\partial_j v_i + \partial_i v_j - \frac{2}{3} \delta_{ij} \partial_k v_k \right) \tag{123}$$

Viscous stress tensor

where η is a parameter called **dynamic viscosity** and in general may be a function of fluid quantities such as density and pressure, $\eta = \eta(\rho, P, \dots)$. Its dimensions are

$$[\eta] = \frac{\text{mass}}{\text{length} \times \text{time}}. \tag{124}$$

Other properties of the viscous stress tensor worth mentioning are that it is symmetric, $\sigma_{ij} = \sigma_{ji}$, and that it has zero trace, $\sigma_{ii} = 0$, which is a consequence of condition (5) above (the trace of σ_{ij} is proportional to $\mathbf{\nabla} \cdot \mathbf{v}$, which is associated with expansions).

Substituting (122) where σ_{ij} is given by (123) back into (120) and performing a few steps[9] one arrives at:

$$\partial_t v_i + v_j \partial_j v_i = -\frac{\partial_i P}{\rho} + \frac{1}{\rho} \partial_j \left(\sigma_{ij} \right) \tag{125}$$

This equation replaces the Euler equation for a fluid in which viscous processes occur. If we further assume that $\eta =$ constant so that we can take it out of the derivatives we obtain the **Navier-Stokes** equation[10]

$$\boxed{\partial_t \mathbf{v} + \left(\mathbf{v} \cdot \mathbf{\nabla} \right) \mathbf{v} = -\frac{\mathbf{\nabla} P}{\rho} - \frac{\eta}{\rho} \mathbf{\nabla}^2 \mathbf{v} + \frac{\eta}{3\rho} \left[\mathbf{\nabla} \left(\mathbf{\nabla} \cdot \mathbf{v} \right) \right]} \tag{126}$$

Navier-Stokes equation

[9] You will need to use the continuity equation (6).

[10] Sometimes what is called the Navier-Stokes equation is (126) for an incompressible fluid, i.e. assuming $\mathbf{\nabla} \cdot \mathbf{v} = 0$, while some other times it is (126) including an additional term that accounts for the *second viscosity* briefly mentioned above.

where $\boldsymbol{\nabla}^2\mathbf{v} = \partial_j\partial_j v_i$ and $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) = \partial_i(\partial_j v_j)$.

We have seen in Section 1.5 that the continuity and Euler equations alone are not enough to form a complete system of equations that determine the evolution in time of the fluid. Analogously, the continuity equation plus the Navier-Stokes equation are not a complete system of equations. Among the possible ways to complete the equations discussed in Section 1.5, the incompressible approximation and the isothermal one remain valid possibilities.[11] However, the adiabatic approximation cannot be valid anymore because viscous processes will dissipate energy which according to the first law will be converted into internal energy. For a viscous fluid the conservation of entropy equation (23) (or equivalently 22) cannot hold anymore. To see how these should be replaced, we need to find out what is the dissipation due to viscous processes and then use the first law of thermodynamics to equate them with the increase in the internal energy of fluid elements.

Dissipation means loss of mechanical energy. For a non-viscous fluid the change in mechanical energy was expressed by (71). We can derive an analogous equation for the case of a viscous fluid. Taking the dot product between $\mathbf{v}$ and (125) and after manipulations similar to the non viscous case (exercise) one finds that

$$\partial_t\left[\frac{\rho\mathbf{v}^2}{2}\right] + \boldsymbol{\nabla}\cdot\left[\left(\frac{\rho\mathbf{v}^2}{2}+P\right)\mathbf{v} - \mathbf{T}\right] = P(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mathcal{D} \qquad (127)$$

where the new terms with respect to (71) are coloured in red and are given by

$$T_i = \sigma_{ij}v_j. \qquad (128)$$

and

$$\mathcal{D} = \sigma_{ij}\left(\partial_i v_j\right) \qquad (129)$$

It can be shown that $\mathcal{D} \geq 0$ (Problem 3). To interpret the various terms, consider what happens when we integrate (127) over some arbitrary closed volume V that is *fixed* in space and bounded by a surface S. After using the divergence theorem we obtain

$$\partial_t\left(\int_V \mathrm{d}V\,\frac{\rho\mathbf{v}^2}{2}\right) = -\oint_S \mathrm{d}\mathbf{S}\cdot\left[\left(\frac{\rho\mathbf{v}^2}{2}+P\right)\mathbf{v} - \mathbf{T}\right] + \int_V \mathrm{d}V\,P\,(\boldsymbol{\nabla}\cdot\mathbf{v}) - \int_V \mathrm{d}V\,\mathcal{D}. \qquad (130)$$

[11] In the isothermal case one must assume that the processes that keep the temperature constant are faster than any other process that may change the temperature of fluid elements, such as viscous dissipation.

The term on the LHS is the total change of kinetic energy inside the volume. The first term on the RHS is the contribution to this change due to fluxes through the surface S, which is zero when integrated over all space. Kinetic energy that outflows due to this term, partly because of $\mathbf{T}$, is not lost, but is simply gone into an adjacent volume. The second term on the RHS is the same as in the non viscous case and has been interpreted as the contribution due to the expansion work done by the gas. The last term containing $\mathcal{D}$ can be interpreted as energy lost due to viscous dissipation. Note that in contrast to the energy lost due to $\mathbf{T}$, the energy lost due to $\mathcal{D}$ does not go into a neighbouring fluid element, because $\mathcal{D}$ has always the same sign, $\mathcal{D} \geq 0$. Integrating over adjacent volumes does not cancel what happens in a given one, only adds up. Therefore, we *interpret* $\mathcal{D}$ as the energy dissipated per unit volume due to viscous dissipation. Note that this is not a "theorem", in the sense that this interpretation cannot be derived purely mathematically, it is an extra physical assumption that we make based on the equations.

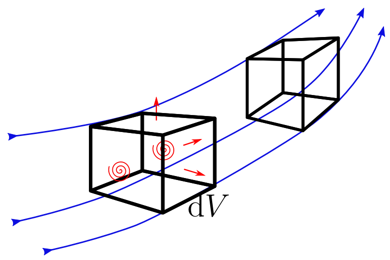

Figure 7: During the motion of a fluid element in a viscous fluid, its temperature changes because of three distinct effects: i) adiabatic compression and expansion ii) dissipation due to viscosity (red swirls) iii) thermal conduction with neighbouring fluid elements (red arrows). Schematic blue lines represent streamlines.

Now that we know how to quantify viscous dissipation, we need to use the first law to equate it to the heat gained by a fluid element. Consider a small fluid element of volume $\mathrm{d}V$, mass $\mathrm{d}M = \rho\,\mathrm{d}V$, internal energy $\mathcal{U}\,\mathrm{d}M$ and entropy $s\,\mathrm{d}M$, and follow it as it moves through the fluid. The second law of thermodynamics states that for a reversible process

$$Ds = \frac{D\mathcal{Q}}{T} \tag{131}$$

where $D\mathcal{Q}$ is the heat per unit mass absorbed by the fluid element. In the presence of viscous dissipation this is $D\mathcal{Q} = (\mathcal{D}/\rho)Dt$, so one should replace (22) with

$$\rho T \frac{Ds}{Dt} = \mathcal{D}. \tag{132}$$

It this all? No. There is still one process that is closely related to viscous dissipation that we have not accounted for. This is **thermal conduction**. With the physics we have included so far in our equations, a fluid element can heat up because it is compressed, or because energy is dissipated within it. The only way it has to to transfer energy gained because of viscous dissipation to an adjacent fluid element is through adiabatic expansion. But if a fluid element is much hotter than the one next to it, and the fluid is at rest, it will remain hotter forever according to our equations! Not very sensible. Thermal conduction takes care of this and provides a way to transfer energy between neighbouring fluid elements without involving macroscopic fluid motion. Hence we will not need to modify (125) or the Navier-Stokes equation (126) to include it.

Let us call $\mathbf{F}$ the heat flux density due to thermal conduction, i.e. the energy transferred per unit area and time. We expect it to be related to temperature variations through the fluid. It is usually a good approximation to assume that $\mathbf{F}$ is proportional to the gradient of T:

$$\mathbf{F} = -\kappa \boldsymbol{\nabla} T \tag{133}$$

Where $\kappa > 0$ is a coefficient called *thermal conductivity*, and in general $\kappa = \kappa(T, \rho, \dots)$. The minus sign comes from the requirement that heat must flow from hotter to colder fluid elements. By considering a small fluid element $\mathrm{d}V$, you can show that the amount of heat conducted into it per unit time and volume is given by $\boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T)$. Hence in presence of both viscous dissipation and thermal conduction, equation (22) should be replaced with

General equation of heat transfer.

$$\boxed{\rho T \frac{Ds}{Dt} = \mathcal{D} + \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T)} \tag{134}$$

This equation replaces (22) for a fluid with viscous dissipation and thermal conduction (see Fig. 7). The first term on the RHS is the heat dissipated into the fluid element by viscosity, the second is the heat driven into it by thermal conduction. It is possible to show that this equation together with the equation of motion (125), (123) and the continuity equation (6) imply that the total entropy of the *whole* fluid always increases (see for example [2]). Using the same three equations, one can show that

$$\partial_t \left(\frac{1}{2} \rho v^2 + \frac{P}{\gamma - 1} \right) + \boldsymbol{\nabla} \cdot \left[\left(\frac{\rho \mathbf{v}^2}{2} + P + \frac{P}{\gamma - 1} \right) \mathbf{v} + \mathbf{T} + \kappa \boldsymbol{\nabla} T \right] = 0 \tag{135}$$

Which is a statement of the conservation of energy for a viscous fluid with thermal conduction. Note that we have not *derived* energy conservation, we have *imposed* it by requiring that energy dissipated must go into internal energy of fluid elements according to the first law of thermodynamics.

Thermal conduction is important in the interior of stars. Is viscosity important in astrophysics? Rarely, and usually not quite in the sense in which we discussed it in the previous section. ISM has substructure, and we can average over this substructure and treat it phenomenologically as some kind of viscosity. However this behaves differently from microscopic viscosity in, say, air, and discussed in this section. We will see this when we study accretion discs.

1.14 The Reynolds number

When is viscosity important? To find out, we need to compare viscous forces to other forces that are present in the fluid. Consider (125) where σ_{ij} is given by (123). The second term on the RHS are viscous forces, while the LHS represents the total force acting on the fluid. Without further information about the particular situation at hand, there is no way to know which one is bigger. However, for a given situation we can often estimate the relative contribution of these two terms and decide whether we can neglect one of them. Suppose that in a particular situation fluid quantities vary on a typical length scale L and the flow has a typical velocity V; for example, if we have an object moving in the fluid with certain size and velocity this would give us some characteristics values. Then we can crudely approximate the spatial derivatives as $\boldsymbol{\nabla} \simeq L^{-1}$. The order of magnitude of the second term on the RHS of (125), which represent viscous forces, is then

$$\frac{\eta V}{\rho L^2}, \tag{136}$$

while the order of magnitude of the term on the LHS is

$$\frac{V^2}{L}. \tag{137}$$

The ratio between the latter and the former is a *dimensionless* quantity called the **Reynolds number**:

$$\boxed{\mathrm{Re} = \frac{VL}{\nu}} \tag{138}$$

Reynolds number.

where $\nu \equiv \eta/\rho$ is called **kinematic viscosity**, while η was called **dynamic viscosity**. The Reynolds number quantifies the relative importance of viscous forces and total (also called inertial, because they are related with the total acceleration) forces.

A high Reynolds number corresponds to a flow in which viscous forces are negligible. Such a flow may be smooth and steady, but, perhaps counter-intuitively, is more often *turbulent*. Turbulence occurs almost certainly if $\mathrm{Re} \gtrsim 10^4$. A low Reynolds number, on the other hand, corresponds to a viscosity-dominated flow, in which dissipational effects damp out turbulence before it can become established.

In the vast majority of astrophysical flows, the Reynolds number is very high, and viscosity can be neglected. For example, in the interstellar medium the typical range is $\mathrm{Re} \sim 10^5$ to 10^{10}. However, in certain situations a somewhat different type of viscosity plays a role. Astrophysical flows usually show a great deal of substructure. We sometimes want to average over this substructure and consider fluid elements that are much bigger than it (note that this violates one of the assumption that we made at the beginning for the validity of the fluid approximation!). For example, in accretion discs, the fluid is turbulent on small scales, but can be considered smooth when averaged over larger scales. Thus we can consider fluid elements that average out this small scale turbulence, and account for the effects of this small-scale turbulence through a phenomenological viscosity which is created not by single particles crossing fluid elements, as was the case in the picture of Section 1.13, but by "macro-particles" formed by clouds and subclouds. This type of viscosity is not completely equivalent to viscosity as discussed in the previous section, and care must be taken.

What about the opposite regime, i.e. very low Reynolds numbers? This gives us the chance for a little excursus. You may have never thought about it, but there is a whole world where this regime is relevant. Since microorganisms such as bacteria are small, their Reynolds numbers are extremely small. For a bacteria, swimming through water is like for a human swimming in a pool of molasses while only being allowed to move at 1 cm/min, like the hands of a clock. The equations approximated for this regime are completely different from the equations that are appropriate in most astrophysical applications. In water, we can assume incompressibility and at bacteria scales we can through away the LHS in Eq. (126) to obtain

$$0 = -\boldsymbol{\nabla} P + \eta \boldsymbol{\nabla}^2 \mathbf{v} \tag{139}$$

Note that time does not appear in this equation! The forces on a bacteria will be determined entirely by the instantaneous velocity patterns on the surface of its body. It is a very interesting branch of modern physics (and biology) to study how these microorganism can swim and move. If you want to take a break from astrophysics and are looking for a treat, you can read the wonderful article that popularised it all: *Life at low Reynolds number* by E.M. Purcell, American Journal of Physics (1977).

1.15 Adding radiative heating and cooling

The most common reason why a fluid absorbs or releases heat in astrophysics is not viscosity or thermal conduction. Instead, it is **radiative processes**. For example, if nearby a fluid element is a massive star, its strong UV radiation will heat it. Cosmic rays permeate the universe and can be absorbed by fluid elements, heating them up. In the interstellar medium, molecules collide, get excited and then release a photon, which then leaves the fluid element, leaving it cooler; this photon could be catched later by another fluid element, but if the system is **optically thin** it escapes from the entire system. Electrons revolving around magnetic fields are accelerated charged particles, and as such emit radiation called synchrotron radiation, cooling the system. Many other radiative processes play a role in astrophysics.

To take into account these processes, we define a **heating rate per unit volume**, with units

$$\dot{Q} = \frac{\text{energy}}{\text{time} \times \text{volume}} \tag{140}$$

which is usually divided into heating and cooling terms

$$\dot{Q} = \Gamma - \Lambda \tag{141}$$

Note that Γ and Λ here are the heating and cooling rates *per unit volume*, while Γ/ρ and Λ/ρ are their counterparts *per unit mass*. Different conventions are used in the literature, so one must be careful.

For a system in which radiative processes are present, (22) cannot hold anymore, and according to the first law of thermodynamics must be replaced by

$$\rho T \frac{Ds}{Dt} = \dot{Q} \tag{142}$$

In case viscous dissipation and thermal conduction are also important, one should also add all the corresponding terms that are present on the RHS of (134).

Often, but not always, the system can be considered optically thin (after all, most of the things we can see *are* optically thin: we cannot see inside optically thick things, like the interior of the Sun! So we are strongly biased towards seeing optically thin things). This simplifies things a lot, because we do not need to keep track of all photons to see if they are eventually reabsorbed or – in case they come from an external source – whether they can reach the inner parts of a system which are better shielded from the outside world. Accounting for both these things usually depends in a complicate way on the geometry of the system.

Sometimes, if radiative processes are very fast and act on a much shorter timescale the dynamical processes, they may effectively keep the medium isothermal. In that case, we may simply replace (142) with the isothermal equation of state (17).

As a simple example of the form the heating and cooling rate can take, consider the interstellar medium as heated by an external energy source to which our medium is optically thin (for example, cosmic rays) and cooled by collisions which excite internal energy states of molecules which then emit photons that escape the system. In this case one can say $\Gamma = \rho \times \text{constant}$, where the constant is the strength of the external heating, and $\Lambda = \rho^2 f(T)$, where $f(T)$ is a function of temperature and the factor of ρ^2 comes from the fact that the number of collisions *per unit volume* is proportional to the density squared (the number of collisions increases if there are more particles, and if particles are closer).

1.16 Summary

The equations of fluid motion are

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{143}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla}\Phi - \partial_j (\sigma_{ij}) \tag{144}$$

where

$$\sigma_{ij} = \eta \left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\delta_{ij}\partial_k v_k \right) \tag{145}$$

These alone are not enough to uniquely specify the time evolution and must be complemented by another equation. Possible choices include:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0 \qquad \text{incompressible} \tag{146}$$

$$P = P(\rho) \qquad \text{barotropic} \tag{147}$$

$$\rho T \frac{Ds}{Dt} = \mathcal{D} + \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T) + \rho \dot{Q} \qquad \text{equation of heat transfer} \tag{148}$$

For an ideal adiabatic fluid with no viscosity, no thermal conduction and no radiative heating and cooling processes the equation of heat transfer reduces to

$$\frac{D}{Dt}\left(\log P \rho^{-\gamma}\right) = 0. \tag{149}$$

2 Magnetohydrodynamics

2.1 Basic equations

Most of the fluids in the universe are electrically conductive. For example, stars are made of hot, almost completely ionised gas. The interstellar medium can often be considered as composed almost entirely of neutral particles, plus a small population of charged particles which is enough to make it an effective conductor (after all, salty water is a good conductor despite only a small fraction of ions being present). The interior of planets, such as the Earth, is often made of molten metals.

It may seem at first sight that all these different situations should be treated differently. A fully ionised gas should be treated as a two-component fluid, while a weakly ionised gas (a mix of negative, positive and neutral particles) should be treated as a three-component fluids. This is because the Lorentz force acts differently on positive, negative and neutral particles, so these should be treated as different components. Remarkably, under the assumption that mean free path is small enough, collisions redistribute the effects of the Lorentz force so that its effects on one component are shared among all other components within the same fluid element, and all these different situations can be treated under the same one-component theory. This theory is called **magnetohydrodynamics**, and is obtained by coupling the fluid equations to the equations of electromagnetism. It is a theory of electrically conducting fluids.

To derive such a theory an obvious starting point is Maxwell's equations. A difficulty arises because Maxwell's equations are Lorentz invariant, while the equations of hydrodynamics considered so far are Galilean invariant. To couple the two consistently, we either need to make the former Galilean invariant, or the latter Lorentz invariant. Here, we consider a non relativistic theory, and choose the first option. Therefore in our equations we neglect all the terms of order $(v/c)^2$, where v is the speed of the fluid. It is possible to derive a theory of relativistic magnetohydrodynamics, but we will not consider it in this course.

We assume that the fluid is almost neutral, i.e. the net charge density is very small,[12] but large currents can be present because charges of opposite sign can flow in different directions. This condition can be

[12] Why don't we assume that the net charge is exactly zero? Recall from courses in elementary electromagnetism that a wire which looks neutral in one frame does not look *exactly* neutral in another frame which moves with respect to the first. More precisely, consider an infinitely long wire and assume that in a certain frame there are positive charges with charge density per unit length λ and velocity v_+ and negative charges with charge density per unit length $-\lambda$ and velocity v_-, so that the total current is $I = \lambda(v_+ - v_-)$. The wire is neutral in this frame. The wire is not neutral in another frame which moves with respect to the first in a direction parallel to the wire. The reason is that relativistic length contraction changes the average distance between charges and so it changes their densities per unit length. Therefore in our fluid we cannot expect the charge to be exactly zero, but only to be small according to (150).

written

$$|\rho_{\mathrm{e}}\mathbf{v}| \sim \left(\frac{v^2}{c^2}\right)|\mathbf{J}| \ll |\mathbf{J}| \tag{150}$$

where ρ_{e} is the electric charge, $\mathbf{v}$ is the typical velocity of the fluid and $\mathbf{J}$ is the current density, which has units of current per unit area. This is the same situation that we have in familiar conductors, such as electrical wires in our houses. This assumption is reasonable because any local net charge would be soon neutralised in a good conductor. One of the consequences of this assumption is that electric fields generated by charges are much smaller than magnetic fields generated by currents. This is why we talk about *magneto* hydrodynamics rather than *electromagneto* hydrodynamics. Thus, if an astrophysical object displays large magnetic fields, it is an indication that one should magnetohydrodynamics rather than hydrodynamics.

Enough said, let's start with writing down some equations. Maxwell's equations in CGS units are:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho_{\mathrm{e}} \tag{151}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{152}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{B} \tag{153}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial_t\mathbf{E} \tag{154}$$

First, we need to see which terms can be neglected in our non relativistic approximation. In a highly conducting fluid, we expect the fields to vary on the same typical length L and time T of the fluid, where $L/T = v$ is the typical speed of the fluid, because any variation on smaller scales would be quickly smeared out by a rearrangement of charges, while changes on the typical scales of the fluid are maintained by the fluid motions. Approximating $\boldsymbol{\nabla} = L^{-1}$ and $\partial_t = T^{-1}$ the third Maxwell equation (153) says that the electric field is a factor v/c smaller than the magnetic field:

$$E \sim \frac{v}{c}B \tag{155}$$

Using this we see that the term $(1/c)\partial_t\mathbf{E}$ in the fourth Maxwell equation (154), called the *displacement current*, can be neglected, because is of order v^2/c^2 compared to the term on the LHS of the same equation. Hence (154) can be approximated as

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} \tag{156}$$

Which implies $B \sim (L/c)J$ and one can check that under our assumption $\rho v \sim (v/c)^2 J$ the first Maxwell equation is consistent with these approximations.

Now we need to couple Maxwell's and the fluid equations. First, let us find the force due to electromagnetic fields on a fluid element. The total Lorentz force per unit volume in the non relativistic case is

$$\mathbf{F}_{\mathrm{L}} = \rho_{\mathrm{e}}\mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B} \tag{157}$$

This is obtained by summing the contributions on the positive and negative charges separately. These contributions acts on the charge carriers, which may only constitute a small part of the mass of a fluid element (for example in a gas whose mass is mostly in a neutral component, as is often the case for the interstellar medium), not on the fluid element as a whole. Moreover, the forces on the different components have in general different directions! For example, the electric force is in opposite directions for negative and positive particles. Why then do we treat the total force as if it's acting on the fluid element as a whole? As mentioned at the beginning of the section, it is thanks to collisions. For the fluid approximation to be valid, the mean free path of the charges must be small compared to the size of a fluid element. Hence, charge carriers must share their momentum before leaving a fluid element. The net result is that we can treat the total Lorentz force as if it's acting on the fluid elements as a whole.

Note also that in a magnetised plasma, charged particles revolve around magnetic fields with a radius given by the Larmor radius. If this radius is small enough this might give a fluid-like behaviour even in absence of collisions. Hence, when you read "mean free path" in a magnetised fluid, sometimes people mean the minimum between the one given by collisions and the Larmor radius. However the Larmor radius only applies to the direction perpendicular to the magnetic field lines, and charges are free to move in the direction parallel to the magnetic field, so one must be careful (see for example [9] for more details).

Thanks to our assumption (150) and using (155) we see that the electric term in (157) is of order $(v/c)^2$ compared to the magnetic term, hence the Lorentz force can be approximated by

$$\mathbf{F}_{\mathrm{L}} = \frac{1}{c}\mathbf{J} \times \mathbf{B} \tag{158}$$

Thus, the Euler equation (15) with the addition of this force becomes:

$$\rho\frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} P - \rho\boldsymbol{\nabla}\Phi + \frac{1}{c}\mathbf{J} \times \mathbf{B} \tag{159}$$

using equation (156) and the following identity which can be proved using (47):

$$(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{2}\boldsymbol{\nabla} B^2 + (\mathbf{B} \cdot \boldsymbol{\nabla})\,\mathbf{B} \tag{160}$$

we can rewrite (159) as

MHD Euler equation.

$$\boxed{\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) - \boldsymbol{\nabla}\Phi + \frac{1}{4\pi}\,(\mathbf{B} \cdot \boldsymbol{\nabla})\,\mathbf{B}} \tag{161}$$

This replaces Euler equation (15) in magnetohydrodynamics. The term B^2 clearly acts like a sort of pressure. We will analyse the physical meaning of the magnetic force in more detail in the next section.

We have seen how the conducting fluid reacts to the presence of the fields, but we have said nothing about how the fields are affected by the motion of the fluid. Fields respond to $\mathbf{J}$. We need to relate $\mathbf{J}$ to motions of the fluid. To do this, we assume that the (non relativistic) Ohm's law is valid:[13]

Ohm's law.

$$\mathbf{J} = \sigma\left(\mathbf{E} + \frac{1}{c}\,\mathbf{v} \times \mathbf{B}\right) \tag{162}$$

where σ is the **electrical conductivity**. Note that in this equation $\mathbf{v}$ is the velocity of the fluid! Substituting this into (156) we find[14]

$$\mathbf{E} = \frac{c}{4\pi\sigma}\,\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\,\mathbf{v} \times \mathbf{B} \tag{163}$$

then substituting into (153) we find:

Induction equation.

$$\boxed{\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) - \boldsymbol{\nabla} \times \left(\frac{c^2}{4\pi\sigma}\boldsymbol{\nabla} \times \mathbf{B}\right)} \tag{164}$$

[13] If the electrons are the charge carriers, then one should not put $\mathbf{v}$ in Ohm's law, but $\mathbf{v}_-$, the velocity of the electrons, which is slightly different because they have drift velocities (if the positive and negative charges had exactly the same velocity, there would be no current!). For example, considering the case $n_i = n_e = n$ where n is the number of carrier particles, (i.e., the number of ions is the same as the number of electrons) and using $\mathbf{J} = nq(\mathbf{v}_+ - \mathbf{v}_-)$ where q is the charge of one particle and assuming that $\mathbf{v} \simeq \mathbf{v}_+$, i.e. the positive charges constitute the bulk of the mass (or move with the bulk of the mass if the latter is made by a neutral component), one obtains $\mathbf{J} = \sigma(\mathbf{E} + (\mathbf{v}_-/c) \times \mathbf{B}) = \sigma(\mathbf{E} + [\mathbf{v} + (\mathbf{v} - \mathbf{v}_-)/c] \times \mathbf{B}) = \sigma(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B} + [\mathbf{J}/(nqc)] \times \mathbf{B})$. The extra term is called **Hall effect**. If furthermore one assumes that also the ions can move differently from the neutrals, then one obtains an additional term called **ambipolar diffusion**. These extra terms are usually small and we neglect them in this course.

[14] Note that to be consistent with the approximation $E \sim (v/c)B$ we need $\sigma \gtrsim c^2/(vL)$. The conductivity cannot be too small.

which for historical reasons[15] is called the **induction equation**.

In the limit of $\sigma \to \infty$ we obtain the so called **ideal magnetohydrodynamics**, and the induction equation becomes:

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \tag{165}$$

Induction equation (ideal MHD).

This corresponds to the limit in which our fluid is a perfect conductor. Taking the limit the limit $\sigma \to \infty$ in equation (163) we find:

$$\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} = 0 \tag{166}$$

The interpretation of this equation is simple. Recall from elementary courses in electromagnetism that the electric field $\mathbf{E}$ vanishes in a conductor at rest. In a conductor that moves, the total Lorentz force, not the electric field, must vanish, which is what this equations says.

A word of caution. In astrophysics σ is often large enough that ideal MHD is a useful approximation. Note however that in simple models like the Drude model of electric conduction σ is essentially related to collisions between electrons and more massive particles. $\sigma \to \infty$ is then the limit of no collisions, which means infinite mean free path. This is in tension with our assumption that the mean free path is small with respect to the size of a fluid element. Thus one must be careful in the way one takes this limit, and what is kept constant.[16] As remarked above one may consider the mean free path to coincide with the Larmor radius

[15] A similar equation was used to describe the generation of voltages by changing magnetic fields in coils.

[16] For example, in a simple Drude–like model there is a friction force due to collisions proportional to their relative velocity. The friction force per unit volume is given by:

$$\mathbf{F}_{\mathrm{fr}} = -\alpha \frac{n m_{\mathrm{e}}}{\tau} (\mathbf{v}_+ - \mathbf{v}_-) \tag{167}$$

where m_{e} is the mass of the electron, τ is the time between collisions, n is the electron number density and α is a dimensionless parameter of order unity. In the same model the conductivity is:

$$\sigma = \frac{n e^2 \tau}{\alpha m_{\mathrm{e}}}. \tag{168}$$

Using $\mathbf{J} = ne(\mathbf{v}_+ - \mathbf{v}_-)$ and imposing that the friction force must be of the order of $\mathbf{J} \times \mathbf{B}/c$, which is required for efficient coupling between the various species and the MHD approximation to be valid, one obtains

$$\frac{nec}{\sigma} \sim B \tag{169}$$

If we want B to remain finite in the $\sigma \to \infty$ limit, we need to take n to infinity!

rather than with the value derived from collisions, but since this only works in the direction perpendicular to the magnetic field it is less clear that the theory remains valid and it is also less clear how the momentum transfers works in this case. Another way by which electron and ions can transfer energy between them and with a possible neutral component is through collective effects (think of the sea of electrons in a metal), discussion of which is outside the scope of the present notes. Thus while it is useful to consider ideal MHD in situations in which the timescale associated with magnetic diffusion is small compared to the dynamical timescale of interest, the limit of pure ideal MHD must be considered with caution.

The Euler MHD equation (159), the continuity equation (6) and the induction equation (164), plus one further condition, for example the incompressibility condition (27), or a barotropic condition (29), or an energy equation analogous to (22), form a complete system of equations for the quantities ρ, $\mathbf{v}$, $\mathbf{B}$. In ideal MHD it can be shown (see Section 2.6) that there are no dissipations related to $\mathbf{B}$, and therefore one could take (22) to be valid. In the presence of a finite σ instead, one must account for the associated dissipations and write an equation analogous to (134) (see equation 202).

Note that $\mathbf{B}$ is the only new quantity that appears in the final set of equations with respect to the hydrodynamics case. The electric field never enters the final set of equations. However it does not mean that there isn't an electric field, or that it is negligible. But it is given by (163) and therefore can be always written in terms of $\mathbf{B}$. The electric field is important in the transfer of energy between fields and fluid (see Section 2.7). The current $\mathbf{J}$ also does not appear explicitly, but can be calculated once we know $\mathbf{B}$ from (156).

What about the second Maxwell equation, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$? Note that if this condition is satisfied at the initial time, taking the divergence of (164) shows it will be always satisfied. Hence we only need to impose it in our initial conditions, and it constitutes a constraint on how we can set them up.

We can check the self–consistency of our approximations by using the electric field given by (163) and then the first Maxwell equation (151) to calculate the electric charge and see whether (150) is satisfied. Taking the divergence of (163), the first term vanishes because the divergence of the curl is identically zero, so we find $\rho_{\mathrm{e}} \sim \boldsymbol{\nabla} \cdot \mathbf{E} \sim vB/(Lc)$. Then using (156) we find $B \sim LJ/c$, hence $\rho_{\mathrm{e}} v \sim (v/c)^2 J$, which is (150), and everything is ok. Note that apart from such checks, the first Maxwell equation (151) is *not* used in MHD to derive any conclusion and can be taken as a definition of the charge density. From a purely mathematical point of view, the electric field can be considered as *defined* by (163).

2.2 Magnetic tension

We have seen that the magnetic force in the MHD Euler equation (161) is

$$\mathbf{F}_\mathrm{L} = \frac{1}{c}\mathbf{J}\times\mathbf{B} = -\frac{1}{8\pi}\boldsymbol{\nabla}B^2 + \frac{1}{4\pi}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{B} \tag{170}$$

The first term clearly acts like a sort of magnetic pressure. The second term vanishes when the magnetic field does not change along its own direction.

There is a way to rewrite (170) that makes its interpretation more clear. Write the magnetic field as

$$\mathbf{B} = B\hat{\mathbf{s}} \tag{171}$$

where $\hat{\mathbf{s}}(\mathbf{x},t)$ is the unit vector in the direction of $\mathbf{B}$. Then one can rewrite the Lorentz force as

$$-\frac{1}{8\pi}\boldsymbol{\nabla}B^2 + \frac{1}{4\pi}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{B} = -\frac{1}{8\pi}\boldsymbol{\nabla}B^2 + \frac{1}{8\pi}\hat{\mathbf{s}}\left(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla}B^2\right) + \frac{B^2}{4\pi}(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})\hat{\mathbf{s}} \tag{172}$$

$$= -\frac{1}{8\pi}\boldsymbol{\nabla}_\perp B^2 + \frac{B^2}{4\pi}(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})\hat{\mathbf{s}} \tag{173}$$

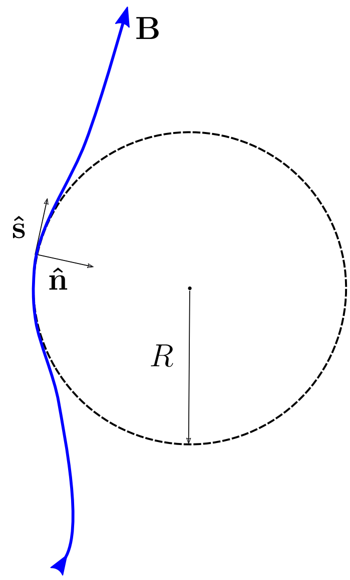

Figure 8: Magnetic tension.

where $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})$ is the projection of the gradient operator in the direction perpendicular to $\hat{\mathbf{s}}$. The vector $(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})\hat{\mathbf{s}}$ is perpendicular to $\hat{\mathbf{s}}$. This is clear since all the other terms in the same equation, the total Lorentz force and the term $\boldsymbol{\nabla}_\perp B^2$, are perpendicular to $\hat{\mathbf{s}}$.[17] Since $(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})$ is the projection of the gradient operator along $\hat{\mathbf{s}}$, $(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})\hat{\mathbf{s}}$ is the rate of change of $\hat{\mathbf{s}}$ along its own direction. Since $\hat{\mathbf{s}}$ cannot change in modulus but only in direction, this must be related to the local curvature of the magnetic field line. It can actually be taken to be the definition of such curvature. For a circle of radius R, $\hat{\mathbf{s}} = \hat{\mathbf{e}}_\phi$, so $(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})\hat{\mathbf{s}} = \hat{\mathbf{e}}_\phi(1/R)\cdot\partial_\phi\hat{\mathbf{e}}_\phi = -1/R\hat{\mathbf{e}}_R$. Hence one finds that

$$(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})\hat{\mathbf{s}} = \frac{1}{R}\hat{\mathbf{n}} \tag{174}$$

[17] It can also be proved by showing that the scalar product between $\hat{\mathbf{s}}$ and $(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})\hat{\mathbf{s}}$ is zero. Deriving $\hat{\mathbf{s}}\cdot\hat{\mathbf{s}} = s_js_j = 1$ we find $s_j\partial_i s_j = 0$. Hence $\hat{\mathbf{s}}\cdot[(\hat{\mathbf{s}}\cdot\boldsymbol{\nabla})\hat{\mathbf{s}}] = s_j[s_i\partial_i s_j] = s_i[s_j\partial_i s_j] = 0$.

where R is the radius of the circle that has the same curvature as the magnetic field line at the given point and $\hat{\mathbf{n}}$ is the direction perpendicular to $\hat{\mathbf{s}}$ in the direction "inside" the curvature (see Fig. 8). Thus in summary the Lorentz force can be rewritten:

$$\mathbf{F}_{\mathrm{L}} = -\frac{1}{8\pi}\boldsymbol{\nabla}_{\perp} B^2 + \frac{B^2}{4\pi R}\hat{\mathbf{n}} \tag{175}$$

Now it is clear how we should interpret these terms. The first term is a **magnetic pressure** that acts only in the direction perpendicular to the magnetic field. A bundle of magnetic field lines does not like to be squeezed. The second term says that the magnetic field does not like to be curved, and there is a restoring force which tries to bring it straight. It is a sort of **magnetic tension**. It is similar to the force that restores a violin string under tension. Waves can propagate along these "magnetic strings", and they are called Alfvén waves (see Section 5.6).

2.3 Magnetic flux freezing

One of the most useful concepts to intuitively visualise what happens to the magnetic field during the flow is that of **flux freezing**. In ideal MHD, the induction equation is given by (165):

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \tag{176}$$

to prove flux freezing we need to "uncurl" this equation to bring it to the form (98). Recall from the electromagnetism course that thanks to the fact that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ one can always represent $\mathbf{B}$ as the curl of a vector field:

$$\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B} \tag{177}$$

where $\mathbf{A}$ is called **vector potential**. This is not defined uniquely. Many possible equally good choices are possible for the vector potential. All the possible vector potentials are related by an equation of the type:

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} f \tag{178}$$

This freedom in the choice of $\mathbf{A}$ is known as gauge invariance. We choose the gauge in the following way. Pick any $\mathbf{A}$ that satisfies (177) at the initial time. Then *define* $\mathbf{A}$ to be the one that satisfies the following equation:

$$\partial_t \mathbf{A} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \boldsymbol{\nabla} f \tag{179}$$

where f is a given function. Is this a legit choice? Yes, because if $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$ at $t = 0$, taking the curl of (179) implies that this will be satisfied at all times (by comparison with 176 the evolution of $\boldsymbol{\nabla} \times \mathbf{A}$ and $\mathbf{B}$ is the same). Hence this defines a good possible gauge for $\mathbf{A}$. Actually by varying f you can obtain all the possible gauges. Equation (179) is exactly the same as (98), hence the theorem proved in that section applies. Therefore we obtain that the magnetic flux (equation 107)

$$\Phi_{\mathrm{B}}(t) = \oint_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} \tag{180}$$

$$= \oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}. \tag{181}$$

is constant as the open area S moves with the fluid, i.e.

$$\frac{D\Phi_{\mathrm{B}}(t)}{Dt} = 0. \tag{182}$$

It is intuitively useful to consider a bundle of magnetic field lines. Since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, field lines never end or start at any point, but they either extend to infinity or form closed loops. One can follow such lines either until they close or to infinity, and divide space into a collection of **flux bundles**. These flux bundles move as if they are "frozen" in the fluid. If two fluid elements are connected by a field line, they will always remain connected by a field line in the limit of ideal MHD. The induction equation is the one that describes how such bundles move with the flow.

When the conductivity is finite, (182) does not hold. In this case, an extra term will describe the "diffusion" of magnetic field due to the finite conductivity of the fluid. Field lines "diffuse" out of flux bundles, and flux bundles cannot be labeled in a time-independent way anymore. If two fluid elements are connected by a field line at a certain time, it is not true anymore that they will always remain connected by a field line. This allows the topology of magnetic field to change in ways that are not possible in ideal MHD, and this is related to the phenomenon of **magnetic reconnection**. In astrophysics, the conductivity is high enough that intuition based on ideal MHD is often useful as a first step, but there are processes in which magnetic diffusion plays an important role, for example magnetic reconnection processes are believed to be behind the sudden releases of energy during solar flares.

(a)
(b)
(c)

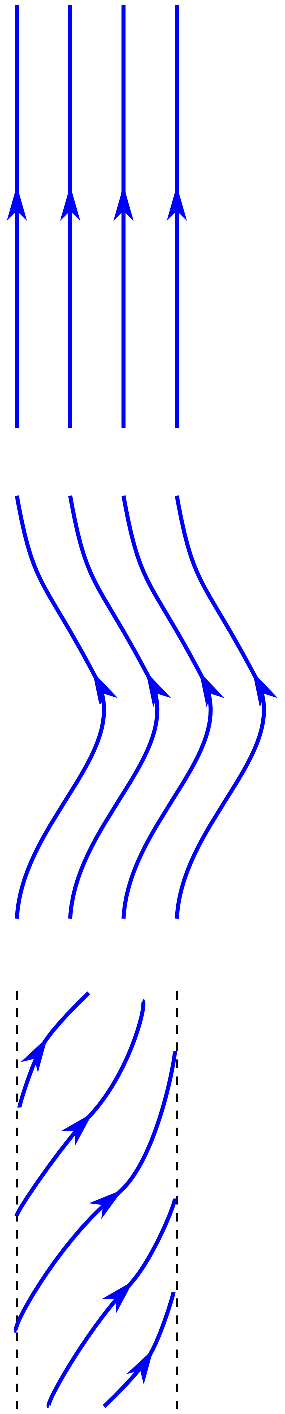

Figure 9: Illustration of flux freezing. (a) A straight column of magnetic field lines. (b) Magnetic configuration after bending the column. (c) Magnetic configuration after twisting the column.

2.4 Magnetic field amplification

According to equation (164), if $\mathbf{B}$ is zero everywhere at the beginning, it will always remain zero. What if there is a small magnetic field at the beginning? Can it be amplified? Consider the induction equation in the limit of ideal MHD (165). Using identity (50) and then substituting $\boldsymbol{\nabla}\cdot\mathbf{v}$ from the continuity equation in the form (73) we can rewrite it as[18]

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \left[\left(\frac{\mathbf{B}}{\rho}\right)\cdot\boldsymbol{\nabla}\right]\mathbf{v} \tag{183}$$

since $\mathbf{B}\cdot\boldsymbol{\nabla}$ projects the gradient operator multiplied by $|\mathbf{B}|$ along the magnetic field lines, this equation means that if $\mathbf{v}$ increases in the direction of the magnetic field then $\mathbf{B}/\rho$ increases. Consider for simplicity the incompressible case in which $\rho =$ constant, so that we can simplify it from equation (183). It is not difficult to devise examples in which a given $\mathbf{v}(\mathbf{x}, t)$ leads to magnetic field amplification (see Problem 2).

In a situation in which the fluid is in turbulent motion, magnetic fields entrained in the fluids are stretched and folded by the fluid motion, and are amplified in the process. Mechanical energy is converted into magnetic fields. This is the basic principle of **magnetic dynamos**. This is how it is believed that the Earth maintains its magnetic field. Magnetic dynamos are thought to be widespread in the universe, for example inside stars, in the interstellar medium, and in the launching of astrophysical jets. The ability of the magnetic fields to "feed" on the fluid flow energy explains why magnetic fields are so ubiquitous in the universe.[19]

[18]These are similar to the steps used in going from equation (91) to (94).

[19]Magnetic fields are indeed ubiquitous in the universe. Most planets, including the Earth, possess magnetic fields. In the Solar system, the intensities of these magnetic fields have been measured by sending spacecrafts carrying magnetometers near every planet. The intensities range from $\sim 3.5\times10^{-3}$ gauss at the poles for Mercury to ~ 0.6 gauss at the poles for Earth to ~ 8 gauss at the poles for Jupiter. The Sun has long been known to possess magnetic fields, with intensity ~ 1 gauss at the poles, which rises to ~ 3000 gauss at sunspots. Other main sequence stars also have magnetic fields, which can be measured through the Zeeman effect: when the atoms in their atmospheres are within a magnetic field, their aborption lines become split into multiple, closely spaced lines, and the spacing depends on the intensity of the magnetic field. White dwarfs have magnetic fields of $\sim 10^7$ to 10^8 gauss, pulsars of $\sim 10^{12}$ gauss and magnetars, a type of neutron star with particularly strong magnetic fields, can reach 10^{15} gauss. Magnetic fields are present in the interstellar medium. The disk of our Galaxy is permeated by a field which is about ~ 1–100×10^{-6} gauss near the Sun and increases to ~ 10–1000×10^{-6} in the Galactic centre regions. This is known i) more indirectly through the observation that dust grains can polarise the light from stars behind them, and can emit themselves polarised infrared light ii) more directly through the observations of the Zeeman splitting of some radio lines, like the 21 cm line of hydrogen iii) through the observation of synchrotron emission, which is produced by electrons revolving around magnetic fields. A number of other astrophysical objects, including supernovae remnants, jets, accretion discs, external galaxies quasars, cluster of galaxies are also known to possess magnetic fields.

Magnetic dissipation, due to finite conductivity, opposes amplifications and damps fields down. Thus in general one has to compare dissipation with amplification to see whether fields will increase or decrease. An equilibrium may be reached, in which dissipation balances amplification in a turbulent medium. Finally, note that we have not discussed how an initial magnetic field is generated. This is the topic of **Astrophysical batteries**. We will not discuss this topic here, but we merely point out that, when even the tiniest current is present, fields can be amplified. A state with no current does not seem a particularly stable one in a turbulent medium.

2.5 Conservation of momentum

Recall from your course in electrodynamics that, using only the Maxwell equations and nothing else, one can prove the following relation:

$$\partial_t([\mathbf{p}_{\text{fields}}]_i) = \partial_j M_{ij} - [\mathbf{F}_{\text{L}}]_i \tag{184}$$

where

$$\mathbf{F}_{\text{L}} = \rho_{\text{e}}\mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B} \tag{185}$$

is the Lorentz force per unit volume,

$$M_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\left(E^2 + B^2\right)\delta_{ij}\right] \tag{186}$$

is known as the **Maxwell stress tensor** and

$$\mathbf{p}_{\text{fields}} = \frac{1}{4\pi c}\mathbf{E} \times \mathbf{B} \tag{187}$$

is interpreted as the momentum stored in electric and magnetic fields.

Equation (184) is a statement of the conservation of momentum of electromagnetic fields. If you integrate it over a volume V, it says that the change in momentum of the fields, $\partial_t(\mathbf{p}_{\text{fields}})$, is due to two things i) outflow of momentum through the surface bounding the volume, which can be calculated from the Maxwell stress tensor ii) momentum that has been given to the charges through the Lorentz force. This equation is derived from the Maxwell's equation alone, and therefore it must be valid in MHD.

We can neglect some terms. Under our approximations we have already seen that we can approximate the Lorentz force as $\mathbf{F}_{\text{L}} \simeq (1/c)\,\mathbf{J}\times\mathbf{B}$ (equation 158). Using $E \sim (v/c)B$ (equation 155) and throwing away terms that are of order $(v/c)^2$ we can approximate the Maxwell stress tensor as:

$$M_{ij} \simeq \frac{1}{4\pi}\left[B_i B_j - \frac{1}{2}B^2\delta_{ij}\right] \tag{188}$$

We also note that $\mathbf{p}_{\text{fields}}$ is of order $(v/c)^2$ compared to other terms in (184). Thus we can rewrite this equation as

$$\partial_j M_{ij} = \frac{1}{c}\,[\mathbf{J}\times\mathbf{B}]_i \tag{189}$$

which can also be proved directly using (156) and that $\nabla\cdot\mathbf{B}=0$. Now take the Euler MHD equation (161) and rewrite it as (follow similar passages as in Section 1.8):

$$\partial_t(\rho v_i) = -\partial_j\left[(\rho v_i)v_j + \delta_{ij}P\right] + \frac{1}{c}\,[\mathbf{J}\times\mathbf{B}]_i \tag{190}$$

Using (189) we can rewrite this as:

$$\partial_t(\rho v_i) = -\partial_j\left[(\rho v_i)v_j + \delta_{ij}P\right] + \partial_j M_{ij} \tag{191}$$

which is a statement of momentum conservation for the fluid. Note that under our approximations the momentum contained in the fields is negligible, but might be present in a more general theory.

2.6 Conservation of energy

Recall from your course in electrodynamics that, using only the Maxwell equations and nothing else, one can prove the **Poynting theorem**, which states that:

$$\partial_t\left(u_{\text{fields}}\right) = -\boldsymbol{\nabla}\cdot\mathbf{S} - \mathbf{J}\cdot\mathbf{E} \tag{192}$$

where

$$u_{\text{fields}} = \frac{1}{8\pi}\left(E^2+B^2\right) \tag{193}$$

is the energy of the fields per unit volume, $\mathbf{J}\cdot\mathbf{E}$ is the rate at which the fields do work on charges per unit volume and

$$\mathbf{S} = \frac{c}{4\pi}\left(\mathbf{E}\times\mathbf{B}\right) \tag{194}$$

is the Poynting vector.

Under our approximations and neglecting terms of order $(v/c)^2$, we can rewrite Poynting theorem as (use 156 to substitute $\mathbf{J}$ and then the expression for $\mathbf{E}$ given by 163):

$$\partial_t \left(\frac{1}{8\pi} B^2 \right) = -\boldsymbol{\nabla} \cdot \mathbf{S} - \mathbf{J} \cdot \mathbf{E} \tag{195}$$

$$= -\boldsymbol{\nabla} \cdot \mathbf{S} - \left(\frac{c}{4\pi} \right)^2 \frac{1}{\sigma} |\boldsymbol{\nabla} \times \mathbf{B}|^2 + v_i \partial_j M_{ij} \tag{196}$$

This equation describes how the energy stored in the magnetic field changes with time. It can also be proved directly by using (156) and (163). The energy stored in the electric field is negligible.

Now let's find the equation that describes how the kinetic energy of the fluid changes. Taking the scalar product between $\mathbf{v}$ and the MHD Euler equation (161), assuming $\Phi = 0$ for simplicity, after some manipulations and using the continuity equation one arrives at:

$$\partial_t \left[\frac{\rho \mathbf{v}^2}{2} \right] + \boldsymbol{\nabla} \cdot \left[\left(\frac{\rho \mathbf{v}^2}{2} + P \right) \mathbf{v} \right] = P(\boldsymbol{\nabla} \cdot \mathbf{v}) + \frac{1}{c} \mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) \tag{197}$$

$$= P(\boldsymbol{\nabla} \cdot \mathbf{v}) - \mathbf{v} \cdot \boldsymbol{\nabla} \left(\frac{B^2}{8\pi} \right) + \mathbf{v} \cdot \frac{1}{4\pi} (\mathbf{B} \cdot \boldsymbol{\nabla}) \mathbf{B} \tag{198}$$

$$= P(\boldsymbol{\nabla} \cdot \mathbf{v}) - v_i \partial_j M_{ij} \tag{199}$$

where M_{ij} is given by (188) and in the last step we have used $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$.

Summing (196) and (199) the term $v_i \partial_j M_{ij}$ cancels out and we find

$$\partial_t \left(\frac{\rho \mathbf{v}^2}{2} + \frac{1}{8\pi} B^2 \right) + \boldsymbol{\nabla} \cdot \left[\left(\frac{\rho \mathbf{v}^2}{2} + P \right) \mathbf{v} + \mathbf{S} \right] = P(\boldsymbol{\nabla} \cdot \mathbf{v}) - \left(\frac{c}{4\pi} \right)^2 \frac{1}{\sigma} |\boldsymbol{\nabla} \times \mathbf{B}|^2 \tag{200}$$

The interpretation of the various terms is now clear. The term $v_i \partial_j M_{ij}$ appears with opposite signs in the change of energy of the fields and in the change of kinetic energy of the fluid. What one gains, the other loses. Hence it is the transfer of energy between these two components. The term $\boldsymbol{\nabla} \cdot \mathbf{S}$, when integrated over a finite volume, denotes the energy stored in fields that outflows from the bounding surface and goes into an adjacent volume.

The term

$$\left(\frac{c}{4\pi} \right)^2 \frac{1}{\sigma} |\boldsymbol{\nabla} \times \mathbf{B}|^2 \tag{201}$$

has always the same sign and therefore represents a loss of magnetic energy. It doesn't go into kinetic energy. Hence it must go into internal energy, and represents the energy dissipated per unit volume due to finite electric conductivity. Using the first law of thermodynamics then one sees that the replacement of (134) in case there is resistive dissipation is (note that we do not derive this equation, we impose it):

MHD equation of heat transfer.

$$\rho T \frac{Ds}{Dt} = \left(\frac{c}{4\pi}\right)^2 \frac{1}{\sigma} \left|\boldsymbol{\nabla} \times \mathbf{B}\right|^2 \tag{202}$$

Using this last equation one can finally obtain a statement of the total conservation of energy:

$$\partial_t \left(\frac{\rho \mathbf{v}^2}{2} + \frac{P}{\gamma - 1} + \frac{1}{8\pi} B^2\right) + \boldsymbol{\nabla} \cdot \left[\left(\frac{\rho \mathbf{v}^2}{2} + P + \frac{P}{\gamma - 1}\right)\mathbf{v} + \mathbf{S}\right] = 0 \tag{203}$$

There are three types of energy in MHD: the kinetic energy of the fluid elements, the internal energy of the fluid elements, and the energy stored in the magnetic field. The three terms in the time derivative on the LHS are these three contributions respectively. Of course, this equation and (202) and therefore (203) are not valid in case we consider an incompressible or isothermal gas, while the equations preceding (202) in this section are always valid.

2.7 How is energy transferred from fluid to fields if the magnetic force does no work?

A magnetic field exerts on a charged particle a force that is directed perpendicular to the velocity of the charged particle. This means that the force exerted by the magnetic field cannot do work on the particle. In our calculations, we have approximated the Lorentz force by keeping only the magnetic contribution, which arises from magnetic forces exerted on charged particles. Since this is the only way that matters interacts with the fields, one might come to the conclusion that fields cannot exchange energy with the fluid. This conclusion is *wrong*. Indeed, we have found that field and fluids *can* exchange energy, even in the limit of ideal MHD. The rate at which kinetic energy is converted into magnetic fields is given by (equation 197):

$$\frac{1}{c}\mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) = -v_i \partial_j M_{ij} \tag{204}$$

This transfer happens through the magnetic force, $\mathbf{J} \times \mathbf{B}$. How is this possible if the magnetic force does no work on particles?

To understand this, let us first consider the situation depicted in Fig. 10 which may be familiar from your electromagnetism course. A circuit is made by a U-shaped conducting wire and a conducting rod of length L which can slide without friction. A uniform, external magnetic field $\mathbf{B}$ points inside the page. The rod is pulled to the right by you at constant velocity $\mathbf{v} = v\hat{\mathbf{e}}_x$. The way to deal with this situation that might look familiar from your EM course is something like the following. You say that there will be an induced current I in the circuit. Hence there will be a "magnetic force" of magnitude $F = BLI$ directed to the left. This force is directed parallel to $\mathbf{v}$, so it certainly looks like it is doing work. But the magnetic force is perpendicular to the velocity of moving charges, so it cannot do work! What's going on here? Is the magnetic force doing work or not? If not, what is it?

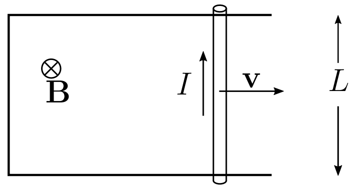

Figure 10: A closed circuit with a conducting rod that slides at constant velocity $\mathbf{v}$ immersed in a constant magnetic field pointing into the page.

For simplicity, we assume that the moving charges are negative electrons, while positive charges are glued to the bulk of the rod and move with it. The important point to realise is that there are two components to the electrons velocity $\mathbf{u} = v\hat{\mathbf{e}}_x - v_{\text{drift}}\hat{\mathbf{e}}_y$: a horizontal component v to the right and a vertical velocity v_{drift} which gives the current along the rod. This means that the magnetic force $\mathbf{F}_B = e\mathbf{u} \times \mathbf{B} = -evB\hat{\mathbf{e}}_y - ev_{\text{drift}}B\hat{\mathbf{e}}_x$ where $e > 0$ is the charge of an electron points down and to the left as shown in the diagram in Fig. 11. The other forces acting on an electron are the force that you exert on the rod $\mathbf{F}_{\text{you}}$ (which we assume the rod can transmit to the electrons as if it has some kind of "frictionless walls") and the resistive force $\mathbf{F}_R$ that the electrons feel because they are moving at a different velocity with respect to the bulk of the rod and to the positive charges. There must be such a force, because the electrons are moving at a constant velocity, hence the total force on them must be zero.

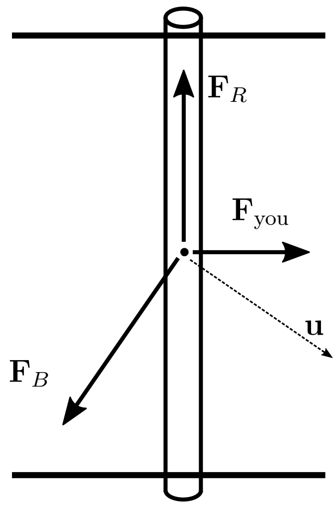

Figure 11: Forces acting on an electron inside the rod.

Which forces do work? The magnetic force clearly does no work because it's perpendicular to $\mathbf{u}$. The force you exert and the resistance force both do work, and if the total force is zero the two rates must be equal and opposite. Thus you do the work, which is entirely dissipated within the rod by the resistive force. The magnetic field only acts as a "mediator", exactly like the normal force is a mediator if you push with a horizontal force a block up a frictionless inclined plane in the presence of gravity, Fig. 12. The normal force does no work, and you are pushing the block horizontally, yet it goes up thanks to the mediation of the normal force which converts your horizontal push into a vertical push against gravity. In the wire, you push the electrons to the right, and they "climb" against the resistive force, vertically down the rod. In this analogy, $\mathbf{N}$ corresponds to $\mathbf{F}_B$ and $\mathbf{F}_g$ to $\mathbf{F}_R$.

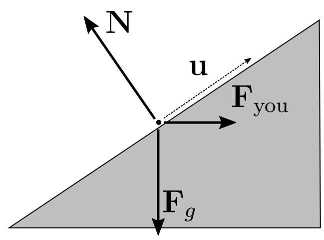

Figure 12: Analogy between forces on electrons in the rod and an inclined plane.

If you now make the MHD connection and imagine that the rod is actually a fluid element immersed in a MHD fluid, and your push is actually the push of neighbouring fluid elements, you see that in this example all the push the fluid element gets from his neighbours would be dissipated and would go into heat. However, in MHD you also have electric fields although they never enter explicitly into the final equations. In ideal MHD, you know that this is given by equation (166). If you extend the rod example and add this electric field acting on the rod and repeat the analysis, you see that the electric force $\mathbf{F}_E = -e\mathbf{E} = e\mathbf{v} \times \mathbf{B} = evB\hat{\mathbf{e}}_y$ acting on an electron *exactly balances* the vertical component of the magnetic force! So you don't need the resistive force anymore to keep the total force on an electron to be zero. In this case, the push on a fluid element by its neighbour does work *against* the electric field, and therefore this energy goes into energy of the fields. This is exactly the same as the inclined plane analogy, by pushing the block horizontally you are actually *increasing* its gravitational potential energy, where gravity plays the role of the electric field! This is how energy is transferred from the fluid kinetic energy to the fields energy in MHD. Magnetic forces do no work, only acts as mediators.

In the case of finite conductivity and in the presence of electric fields you have a mixed situation in which part of your work will go into increasing the energy stored in the fields, and part will be dissipated.

Note that the presence of the electric field it is essential for transferring energy between fields and matter, although it never appears explicitly in the MHD equations. While the electric part of the Lorentz force can be neglected on the fluid element *as a whole*, because the net charge is negligibly small, the electric force *cannot* be neglected when we consider positive and negative charges separately.

In the above we assumed that $\mathbf{F}_{\text{you}}$ can transmit the force you exert on the rod to the electrons as if it has some kind of "frictionless walls". In a fluid, this is the same as assuming that one fluid element exerts forces (such as pressure) on the one adjacent to it only through electrons, which of course is not really true. In the rod, charges can accumulate on the sides giving rise to a horizontal electric field who does this force (this is the Hall effect). In a fluid, one must model how the electrons and ions exchange momentum between them, and possibly with the neutral particles (if present). For example, if one assumes that there is some kind of friction between different components proportional to their relative velocity (as in Drude model of electric conduction), one finds that electrons must have a horizontal component of v_{drift} which must be accounted for, otherwise the friction force would be zero and the force balance would not be possible. Other collective forces may be present, for example in a metal, felt by the "sea of electrons". However, these topics lie outside the scope of the present lecture notes, and the reader is referred to a more specialised treatment.

2.8 Summary

The MHD equations are

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{205}$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} \left(P + \frac{B^2}{8\pi} \right) - \rho \boldsymbol{\nabla} \Phi + \frac{1}{4\pi} \left(\mathbf{B} \cdot \boldsymbol{\nabla} \right) \mathbf{B} \tag{206}$$

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) - \boldsymbol{\nabla} \times \left(\frac{c^2}{4\pi\sigma} \boldsymbol{\nabla} \times \mathbf{B} \right) \tag{207}$$

Similarly to the purely hydrodynamic case, these alone are not enough to uniquely specify the time evolution and must be complemented by another equation. Possible choices include:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0 \qquad \text{incompressible} \tag{208}$$

$$P = P(\rho) \qquad \text{barotropic} \tag{209}$$

$$\rho T \frac{Ds}{Dt} = \left(\frac{c}{4\pi} \right)^2 \frac{1}{\sigma} \left| \boldsymbol{\nabla} \times \mathbf{B} \right|^2 \qquad \text{equation of heat transfer} \tag{210}$$

In the limit of ideal MHD, $\sigma \to \infty$, the induction equation reduces to

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \tag{211}$$

and the equation of heat transfer reduces to

$$\frac{D}{Dt} \left(\log P \rho^{-\gamma} \right) = 0. \tag{212}$$

3 Hydrostatic equilibrium

3.1 Polytropic spheres

Consider the hydrostatic equilibrium of a self–gravitating sphere with a polytropic equation of state:

$$P = K \rho^{\gamma} \tag{213}$$

Polytropic models have been the primary models for the discussion of stellar interiors for a long time in the early days of astrophysics, when one of the major concerns was the internal structure of the Sun.[20] This framework has been eventually superseded by the modern equations of stellar structure, which include the important effects of energy transport within the star.[21] This also happened thanks to the availability of computers who could finally solve these equations in a practical way. Nowadays, polytropic models are still useful in describing the structure of white dwarfs and the dense degenerate cores that appear in some stages of stellar evolution. They are also a reasonable approximation in fully convective stars. The main reason why we study them is because they are instructive, elegant and simple.

The equation of hydrostatic equilibrium is:

$$\frac{1}{\rho}\boldsymbol{\nabla} P = -\boldsymbol{\nabla}\Phi \tag{214}$$

where Φ is the potential self–generated by the sphere, given by

$$\boldsymbol{\nabla}\Phi = \frac{GM(r)}{r^2} \tag{215}$$

and $M(r)$ is the mass within radius r,

$$M(r) = \int_0^r \rho\, 4\pi r^2 \mathrm{d}r \tag{216}$$

Substituting (213) and (215) into (214) we get

$$\gamma K r^2 \rho^{\gamma-2}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -GM(r) \tag{217}$$

and differentiating this last equation

$$\gamma K \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho^{\gamma-2}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -G\, 4\pi\rho r^2 \tag{218}$$

This equation can be reduced to dimensionless form by writing

$$\gamma = 1 + \frac{1}{n} \tag{219}$$

$$\rho = \rho_c \theta^n \tag{220}$$

$$r = a\xi \tag{221}$$

[20]The theory was initiated by Lane in 1869 and reached its most complete form in Emden's book *Gaskugeln* published in 1907. Then it has continued to be be used until the 1950s.

[21]Remember also that at the beginning of 1900 people believed that stars derive their energy of radiation from work done by gravity as they contract.

where n is the **polytropic index** (see also discussion below equation 29), ρ_c is the central density, θ is a new dimensionless variable which replaces ρ, a is a characteristic length to be determined and ξ is a dimensionless radius. Substituting in (218) we find:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\left[\frac{4\pi G\rho_c^{1-1/n}a^2}{(n+1)K}\right]\theta^n \tag{222}$$

To make equations look nice we choose a such that the quantity between parentheses is unity, i.e.

$$a^2 = \frac{(n+1)K}{4\pi G\rho_c^{1-1/n}} \tag{223}$$

note that a has dimensions of length and is the typical length-scale of the problem. We obtain

$$\boxed{\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n} \tag{224}$$

Lane-Emden equation.

this is known as the **Lane-Emden equation**.

This equation is usually solved under the following boundary conditions:

$$\theta(0) = 1 \quad \text{by the definition of } \theta \text{ in equation (220).} \tag{225}$$

$$\theta'(0) = 0 \quad \text{because we require no density “cusp” at the center.} \tag{226}$$

where the superscript $'$ indicates derivative with respect to ξ.

The Lane-Emden equation can be solved analytically only in the particular cases $n = 0, 1$ and 5, not for general n (see Problem 1). In all other cases, it must be solved numerically. Fig. 13 shows the solution of the Lane-Emden equation for a bunch of values of n. It is found that for $n < 5$ the solutions decrease monotonically and have a zero at $\xi = \xi_{\text{max}}$, i.e. $\theta(\xi_{\text{max}}) = 0$. At this radius the density vanishes and the hypothetical star “ends”. For $n > 5$ the solutions are infinite, i.e. $\xi_{\text{max}} = \infty$, the density never vanishes the star formally extends to infinity! Thus, if such a star is truncated at some radius and outside there is void, the outer layers will start expanding unless there is a confinement pressure. The need of a confinement pressure to prevent the gas of the outer layers from flying away is at the origin of stellar winds (see Section 4.1).

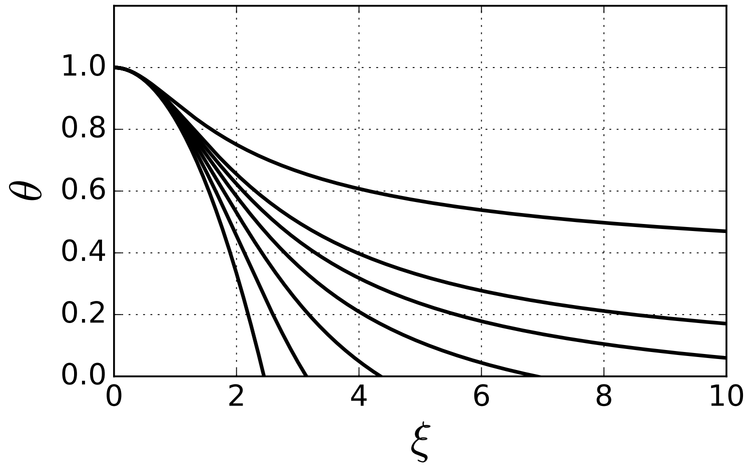

Figure 13: Solutions of the Lane-Emden equation (224) for different values of n. From bottom to top: $n = 0, 1, 2, 3, 4, 5, 10$. For $n < 5$ the solutions cross the line $\theta = 0$ at increasingly large radii, while they never cross it for $n \geq 5$.

We can find a nice expression for the total mass of the star:

$$M = \int_0^R \rho\, 4\pi r^2 \mathrm{d}r \tag{227}$$

$$= \int_0^{\xi_{\max}} (\rho_c \theta^n)\; 4\pi(a\xi)^2 \mathrm{d}(a\xi) \tag{228}$$

$$= 4\pi\rho_c a^3 \int_0^{\xi_{\max}} \theta^n\, \xi^2 \mathrm{d}\xi \tag{229}$$

$$= -4\pi\rho_c a^3 \int_0^{\xi_{\max}} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) \mathrm{d}\xi \qquad \text{By Eq. (224)} \tag{230}$$

$$= 4\pi\rho_c a^3 \xi_{\max}^2 \left|\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=\xi_{\max}} \tag{231}$$

$$= 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n}\, \xi_{\max}^2 \left|\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=\xi_{\max}} \tag{232}$$

The radius of the star is given by

$$R = a\xi_{\max} = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n} \xi_{\max} \tag{233}$$

n	$\xi_{\max}^2 \left\|\mathrm{d}\theta/\mathrm{d}\xi\right\|_{\xi=\xi_{\max}}$
0	4.8960
1/2	3.7884
1	3.1416
3/2	2.7141
2	2.4111
5/2	2.1872
3	2.0182
7/2	1.8906
4	1.7972
9/2	1.7378
5	1.7320

Table 1

Once we know the value of $\xi_{\max}$ and $\xi_{\max}^2 \left|\mathrm{d}\theta/\mathrm{d}\xi\right|_{\xi=\xi_{\max}}$ we can calculate the mass of the star and its radius as a function of its central density. Table 1 gives values of the quantity $\xi_{\max}^2 \left|\mathrm{d}\theta/\mathrm{d}\xi\right|_{\xi=\xi_{\max}}$ for various values of n (for $n > 5$ it is infinite!).

Eliminating ρ_c from (232) and (233) gives the mass–radius relation for a polytrope:

$$M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)} \xi_{\max}^{(3-n)/(1-n)} \xi_{\max}^2 \left|\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=\xi_{\max}} \tag{234}$$

White dwarfs are sustained by a gas of degenerate electrons (the pressure of the atomic nuclei is negligible in white dwarfs. The density is not high enough to make them degenerate). A gas of degenerate electrons has a polytropic equation of state both in the non–relativistic limit, in which $n = 3/2$ ($\gamma = 5/3$), and in the ultra–relativistic limit, in which $n = 3$ ($\gamma = 4/3$) (see for example the book by Shapiro & Teukolsky, *The physics of compact objects*).

Imagine constructing white dwarfs by slowly increasing the value of the central density ρ_c. At first the gas is non–relativistic and the larger the value of ρ_c the larger the stellar mass M ($n = 3/2$ in equation 232). At the same time, the radius of the star decreases (equation 234). Hence the electrons become more and more packed, until they become relativistic and $n \to 3$. For this value of n, we see from (232) that the mass of a star *does not depend on the central density* ρ_c! Hence the mass of the star tends to a finite value as $\rho_c \to \infty$. The mass of a white dwarf cannot exceed this value. This limiting mass is called the Chandrasekhar Mass, and it can be evaluated using the value $\xi_{\rm max}^2 \left|{\rm d}\theta/{\rm d}\xi\right|_{\xi=\xi_{\rm max}} = 2.0182$ for $n = 3$ from Table (1) and that for a gas of relativistic degenerate electrons the equation of state is:

$$P = \frac{\hbar c (3\pi^2)^{1/3}}{4} n_{\rm e}^{4/3} \tag{235}$$

where $n_{\rm e}$ is the number density of electrons, $\hbar$ is the Planck constant and c is the speed of light. The total density ρ is related to the number of electrons by $\rho = n_{\rm e}\mu_{\rm e}m_{\rm p}$, where $m_{\rm p}$ is the mass of the proton and $\mu_{\rm e}$ is the average molecular weight per electron in units of the proton mass, which depends upon the chemical composition of the star. A typical value is $\mu_{\rm e} = 2$. Using this we calculate the value of K as

$$K = \frac{\hbar c (3\pi^2)^{1/3}}{4} \left(\frac{1}{\mu_{\rm e} m_{\rm p}}\right)^{4/3} \tag{236}$$

Hence the Chandrasekhar mass is

$$M_{\rm Ch} = \frac{(3\pi)^{1/2}}{2} \left(\frac{\hbar c}{G}\right)^{3/2} \left(\frac{1}{\mu_{\rm e} m_{\rm p}}\right)^{2} \xi_{\rm max}^2 \left|\frac{{\rm d}\theta}{{\rm d}\xi}\right|_{\xi=\xi_{\rm max}} \simeq 1.44 M_\odot \tag{237}$$

The existence of this upper mass limit for white dwarfs is what leads to the production of neutron stars and black holes.

3.2 The isothermal sphere

The analysis of Section 3.1 does not work for the particular case $\gamma = 1$, which corresponds to $n \to \infty$. This case must be treated separately. $\gamma = 1$ corresponds to the $T =$ constant in the ideal gas equation of state (16). For this reason, the solutions obtained in this case are called **isothermal spheres**.

Writing the equation of state as

$$P = c_{\rm s}^2 \rho \tag{238}$$

where $c_{\rm s}^2 = kT/\mu$ is a constant, substituting into the condition of hydrostatic equilibrium (214) and using the gravitational potential given by (215) we find:

$$r^2 \frac{c_{\rm s}^2}{\rho} \frac{{\rm d}\rho}{{\rm d}r} = -GM(r) \tag{239}$$

And differentiating this equation we find (use 216 for the right hand side)

$$c_{\mathrm{s}}^2 \left(r^2 \frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} \right) = -4\pi G r^2 \rho \tag{240}$$

This equation can be reduced to dimensionless form by writing

$$\rho = \rho_c \exp\left(-\psi\right) \tag{241}$$

$$r = \left[\frac{c_{\mathrm{s}}^2}{4\pi G \rho_c} \right]^{1/2} \xi = a\xi \tag{242}$$

we obtain

$$\boxed{\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left(\xi^2 \frac{\mathrm{d}\psi}{\mathrm{d}\xi} \right) = \exp\left(-\psi\right)} \tag{243}$$

Emden equation, isothermal case.

This is the analog of the Lane-Emden equation for the isothermal case. The boundary conditions are

$$\psi(0) = 0 \quad \text{if } \rho_c \text{ is the central density in equation (241).} \tag{244}$$

$$\psi'(0) = 0 \quad \text{if we require no density “cusp” at the center.} \tag{245}$$

The solution with these boundary conditions is shown in Fig. 14. It is found that this solution has the following properties:

i ψ is monotonically increasing, so that the density is monotonically decreasing with radius (equation 241).

ii Near the origin the solution behaves as

$$\psi \simeq \frac{\xi^2}{6} \qquad (\xi \to 0). \tag{246}$$

iii Far from the origin the solution behaves as

$$\psi \simeq \log\left(\frac{\xi^2}{2}\right) \qquad (\xi \to \infty). \tag{247}$$

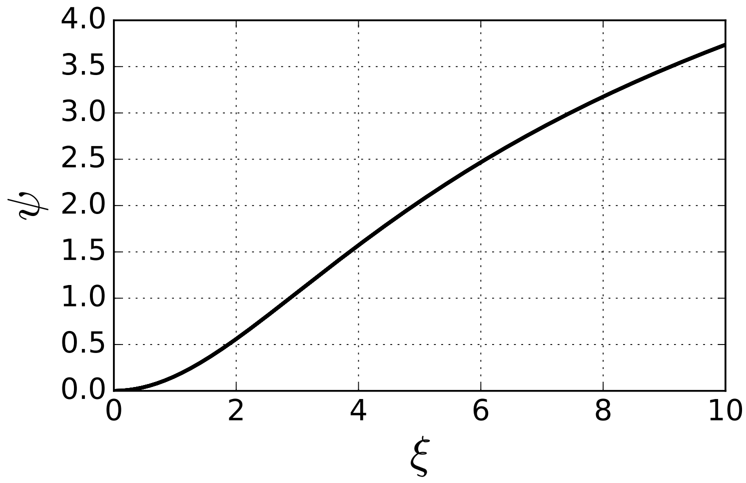

Figure 14: Solution of the equation (243).

Proof of the properties To give you a taste of how astronomers used to deal with such equations before computers became cheap, let us prove these statements. Property (ii) can be proved by assuming an expansion of the form

$$\psi = a\xi^2 + b\xi^3 + c\xi^4 + \ldots \tag{248}$$

in the vicinity of $\xi = 0$ and, after substituting in (243), equating the coefficients of equal powers of ξ. To prove property (i) note that according to (243) the quantity

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right) \tag{249}$$

is always positive. Hence, the quantity $\xi^2\mathrm{d}\psi/\mathrm{d}\xi$ is monotonically increasing and in particular if it starts positive it remains positive, which means that $\mathrm{d}\psi/\mathrm{d}\xi$ is always positive, which means that ψ is monotonically increasing. To prove property (iii) let us first prove that $\psi \to \infty$ as $\xi \to \infty$. Assume by absurdum that $\psi \to A$ where A is a positive number (this is the only other possibility, ψ must tend to something because we have just shown that it is monotonically increasing). Then it exists a ξ_0 such that for $\xi > \xi_0$ the right hand side of equation (243) is always greater than a positive constant B and we can write:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right) > \xi^2 B \tag{250}$$

Integrating this equation we find that $\mathrm{d}\psi/\mathrm{d}\xi \to \infty$ as $\xi \to \infty$, which implies $\psi \to \infty$, hence the absurdum. This proves that $\psi \to \infty$ as

$\xi \to \infty$. Now perform the following substitution in equation (243)

$$t = \log \xi \tag{251}$$
$$x = -\psi + 2t - \log 2 \tag{252}$$

we obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} + 2\exp x - 2 = 0 \tag{253}$$

This is just the equation of a damped one–dimensional particle which moves in the potential $V(x) = 2\exp(x) - 2x$. The term $\mathrm{d}x/\mathrm{d}t$ is a friction proportional to speed. The potential $V(x)$ has one global minimum at $x = 0$ and tends to $+\infty$ as $x \to \pm\infty$. As $t \to \infty$ a particle, whatever its initial conditions, will just sit still at the minimum of $V(x)$. Hence as $t \to \infty$ we have that $x \simeq 0$. Going back to the original variables through (251) and (252) this means that $\psi \simeq \log(\xi^2/2)$ as $\xi \to \infty$, which is property (iii).

Note that (247) implies that at large radii the density behaves as

$$\rho \simeq \frac{2\rho_c}{\xi^2} \qquad (r \to \infty) \tag{254}$$

Hence *an isothermal sphere of infinite radius has infinite mass*!

Note also that the function

$$\psi = \log\left(\frac{\xi^2}{2}\right) \tag{255}$$

is a solution of (243) in itself, although it does not respect the boundary conditions above. The density profile corresponding to this solution is called the **singular isothermal sphere**. The corresponding density profile is $\rho = 2\rho_c/\xi^2$ and diverges at the origin. Obviously in this case ρ_c does not have the meaning of central density. Note that despite the divergence, the mass in a neighbourhood of the origin is finite. As $\xi \to \infty$ the regular solution discussed above oscillates around this singular solution.

3.3 Stability of polytropic and isothermal spheres

In the previous two sections we have found the hydrostatic equilibrium of polytropic and isothermal spheres. We have done this for a polytropic index going from $n = 0$ to $n = \infty$. The limit $n = 0$ corresponds to an incompressible fluid ($\gamma = \infty$), $n = \infty$ to an isothermal fluid ($\gamma = 1$).

A natural question is whether these spheres could collapse under their own gravity. Gravity gets stronger at small radii, and if the supporting pressure is too weak, one may conceive a runaway collapse. If we take one of the states calculated in the previous sections and give it a little kick, will it just crumble under its own gravity, or will it be able to support it? In other words, are these stable or unstable equilibrium states?

Let us start with the case of a ball of incompressible fluid. Imagine such a ball, self–gravitating and floating in vacuum. It is intuitive that such a ball cannot collapse. Any gravitational force, however strong, can be compensated by the pressure force. If you give the sphere a small blow, you will find little waves running on the surface of the sphere, but no collapse.

When $n > 0$ the answer is not so clear anymore. For an incompressible fluid ($n = 0$), the tiniest increase in density means an infinite increase in pressure. As n is increased, the equation of state "softens", in the sense that for the same increase in density (say, a factor of 2) you get a smaller increase in pressure.

So, what happens for our spheres? If one carries out a linear stability analysis[22] one finds that for $n < 3$ ($\gamma < 4/3$) the system is stable, while for $n > 3$ ($\gamma > 4/3$) the system is unstable. It is assumed here that pressure and density fluctuations are connected by the relation (213). This result can be obtained through a rigorous analysis, but there is a simple heuristic argument that explains it. The argument goes as follows.

In an adiabatic flow (i.e., when 22 is satisfied), in the absence of dissipation, each fluid element is like a little hermetic bag that can be compressed and expanded. It can store internal energy when gets compressed and later release it again when is re–expanded, like a little spring. A big ball of gas is made by many of these little springs and so in this sense is just a giant spherical spring. If this ball–spring is stiff enough, it can resist collapse against gravity. To find out whether a ball for a given γ can resist let's ask: as the gas contracts, can the decrease in gravitational potential energy keep up with the increase of internal potential energy stored in our ball–spring?

Consider a constant ball of gas of constant mass M, constant density ρ and volume V. The gravitational energy of a ball can be roughly

[22]Like for any other steady state, one can find the linear stability of polytropic and isothermal spheres by the following steps: i) linearise the fluid equations around the equilibrium state ii) find the eigenmodes of oscillation of the system around the equilibrium state. If at least one eigenfrequency of these modes is such that the mode increases exponentially in time, the system is unstable.

estimated as

$$U_g = \frac{GM^2}{R} \tag{256}$$

while the total internal energy stored in the ball (which corresponds to the "internal energy of the spring") as (see equation 19)

$$U_i = \frac{P}{\gamma - 1} V \tag{257}$$

where $V = 4/3\pi R^3$ and the pressure is given by $P = K\rho^\gamma$. Writing $\rho = M/V$ and putting all together we find

$$U_i = \frac{KM^\gamma}{\gamma - 1} \left[\frac{4\pi}{3}\right]^{1-\gamma} R^{-3(\gamma-1)} \tag{258}$$

Hence the ratio between internal and gravitational energy is

$$\frac{U_i}{U_g} \propto R^{-3\gamma+4} \tag{259}$$

You see that if $\gamma > 4/3$, the internal energy grows faster than the gravitational energy as $R \to 0$. This means that as the ball contracts, gravity cannot supply enough energy. There cannot be collapse. The ball–spring bounces back. When $\gamma < 4/3$, gravity can supply all the energy needed, and the left–over goes into kinetic energy of the collapsing gas. This heuristic argument explains the stability results stated above.

These results have a connection with the theory of stellar pulsation. Variable stars like **cepheids** and **RR Lyrae** are radially pulsating stars. One can calculate the modes of radial oscillations which are analogous to standing sound waves in an organ pipe. In a star, the pipe is analogous to a cylinder capped at one end (the centre of the star) but open at the other (the surface of the star). As one might suspect based on the results of this section, it turns out that pulsating stars are stable only if $\gamma > 4/3$, while they are unstable if $\gamma < 4/3$. If you are interested in the topic, have a look at the book by J. P. Cox, *Theory of Stellar Pulsation*, Princeton University Press, (1980).

Further discussion. We can move one step further. We have assumed above that pressure and density of the perturbations are connected by the relation (213), where γ and K are the same constants used to calculate the equilibrium state. However, this need not be necessarily the case. We can imagine setting up an isothermal sphere, and when we give it a little kick its subsequent evolution will be such that each fluid element evolves adiabatically according to (22). In this case there is no global $P = K\rho^{\gamma}$ relation, but the constant K will be different for each fluid element. I.e., this is an adiabatic but not isentropic flow. Physically this means that when the sphere starts collapsing, the temperature of the fluid elements will increase because they are compressed. Could this rise in temperature halt collapse? Yes, if it's high enough. When is it high enough? The answer is $\gamma > 4/3$. To understand why, we can in fact use again the argument above. The only difference is that now the ball is made by many little springs that are not all equals, because they have different scalings K. However, they have the same γ, which is what matters. Hence, the way in which the internal energy scales is the same, and the argument is still valid.

One more thing. We have seen that the isothermal sphere is stable against adiabatic perturbations if $\gamma > 4/3$. This is because, contrary to the case of purely isothermal perturbations, when the sphere starts contracting, the temperature of fluid elements rises and the increased pressure is enough to halt the collapse. However, what if there was something able to remove some heat from these fluid elements, so that their temperature (and therefore pressure) drops a bit? Could this reactivate the collapse even for $\gamma > 4/3$? The answer is yes. This may happen for example when a molecular cloud collapses to make a star. In this case, the extra heat may be removed in the form of photons that escape the system (radiative losses). There is another interesting case where this may happen. During a typical collapse, the central parts are contracted more and therefore they get hotter than the outer parts. If there is a little bit of thermal conduction, some heat will be transported from the inner parts to the outer parts. The inner parts will contract a bit further, they will get even hotter, and the extra heat will be conducted to the outer parts, and so on. Eventually, this process leads to collapse. Heat continually flows out from the central parts to the outer parts. This type of collapse is called **gravothermal catastrophe**. It is thought to be relevant in the dynamics of globular clusters, where single stars are considered gaseous particles, although the analogy only extends so far since the mean free path for a star is larger than the size of the system and so the fluid approximation breaks down. In these examples, the speed of collapse is limited by the ability to remove heat from the centre, i.e. by how fast thermal conduction or radiative losses are. If these are slow, the collapse may happen through a sequence of equilibrium states, and the timescale for collapse is not a dynamical time scale, it is the scale of heat removal.

Finally, let us comment the following question: are the isothermal spheres a *thermodynamic* equilibrium? At first, it seems to be so; an isothermal sphere is a sphere of constant temperature. Let us refine the question: if you release N particles interacting only through Newtonian gravity in a spherical box with reflecting walls and let the particles relax for an infinite amount of time, do they settle into a truncated isothermal sphere? The short answer is no, they will eventually collapse to a point. So isothermal spheres are *not* a thermodynamical equilibrium, at least in a strict sense. They are only metastable states even in the purely fluid sense, essentially because of the gravothermal catastrophe explained above: any real such fluid with a non–zero thermal conduction will end up with a centre that contracts indefinitely while the outer parts receive the excess heat. However the fluid model is just an approximation of the N body problem, because it cannot address what happens on scales smaller than fluid elements, and fluid elements cannot really exchange particles. A careful treatment of the N body problem would require a **thermodynamics of self–gravitating systems**. Unfortunately, such a theory has not been successfully developed yet. One of the main problems is that gravity is a long–range force, so if you split a system in two you cannot neglect their interaction energy, and the majority of the results derived in standard thermodynamics are not valid. However, it is experimentally found in simulations that if you release N particle, they spend very long times in quasi–equilibrium configurations before collapsing to a point, and these quasi–equilibrium states are more or less independent of the initial conditions (given the total energy). The Navarro, Frenk & White profiles are an example in the collisionless limit. This suggests that there should be a simple statistical argument from which these equilibrium states can be derived. However, this is an open problem. For an introduction to this interesting problem, you can look for example at the review by T. Padmanabhan *Statistical mechanics of gravitating systems*, Physics reports, (1990) and papers that cite this article.

4 Spherical steady flows

4.1 Parker wind

The Sun has a **solar wind** which blows out of its surface. Material from the million degree outer atmosphere of the Sun, the corona, continually expands, eventually reaching supersonic velocities of $\sim 1000\,\mathrm{km\,s^{-1}}$. The

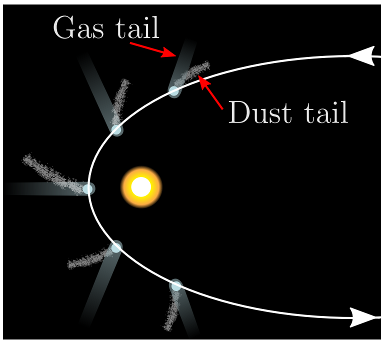

Figure 15: Schematic illustration of a comet orbit. Gas and dust tails have different directions. The gas tail points in the direction away from the Sun because of the solar wind.

solar wind causes comets to have a tail of gas (Fig. 15) pointing away from the Sun. When the solar wind reaches the Earth it impacts against its magnetic field, making it flapping. The solar wind sweeps out through the solar system, eventually being halted at distances of the order of 10^2 AU by the feeble pressure of the interstellar medium. What is the mechanism responsible for the solar wind?

Biermann (1948) was the first to point out that the observed tails of comets could be explained by gas streaming at high speed outwards from the Sun. Astronomers thought that, if such an outflow existed, magnetic fields could be a key ingredient for an explanation. However, observations suggested that the solar wind blows regardless of the existence or absence of flares or the general level of magnetic activity of the Sun. This clue indicated that the wind in its simplest form could be understood as a purely hydrodynamical phenomenon.

At the time, astronomers thought that the tenuous atmosphere of a star like the Sun was essentially in a hydrostatic equilibrium without any radial motion. Parker (1958) realised that, if the temperature does not decrease fast enough with radius, the pressure at infinity does not vanish and therefore an external confinement pressure is required to keep a static atmosphere from flying away. He showed that the interstellar medium was not dense enough to provide this confinement pressure, and argued that the only alternative was to admit that radial motions are present. He went on to show that these motions must become supersonic after a certain radius. Hence, much to the surprise of his contemporaries that initially did not really believe him, he showed that the presence of a supersonic expanding wind is not only possible, but is dynamical necessary. Here we review his theory. Magnetic fields refine this picture, but are not needed to understand the solar wind in its simplest form.

During our discussion we will assume that the gas is isothermal (see equation 17) with sound speed c_s = constant. This is only approximately true in the Sun's corona. In reality the temperature decreases with radius. However, the essence of the physical argument can be understood within this simple approximation.

Necessity of radial motions First, let us show that for a spherical stellar atmosphere *in vacuum* radial motions are not only possible but necessary. Assume there are no radial motions and that the gravitational potential is dominated by the central star (in other words, the contribution of the atmosphere to the potential is negligible). Then hydrostatic equilibrium requires

$$0 = -\frac{c_\mathrm{s}^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2}. \tag{260}$$

Integrating this equation we find

$$\rho = \rho_0 \exp\left(\frac{GM}{c_\mathrm{s}^2 r}\right), \tag{261}$$

where ρ_0 is an integration constant, which cannot be zero otherwise $\rho = 0$ everywhere. Taking the limit $r \to \infty$ we see that ρ, and therefore the pressure, does not vanish at infinity. Hence, without a confinement pressure this type of equilibrium is not possible, and we must admit the presence of radial motions.

We have discussed this in the isothermal case. What if we drop this assumption? Turns out that if the temperature drops too fast with radius, the pressure *can* go at zero at infinity, and this result does not hold anymore. How fast does the temperature drops in the Sun's atmosphere? Parker estimated this by calculating how the heat produced in the solar corona is dispersed through thermal conduction within the gas, and showed that the temperature drops slowly enough that when the Sun's atmosphere meets the interstellar medium, the gas pressure is still high enough that the interstellar medium cannot contain it. Hence the result holds even in the non isothermal case. We don't go into details here, and refer the reader to the original articles (see end of the section).

Parker wind solution We assume spherical symmetry and that the gas is in a steady state ($\partial_t = 0$). All quantities are functions of r only. We assume a velocity field of the type:

$$\mathbf{v} = v(r)\hat{\mathbf{e}}_r. \tag{262}$$

Under steady state conditions the mass outflow rate $\dot{M}$ does not depend on r and is given by ρv times the surface area of a sphere:

$$\dot{M} = 4\pi r^2 \rho v = \text{constant} \tag{263}$$

The Euler equation (15) in spherical coordinates gives:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{c_\mathrm{s}^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2} \tag{264}$$

Isolating ρ from (263) and substituting into (264) gives:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = c_\mathrm{s}^2\left(\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{2}{r}\right) - \frac{GM}{r^2} \tag{265}$$

Defining the radius

$$r_{\rm s} = \frac{GM}{2c_{\rm s}^2} \tag{266}$$

and rearranging, we can rewrite (265) as

$$\frac{{\rm d}v}{{\rm d}r}\left(v - \frac{c_{\rm s}^2}{v}\right) = \frac{2c_{\rm s}^2}{r}\left(1 - \frac{r_{\rm s}}{r}\right) \tag{267}$$

which defining the dimensionless variables

$$\tilde{r} = \frac{r}{r_{\rm c}}, \qquad \tilde{v} = \frac{v}{c_{\rm s}}, \tag{268}$$

can also be written in dimensionless form

$$\frac{{\rm d}\tilde{v}}{{\rm d}\tilde{r}}\left(\tilde{v} - \frac{1}{\tilde{v}}\right) = \frac{2}{\tilde{r}}\left(1 - \frac{1}{\tilde{r}}\right) \tag{269}$$

We need to study the solutions to this equation. Integrating it we obtain

$$\boxed{\tilde{v}^2 - \log\tilde{v}^2 = 4\log\tilde{r} + \frac{4}{\tilde{r}} - C} \tag{270}$$

where C is a constant of integration. Once C is known, this equation allows us in principle to find $\tilde{v} = \tilde{v}(\tilde{r})$, the velocity as a function of radius. Each value of C corresponds to a different solution (actually to two solutions, as we will see in a moment). How do we determine C?

Let us plot the solutions for different values of C, see Fig. 16. Given C, for each value of $\tilde{r}$ there are either two or none values of $\tilde{v}^2$ which satisfy equation (270). Hence for each value of C there are two possible "branches", i.e. two possible solutions $\tilde{v}(\tilde{r})$ with $\tilde{v} > 0$.[23] Indeed, we see in Fig. 16 each value of C corresponds to two solutions.

The solutions with $C = 3$ form an X-shaped cross that divides the space in Fig. 16 in four parts. Solutions with $C < 3$ live above and below the cross, while solutions with $C > 3$ live left and right. In the figure, $C = 2$ is an example of the former, while $C = 4$ is an example of the latter.

Note that *if* a transition from subsonic to supersonic occurs (i.e. $\tilde{v} = 1$ somewhere), it *must* happen at $\tilde{r} = 1$. This is seen considering equation (269). At a sonic point the left hand side is zero and therefore the right hand side must too. The only other option is that at the sonic

[23]There are two symmetrical branches with $v < 0$, but we don't consider them here! These are accreting solutions that correspond to the Bondi problem, see 4.2.

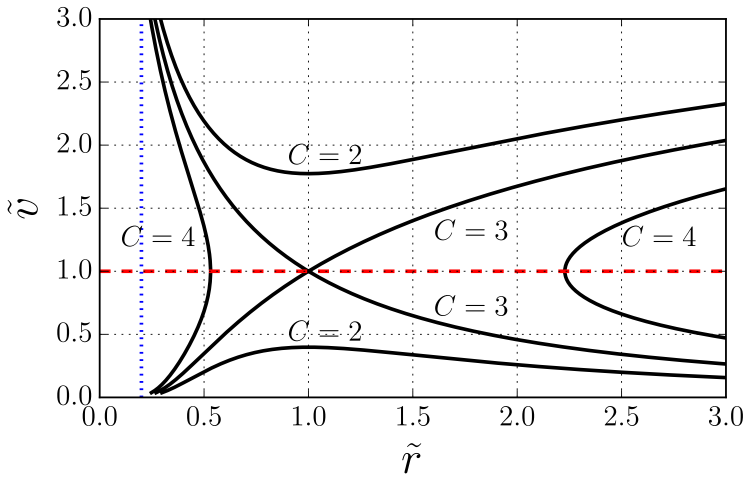

Figure 16: Solutions to equation (270) for different values of C. The parker wind solution corresponds to the solution with $C = 3$ which starts subsonic at low radius. The red dashed horizontal line indicates the value of the sound speed. The blue dotted horizontal line corresponds to the Sun's radius.

point $d\tilde{v}/d\tilde{r} = \infty$, which is what happens in the $C = 4$ solutions. But this is not physical and cannot happen in a real flow. Substituting $\tilde{v} = 1$ and $\tilde{r} = 1$ in equation (270) we also see that the two transonic solutions correspond to $C = 3$.

To proceed and see which solutions are meaningful, we need to place the Sun's radius in the diagram of Fig. 16. Is the Sun surface bigger or smaller than the critical radius? Let's estimate it. The sound speed in the solar corona is

$$c_s = \sqrt{\frac{kT}{m_H}} \sim 10^5 \mathrm{m\,s^{-1}}, \tag{271}$$

where $k = 1.38 \times 10^{-23} \mathrm{JK^{-1}}$ is the Boltzmann constant, $m_H = 1.67 \times 10^{-27}\,\mathrm{kg}$ is the mass of a hydrogen atom and $T \sim 10^6\,\mathrm{K}$ is the temperature of the solar corona. Using the gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, the mass of the Sun $M \simeq 2 \times 10^{30}\,\mathrm{kg}$ and the radius of the Sun $R_\odot \simeq 6.957 \times 10^8\,\mathrm{m}$ we find

$$r_c = \frac{GM}{2c_s^2} \sim 5R_\odot \tag{272}$$

the critical radius is a few times the radius of the Sun. Hence $\tilde{r}_\odot = R_\odot/r_c \simeq 0.2$. This is indicated as the vertical blue dashed line in Fig.

16.

Solutions like the $C = 4$ solutions in the figure are not of interest to us. We need solutions that extend from the Sun's radius to infinity, while each of them does not exist for a certain range of $\tilde{r}$ around the critical radius. The only way in which these solutions could have sense is that the Sun's radius is bigger than the critical radius, thereby intersecting the right branch at some radius, but this is not the case.

Solutions like the upper $C = 2$ solution are also not of interest to us. At the Sun's surface, the gas velocity must be subsonic, while in these solutions it is highly supersonic. Hence we discard these solutions. For a similar reason, we discard the solution with $C = 3$ which crosses the red dashed line from above.

The lower solutions with $C = 2$ are also not ok, because the pressure at infinity does not vanish and hence they require a pressure confinement, the same problem of the purely hydrostatic solution. (Indeed, even if these solutions contain some motions, these are small and their density profile is similar to hydrostatic equilibrium, which is why they are called *settling solutions*). To see this, note that in these solutions $\tilde{v}$ never exceeds the sound speed. Integrating equation (264) we find:

$$\frac{v^2}{2} = -c_{\rm s}^2 \log \rho + \frac{GM}{r} + \text{constant} \tag{273}$$

If by absurdum $\rho \to 0$ in the limit $r \to \infty$, then the logarithm would diverge. But no other term in this equation can diverge, because the velocity is limited from above! Hence the density must remain finite.

The only solution left is the $C = 3$ solution which starts subsonic at the Sun's surface, becomes supersonic at $\tilde{r} = 1$ and then increases its velocity without limit. This is known as **Parker wind solution**. Nowadays, we know by direct observations that the Sun's wind is subsonic at small radii and supersonic at large radii, which confirms this is the correct solution.

Taking the limit $r \to \infty$ in equation (270) we see that for this solution the velocity at infinity goes like

$$\tilde{v}^2 \sim 4 \log \tilde{r} \tag{274}$$

substituting into (263) we see that the density at infinity goes like

$$\rho \propto \frac{1}{r^2 \log r} \tag{275}$$

Hence this solution does not require confinement pressure. This again confirms this is the correct solution.

The Parker solar wind problem has an analogy with the **De Laval Nozzle**. In engineering, people are interested in accelerating gas from subsonic to supersonic velocities using tubes with varying cross-sectional area. It is well known that an ever-narrowing tube does not work. One needs a tube that is first converging and then diverging, and the sonic transition must happen at the point of minimum width. Such a nozzle is called a De Laval Nozzle. The mathematics of this problem is essentially equivalent to the Parker wind problem. The point of minimum width of the nozzle corresponds to the critical radius r_c. A discussion of the De Laval Nozzle can be found for example in [2].

For more details the reader is referred for example to the free article on Scholarpedia by Parker himself, available at `http://www.scholarpedia.org/article/Parker_Wind`, and to the original articles, E. N. Parker, 1958, *Dynamics of the interplanetary gas and magnetic field*, Astrophysical Journal 128, 644–676. and E. N. Parker, 1965, *The dynamical theory of the solar wind*, Space Science Reviews, 4, 666–708.

4.2 Bondi spherical accretion

Astrophysical objects can increase their mass by gravitationally attracting gas from their surroundings. This phenomenon is called **accretion**. It is widespread in the universe. Star formation, planet formation and galaxy formation all occur through accretion processes. Our Galaxy still accretes mass from the intergalactic medium, which is what is believed to fuel its star formation rate and to prevent it from becoming a "read and dead" elliptical galaxy, made only of reddish, old stars. In binary systems, one of the two object often accretes mass from the other. Since approximately 1/2 of all the stars that you see in the Sky with your naked eye are actually binary systems,[24] accretion must be a common event. When something falls on a star or black hole during accretion, its gravitational potential energy decreases, while its kinetic energy increases. This kinetic energy is believed to be the source of energy behind many high-energy astrophysical phenomena in the universe, such as Active Galactic Nuclei, the most luminous objects known.

In this section we study **Bondi (1952) accretion**, one of the simplest types of accretion flow. This is the counterpart of the Parker wind problem in which instead of an outflow we have an inflow. The mathematics of the two problem are very similar. The main difference is in the boundary conditions: in the Parker problem we require $v = 0$ at $r = 0$,

[24] Hence, 2/3 of all the stars are in binaries.

here we require $v = 0$ at $r = \infty$.

Consider an extended gaseous medium whose density and pressure at infinity are P_∞ and ρ_∞. Consider a point mass M immersed in this medium. The point mass may represent for example a star or a black hole whose gravitational field attracts gas from its surroundings. We want to study spherically symmetric steady state accreting solutions of the hydrodynamics equations. We assume that the point mass is at rest with respect to the gas at infinity. Hence, our problem *does not* represent an object that is moving with some velocity through a gaseous medium, for example a star that enters a cloud.

We work in spherical coordinates and assume that all quantities are functions of r only. The velocity field is of the type:

$$\mathbf{v} = v(r)\hat{\mathbf{e}}_r. \tag{276}$$

The mass accretion rate $\dot{M}$ is given by ρv times the surface area of a sphere:

$$\dot{M} = -4\pi r^2 \rho v, \tag{277}$$

Under steady conditions ($\partial_t \equiv 0$), mass conservation requires that the $\dot{M}$ is constant and does not depend on r. This result can also be derived from the continuity equation (6). We assume that pressure and density of the gas are related by a simple polytropic equation of state:

$$P = K\rho^\gamma \tag{278}$$

The speed of sound, which plays an important role in this problem, is defined by:

$$c^2(r) = \frac{\mathrm{d}P}{\mathrm{d}\rho} = \gamma K \rho^{\gamma-1} \tag{279}$$

(Do not confuse with the speed of light!). Using (277) it can be rewritten as

$$c^2 = \gamma K \left(\frac{\dot{M}}{4\pi r^2 |v|}\right)^{\gamma-1} \tag{280}$$

The Euler equation (15) in spherical coordinates gives:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho} - \frac{GM}{r^2} \tag{281}$$

We must treat separately the case $\gamma = 1$ and the case $1 < \gamma < 5/3$.

Isothermal case In the isothermal case $\gamma = 1$, equations identical as in the Parker wind section, with $K = c_s^2$. The only difference is that now $\tilde{v} < 0$, while in the Parker wind problem $\tilde{v} > 0$. Starting from (264) and repeating identical steps we obtain (270):

$$\boxed{\tilde{v}^2 - \log \tilde{v}^2 = 4 \log \tilde{r} + \frac{4}{\tilde{r}} - C} \tag{282}$$

where

$$\tilde{r} = \frac{r}{r_c}, \qquad \tilde{v} = \frac{v}{v_c}, \tag{283}$$

and

$$r_c = \frac{GM}{2K}, \qquad v_c = \frac{v}{K^{1/2}}. \tag{284}$$

$\tilde{v}$ appears quadratically in (282), so any solution with $\tilde{v} > 0$ can be immediately translated into one with $\tilde{v} < 0$, as one would expect since the equations of motion are time-reversible which means that a solution seen backwards in time is still a solution. Hence figure 16 applies identical to the present problem!

Let us examine what figure 16 tells us for the Bondi accretion problem. We require $\tilde{v} = 0$ at $\tilde{r} = \infty$. This rules out the $C = 4$ solutions, the upper $C = 2$ solutions, and one of the two $C = 3$ solutions (which was the Parker wind solution) because in these the velocity tends to infinity as $\tilde{r} \to \infty$. The solutions that survive are i) the lower $C = 2$ solutions ii) the $C = 3$ solution that starts with infinite velocity, and tends to zero as $\tilde{v} \to \infty$. These are all good and admissible accreting solutions.

Our problem was formulated in terms of ρ_∞ and P_∞ (and $v_\infty = 0$). What is the correct solution given these parameters? Turns out that there are many compatible values of C, corresponding to the same values of these two parameters but to different accretion rates.[25] We cannot tell the accretion rate only from the parameters in which the problem was formulated. However, given ρ_∞ and P_∞, there is a *maximum* accretion rate, which corresponds to the $C = 3$ solution. To see this, take the limit $r \to \infty$ in equation (277):

$$\dot{M} = -4\pi\rho_\infty \lim_{r\to\infty} (r^2 v) \tag{285}$$

Taking the same limit in (282) we find

$$\tilde{v}^2 \simeq e^C \tilde{r}^{-4} \tag{286}$$

[25] This is not surprising since in the isothermal case the equations of motion are essentially invariant under a rescaling of ρ, hence given C in the dimensionless equation (283) we can still choose this scaling!

which used in (285) together with (283) and (284) gives the accretion rate:

$$\dot{M} = \exp\left(C/2\right) \pi \rho_\infty c_s^{-3} \left(GM\right)^2 \tag{287}$$

Figure 16 tells us that all the accretion rates with $C \le 3$ are possible. The maximum accretion rate is attained in the case $C = 3$ and is

Maximum Bondi accretion rate for the isothermal case.

$$\boxed{\dot{M} = \exp\left(3/2\right) \pi \rho_\infty c_s^{-3} \left(GM\right)^2} \tag{288}$$

Polytropic case In the $\gamma > 1$ case we can rewrite the pressure term in (281) as

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\gamma K \rho^{\gamma-1}}{\gamma - 1}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{c^2}{\gamma - 1}\right) \tag{289}$$

Hence equation (281) can be integrated to give

$$\frac{v^2}{2} + \frac{c^2}{\gamma - 1} - \frac{GM}{r} = \mathcal{C} \tag{290}$$

where $\mathcal{C}$ is an integration constant. Note that this is just Bernoulli's theorem (116). Substituting c^2 from (280) we find:

$$\frac{v^2}{2} + \frac{\gamma K}{\gamma - 1}\left(\frac{\dot{M}}{4\pi r^2 |v|}\right)^{\gamma-1} - \frac{GM}{r} = \mathcal{C} \tag{291}$$

Given the parameters $\mathcal{C}$ and $\dot{M}$, this equation allows us in principle to find $v = v(r)$, i.e. the velocity profile of the gas as a function of the distance from the central object. To study the solutions to this equation, it is useful to rewrite it in dimensionless form as we did for the isothermal case. What characteristic length and velocity should we use to rescale our quantities? In the isothermal case, we used the radius at which the flow becomes supersonic and the sound speed. Can we do the same here? Yes. However, this time the sound speed is not constant, so we must find both the radius at which the flow becomes supersonic and the value of the velocity at that point. Starting from (281), using (278) and then (277) to eliminate ρ, we find:

$$\frac{\mathrm{d}v}{\mathrm{d}r}\left[v - \frac{c^2}{v}\right] = \frac{2}{r}\left[c^2 - \frac{GM}{2r}\right]. \tag{292}$$

When the flow goes from subsonic to supersonic, the square bracket in the left hand side vanishes. Hence the right hand side must vanish too if $\mathrm{d}v/\mathrm{d}r$ remains finite. Solving the system of equations that comes from imposing that the square brackets on both sides vanish simultaneously and using (280) one finds the following characteristic radius and velocity:

$$r_{\mathrm{c}} = (\gamma K)^{\frac{2}{3\gamma-5}} \left(\frac{GM}{2}\right)^{-\frac{\gamma+1}{3\gamma-5}} \left(\frac{\dot{M}}{4\pi}\right)^{\frac{2(\gamma-1)}{3\gamma-5}} \tag{293}$$

$$v_{\mathrm{c}} = (\gamma K)^{-\frac{1}{3\gamma-5}} \left(\frac{GM}{2}\right)^{\frac{2(\gamma-1)}{3\gamma-5}} \left(\frac{\dot{M}}{4\pi}\right)^{-\frac{\gamma-1}{3\gamma-5}} \tag{294}$$

Note that in the isothermal case $\gamma = 1$ these reduce to (284). Note also that they are singular in the case $\gamma = 5/3$, which must be treated separately. Defining the dimensionless variables

$$\tilde{r} = \frac{r}{r_{\mathrm{c}}}, \qquad \tilde{v} = \frac{v}{v_{\mathrm{c}}} \tag{295}$$

We can rewrite (291) as:

$$\boxed{\frac{\tilde{v}^2}{2} + \frac{1}{(\gamma-1)\tilde{r}^{2(\gamma-1)}\tilde{v}^{(\gamma-1)}} - \frac{2}{\tilde{r}} = C} \tag{296}$$

where the old and new integration constants are related by $\mathcal{C} = C/v_{\mathrm{c}}^2$. This equation depends only on one parameter, C. Equation (296) is valid for the case of $1 < \gamma < 5/3$. It is the analog of (282) for the polytropic case.

Equation (296) can be solved analytically only in the case $\gamma = 1.5$. This is the task of Problem 1. We can solve it numerically for the other cases. The results are similar for all cases to the isothermal case studied above, i.e.

1. Given ρ_∞ and P_∞ there are many acceptable solutions with different accretion rates.

2. There is a maximum accretion rate.

For example, figure 17 shows numerical solutions of equation (296) for the cases $\gamma = 1.4$ (corresponding to a diatomic gas) and $\gamma = 1.6$. These diagrams have much in common with the one for the isothermal case shown in figure 16. They both have a cross corresponding to the transonic solution that divides different families of solutions. In the case $\gamma = 1.4$ the cross is similar to the isothermal case, with one transonic solution such that $\tilde{v} \to 0$ and the other such that $\tilde{v} \to \infty$ as $\tilde{r} \to 0$. A family of settling solutions whose velocity tends to zero at small radii lies below the cross. In case $\gamma = 1.6$ both transonic solutions are such that $\tilde{v} \to \infty$ as $\tilde{r} \to 0$. In this case, the settling solutions have a velocity that tends to infinity at small radii while remaining subsonic everywhere (the sound speed also tends to infinity, faster than $\tilde{v}$!). The case $\gamma = 1.5$ is what separates these two types of behaviours.

One can calculate the accretion rate as a function of C, like we did in the isothermal case. In the limit $\tilde{r} \to \infty$ equation (296) gives

$$\tilde{r}^2 |\tilde{v}| \simeq [(\gamma - 1)C]^{1/(\gamma-1)} \tag{297}$$

which substituted into (277) gives in the limit $r \to \infty$:

$$\dot{M} = [(\gamma - 1)C]^{1/(\gamma-1)}\, 4\pi\rho_\infty r_{\mathrm{c}}^2 v_{\mathrm{c}} \tag{298}$$

using (293) and (294) we have

$$r_{\mathrm{c}}^2 v_{\mathrm{c}} = (\gamma K)^{\frac{3}{3\gamma-5}} \left(\frac{GM}{2}\right)^{-\frac{4}{3\gamma-5}} \left(\frac{\dot{M}}{4\pi}\right)^{\frac{3(\gamma-1)}{3\gamma-5}} \tag{299}$$

the constant K can be defined by the properties of the gas at infinity:

$$K = P_\infty \rho_\infty^{-\gamma} \tag{300}$$

and the sound speed at infinity is

$$c_\infty^2 = \gamma \frac{P_\infty}{\rho_\infty}. \tag{301}$$

Combining all together we can rewrite the accretion rate as

$$\dot{M} = [(\gamma - 1)C]^{-\frac{5-3\gamma}{2(\gamma-1)}}\, \pi\rho_\infty c_\infty^{-3} (GM)^2 \tag{302}$$

The value of C corresponding to the critical solution (i.e., the cross) gives the maximum accretion rate. At the critical point, $\tilde{v} = \tilde{r} = 1$. Substituting these into (296) one finds:

$$C_{\mathrm{c}} = -\frac{5 - 3\gamma}{2(\gamma - 1)} \tag{303}$$

The maximum accretion rate is obtained when $C = C_{\mathrm{c}}$:

Maximum Bondi accretion rate for the polytropic case.

$$\boxed{\dot{M} = \left[\frac{5 - 3\gamma}{2}\right]^{-\frac{5-3\gamma}{2(\gamma-1)}} \pi\rho_\infty c_\infty^{-3} (GM)^2} \tag{304}$$

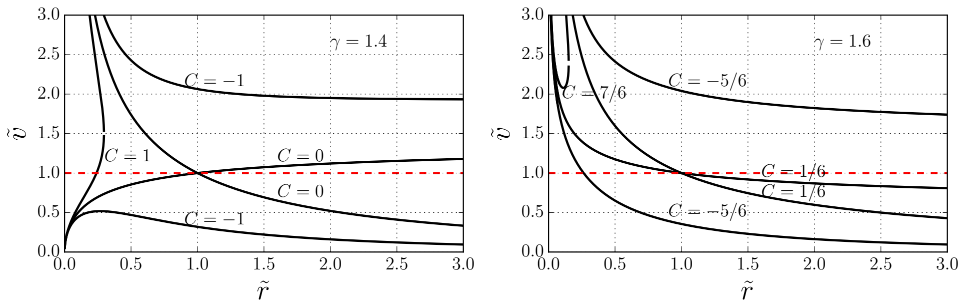

Figure 17: Solutions to equation (296) for different values of C. Left panel is for $\gamma = 7/5$, corresponding to a diatomic gas. Right panel is for $\gamma = 1.6$.

Compare with (288).

Finally, let us examine the behaviour of the critical solutions as $r \to 0$. From (291) we see that for the critical solution[26]

$$v^2 \simeq \frac{GM}{r} \tag{305}$$

which essentially means that the gas is in free fall. In the transonic solutions, the gas starts subsonic at infinity, where it is essentially in hydrostatic equilibrium, and becomes supersonic at $r = r_{\mathrm{c}}$. Once it becomes supersonic, no information can travel back, and the gas is in free fall towards the star. Once it becomes supersonic, it can be brought to subsonic levels only through the mediation of a shock (for example, because of impact with the surface of a star). The sonic point separates these two regimes: inside, free fall, outside, almost hydrostatic equilibrium.

The case $\gamma = 5/3$ must be treated separately (as we did for the isothermal case $\gamma = 1$), because for this value (293) and (294) are singular. Since the results are qualitatively similar to the cases studied above, we do not do it here. We only note that in this case the sonic point of the critic solution occurs at $r = 0$.

We have seen that for the same values of the conditions at infinity, there are many possible solutions with different accretion rates. What determines the accretion rate? Bondi (1952) originally argued that accretion should take place at the maximum rate possible. He thought that solutions with less than the maximum accretion rate would be un-

[26]One must be a bit careful in how the limit is taken and show that the middle term on the LHS is negligible compared to the other two.

stable. However, subsequent linear stability analysis[27] have shown that all solutions are stable. The maximum Bondi rate should be taken as an order of magnitude estimate of the maximum accretion rate possible. (This means that we could have guessed this result just from dimensional analysis! So yes, all these calculations for nothing).

We remark that while it is instructive to study Bondi's accretion problem, in practice accretion most often takes place not in a spherical fashion but through accretion discs (see section **??**). In the Bondi problem zero total angular momentum is assumed. However, in a real situation even a small amount of angular momentum, which is almost always present, can substantially change both the gas flow lines and the value of the accretion rate.

5 Waves

5.1 Introduction

Everyone is familiar with water waves in the sea and sound waves in the air, which we commonly call sound. Waves are disturbances in the fluid quantities such as density and pressure that travel through space. They can transport energy and momentum.

The study of waves in fluids is a very broad topic. The best way to learn it is to see many examples, and construct a personal "library of examples" which is then used to build intuition and predict what happens in a new given situation. In these lecture notes, we start this journey by considering some examples.

Consider a steady state solution $\rho_0(\mathbf{x})$, $\mathbf{v}_0(\mathbf{x})$ [and possibly $\mathbf{B}_0(\mathbf{x})$, $P_0(\mathbf{x})$, etc]. By definition this solution does not depend on time and, if left completely unperturbed, will remain unchanged forever. Now consider travelling disturbances on top of this background flow, i.e. waves. Waves can be roughly divided into three main categories:

- **Linear waves**. This is the case in which the disturbances are very small compared to the corresponding equilibrium values of the steady state solution. For example, if $\rho_0(\mathbf{x})$ is the unperturbed equilibrium density and $\rho(\mathbf{x}, t)$ is the density with the perturbation, in linear theory their difference is small, $\Delta\rho = \rho(\mathbf{x}, t) - \rho_0(\mathbf{x}) \ll \rho_0(\mathbf{x})$.

[27] i.e., like for any steady state, one can linearise the equations around the steady state solution and find the eigenmodes of the system.If among these eigenmodes there is at least one that grows exponentially, the system is unstable

When linear theory is valid, we can linearise the equations of motions around the equilibrium state. We therefore obtain **linear** differential equations that describe the propagation of disturbances. Linear equations are generally much easier to solve than the full non–linear fluid equations, because the **superposition principle** applies. The typical result is that one finds a series of **normal modes** of the system, each of which describes a wave with a definite frequency, and an arbitrary disturbance can be seen as a linear superposition of these basic waves.

It can happen that one or more of the normal modes are such that they grow exponentially in time. In this case, arbitrarily small perturbations will grow until the linear approximation is not valid anymore. In this case the background steady state onto which the disturbances travel is **unstable**: if we prepare the system in such a state, even an infinitesimally small disturbance will eventually grow enough to disrupt the system. The topic of **instabilities** is in itself another large topic (see Section **??**).

Finally, another important thing to remember is that typically, if you wait long enough and you have no viscosity, any wave, no matter how small, will become non-linear and develop shocks. Therefore, linear theory is often valid only for a limited time interval. We will study the steepening of small disturbances into shock waves using an example later in this section.

- **Non linear waves**. In this case, one attempts to follow perturbations of finite amplitude. Thus one must go beyond the linear approximation discussed in the previous item. This case is generally much more difficult mathematically. The superposition principle does not apply, and no general analytical method for their solution exists. Thus, each case must be analysed individually.

- **Shock waves**. A shock wave is a type of highly non linear propagating disturbance characterised by an abrupt, discontinuous change in one or more of the fluid quantities. Mathematically, these solutions are possible because the laws of conservation of mass, momentum and energy that form the basis of the equations for inviscid flow do not necessarily assume continuity of the flow variables. These laws were originally formulated in the form of differential equations simply because it was assumed at the beginning that the flow is continuous, but is possible to have perfectly valid solutions (actually it is often unavoidable) that admit sharp discontinuities.

In a realistic situation, discontinuities are resolved by diffusive processes (viscosity, thermal conduction, resistivity) which become more important at small scales, and discontinuities should be regarded as large but finite gradients in the flow variables across a layer of very small thickness. The inviscid case could in principle be obtained by solving a more complete set of equations which include viscosity to derive the internal structure of a shock and then taking the limit of vanishing viscosity. However, this is often difficult and not necessary in practice, because the matching conditions do not depend on the internal structure of the shock.

Shock waves are easy to excite when the flow velocity is supersonic, which is often the case in astrophysics. They are therefore very common in astrophysics.

We now examine some examples of waves.

5.2 Sound waves

One of the simplest examples of linear waves are small disturbances in a uniform and stationary adiabatic fluid. The equations of motion are:

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{306}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} \tag{307}$$

$$\frac{D}{Dt}\left(\log P\rho^{-\gamma}\right) = 0. \tag{308}$$

Clearly, the following is a steady state solution of these equations:

$$\rho = \rho_0 = \text{constant} \tag{309}$$

$$P = P_0 = \text{constant} \tag{310}$$

$$\mathbf{v} = \mathbf{v}_0 = 0 \tag{311}$$

Now let the system be slightly perturbed, so that

$$\rho = \rho_0 + \rho_1 \tag{312}$$

$$P = P_0 + P_1 \tag{313}$$

$$\mathbf{v} = \mathbf{v}_1 \tag{314}$$

where the quantities with subscript 1 are small with respect to the quantities with subscript 0. We want to **linearise** the equations of motion around the steady state solution by substituting equations (312)–(314) into (306)–(308) and neglecting quadratic and higher order terms in the quantities with subscript 1. This is a very standard step in linear theory, both when we consider linear waves and linear instabilities (see Section **??**), and by the end of this course you should be used to linearise equations around steady state solutions.

Let us consider **adiabatic perturbations**, i.e. perturbations for which the entropy (23) is the same for the perturbed and the initial unperturbed state. Physically, this corresponds to perturbing the fluid without locally heating it, for example it is a valid assumption if we have a moving object generating sound, such as a tuning fork. Thus for the perturbed state we have $P\rho^{-\gamma} = P_0\rho_0^{-\gamma} = \text{constant}$ at $t = 0$. Equation (308) then implies that $P\rho^{-\gamma}$ stays constant for each fluid element, and so $P\rho^{-\gamma} = P_0\rho_0^{-\gamma}$ everywhere at all times. Thus

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \tag{315}$$

i.e.

$$\frac{P_0 + P_1}{P_0} = \left(\frac{\rho_0 + \rho_1}{\rho_0}\right)^{\gamma} \tag{316}$$

$$= \left(1 + \frac{\rho_1}{\rho_0}\right)^{\gamma} \tag{317}$$

$$= \left(1 + \gamma\frac{\rho_1}{\rho_0} + \dots\right) \tag{318}$$

Neglecting quadratic and higher order terms we obtain

$$\boxed{P_1 = c_0^2\rho_1} \tag{319}$$

where

$$c_0 = \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2}. \tag{320}$$

This equation relates density and pressure of adiabatic perturbations.

Now substituting (312)–(314) and (319) into (306)-(307) and neglecting quadratic and higher order terms in the quantities with subscript 1 we obtain the linearised version of the equation of motion of the perturbations:

$$\partial_t\rho_1 + \rho_0\boldsymbol{\nabla}\cdot\mathbf{v}_1 = 0 \tag{321}$$

$$\partial_t\mathbf{v}_1 = -c_0^2\frac{\boldsymbol{\nabla}\rho_1}{\rho_0} \tag{322}$$

To solve these equations, we look for solutions of the following form:

$$\rho_1(\mathbf{x},t) = \tilde{\rho}\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{323}$$

$$\mathbf{v}_1(\mathbf{x},t) = \tilde{\mathbf{v}}\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{324}$$

where $\tilde{\rho}$ and $\tilde{\mathbf{v}} = (\tilde{v}_x, \tilde{v}_y, \tilde{v}_z)$ are *complex* constants. Such solutions are called **plane waves**. $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector and ω is the frequency of the wave. What we really mean when we write equations (323) and (324) is that the physical quantities are given by the real part of these equations. But since all equations are linear, it does not matter if we take the real part at the beginning or at the end of the calculations. And it is very convenient to keep everything complex.

Taking the derivatives of (323) and (324) we have

$$\partial_t \rho_1(\mathbf{x},t) = -i\omega\rho_1 \tag{325}$$

$$\partial_t \mathbf{v}_1(\mathbf{x},t) = -i\omega\mathbf{v}_1 \tag{326}$$

$$\boldsymbol{\nabla}\rho_1 = i\mathbf{k}\rho_1 \tag{327}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v}_1 = i\mathbf{k}\cdot\mathbf{v}_1 \tag{328}$$

Substituting into (321) and (322) we find:

$$-i\omega\tilde{\rho} + i\rho_0\mathbf{k}\cdot\tilde{\mathbf{v}} = 0 \tag{329}$$

$$-i\omega\tilde{\mathbf{v}} = -i\frac{c_0^2}{\rho_0}\mathbf{k}\tilde{\rho} \tag{330}$$

There are two cases in which these equations are simultaneously satisfied.

Sound waves If $\omega \neq 0$ we can divide by ω in the second equation, isolate $\tilde{\mathbf{v}}$ and substitute in the first equation. We find a solution for which the following relations must be simultaneously satisfied:

$$\omega^2 = c_0^2 k^2, \qquad \tilde{\mathbf{v}} = \frac{c_0^2}{\omega\rho_0}\mathbf{k}\tilde{\rho}, \qquad \tilde{\rho}\ \text{arbitrary} \tag{331}$$

this is the case of regular **sound waves** with a **dispersion relation**

$$\omega = \pm k c_0 \tag{332}$$

Hence, sound waves travel with a speed

$$c_0 = \frac{\omega}{k} \tag{333}$$

A sound wave is completely specified once we know $\tilde{\rho}$ and $\mathbf{k}$: ω is obtained from the dispersion relation, and $\tilde{\mathbf{v}}$ is obtained from the second relation in (331).

Vortex waves If $\omega = 0$, equations (329) and (330) admit a solution of the following type:

$$\mathbf{k} \cdot \tilde{\mathbf{v}} = \omega = \tilde{\rho} = 0, \qquad \mathbf{k} \neq 0, \qquad \tilde{\mathbf{v}} \neq 0 \tag{334}$$

Such solutions are called **vortex waves**, for reasons that we will clear up in a moment. These solutions do not evolve in time, and their density perturbation vanishes, i.e. the density is the same as the background state. The velocity is directed perpendicular to the direction of wave vector $\mathbf{k}$, and is also constant along these lines.

What is the physical meaning of these solutions? A velocity field that is constant along parallel stripes in the direction parallel to them is obviously a steady state of the equation of motions! Vortex waves simply corresponds to this type of solutions.[28] Why is it called a "wave" if it does not evolve in time? The reason is that if you consider the same solution in a frame moving at velocity $\mathbf{v}_0$, things are passing by and what you see *looks like* a wave! Indeed these waves become more relevant when we consider propagation of disturbances in a *moving* gas. If we have different parts of a system moving at different velocities, not always we can get rid of the velocity by a change of frame. Why are these waves called vortex? Because the curl is not zero, so they carry vorticity.

5.2.1 Propagation of arbitrary perturbations

Thus we have plane waves solutions. Do we need other solutions? It turns out that these are the only solutions that we need to know, because all other solutions can be expressed as a linear combination of plane waves. For example, suppose that at $t = 0$ you have an arbitrary perturbation given by:

$$\rho_1(\mathbf{x}, t = 0) = f(\mathbf{x}) \tag{335}$$

$$\mathbf{v}_1(\mathbf{x}, t = 0) = \mathbf{g}(\mathbf{x}) \tag{336}$$

How does this perturbation evolve? Let us first do the one-dimensional case, which is simpler, and then turn to the full three-dimensional case.

[28] In some sense, vortex waves are the analog of the normal mode of two masses connected by a spring in vacuum in which the centre of mass simply translates at constant speed.

One dimensional case In this case, we only have sound waves, while vortex waves (334) are not possible because the condition $\mathbf{k}\cdot\tilde{\mathbf{v}} = 0$ with $\mathbf{k} \neq 0$ and $\tilde{\mathbf{v}} \neq 0$ cannot be satisfied. The superposition principle says that the sum of two solutions of (321) and (322) is still a good solution. Hence we can superimpose an infinite number of solutions of the type (323) and (324). The most general superposition of sound waves in one dimension is:

$$\rho_1(\mathbf{x},t) = \int_{-\infty}^{\infty} \tilde{\rho}_a(k) e^{ikx-i\omega t} \mathrm{d}k + \int_{-\infty}^{\infty} \tilde{\rho}_b(k) e^{ikx+i\omega t} \mathrm{d}k \tag{337}$$

$$\mathbf{v}_1(\mathbf{x},t) = \int_{-\infty}^{\infty} \tilde{v}_a(k) e^{ikx-i\omega t} \mathrm{d}k + \int_{-\infty}^{\infty} \tilde{v}_b(k) e^{ikx+i\omega t} \mathrm{d}k \tag{338}$$

where

$$\omega = c_0 k. \tag{339}$$

and

$$\tilde{v}_a = \frac{c_0^2}{\omega\rho_0} k\tilde{\rho}_a \quad ; \quad \tilde{v}_b = -\frac{c_0^2}{\omega\rho_0} k\tilde{\rho}_b \tag{340}$$

There are two terms on the RHS of (337) and (338), one corresponds to the plus sign and the other to the minus sign in the dispersion relation (332). These correspond to waves travelling in opposite directions. Now we need to show that with such a superposition we can satisfy any possible initial condition. At $t = 0$ (337) and (338) take the form

$$\rho_1(\mathbf{x},t=0) = \int_{-\infty}^{\infty} \tilde{\rho}_a(k) e^{ikx} \mathrm{d}k + \int_{-\infty}^{\infty} \tilde{\rho}_b(k) e^{ikx} \mathrm{d}k \tag{341}$$

$$\mathbf{v}_1(\mathbf{x},t=0) = \int_{-\infty}^{\infty} \tilde{v}_a(k) e^{ikx} \mathrm{d}k + \int_{-\infty}^{\infty} \tilde{v}_b(k) e^{ikx} \mathrm{d}k \tag{342}$$

We need to equate this to our initial conditions (335) and (336). Recall that any function $f(x)$ can be decomposed into plane waves using the **Fourier transform** :

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \mathrm{d}k \quad ; \quad \tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} \mathrm{d}x \tag{343}$$

Thus we can write our initial conditions (335) and (336) as

$$\rho_1(\mathbf{x},t=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \mathrm{d}k \tag{344}$$

$$\mathbf{v}_1(\mathbf{x},t=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(k) e^{ikx} \mathrm{d}k \tag{345}$$

Equating (344) and (345) to (341) and (342) and using (340) we get

$$\frac{\tilde{f}}{2\pi} = \tilde{\rho}_a + \tilde{\rho}_b \tag{346}$$

$$\frac{\tilde{g}}{2\pi} = \frac{c_0^2}{\omega \rho_0} k \left(\tilde{\rho}_a - \tilde{\rho}_b\right) \tag{347}$$

This system can be easily solved to get

$$\tilde{\rho}_a = \frac{1}{4\pi}\left(\tilde{f} + \tilde{g}\frac{\omega\rho_0}{c_0^2}\right) \tag{348}$$

$$\tilde{\rho}_b = \frac{1}{4\pi}\left(\tilde{f} - \tilde{g}\frac{\omega\rho_0}{c_0^2}\right) \tag{349}$$

Thus, if we are given the initial conditions (335), (336), to find the evolution in time all we have to do is:

1. Find the Fourier transforms $\tilde{f}(k)$, $\tilde{g}(k)$.
2. Use (348) and (349) to get $\tilde{\rho}_a$ and $\tilde{\rho}_b$.
3. The solution is then given by (337) and (338).

We have thus a systematic way of solving the problem. The same method can be generalised to pretty much all cases of linear waves, even those in which the dispersion relation is not given by (332). This is why the linear waves are much easier to treat than non-linear waves: the superposition principle applies, so we only need to find plane waves solution. All other solutions can be seen as a linear superposition of them.

Note also that using the dispersion relation (339) the general solution (337) can be rewritten as

$$\begin{aligned}
\rho_1(\mathbf{x}, t) &= \int_{-\infty}^{\infty} \tilde{\rho}_a(k) e^{ikx - i\omega t} \mathrm{d}k + \int_{-\infty}^{\infty} \tilde{\rho}_b(k) e^{ikx + i\omega t} \mathrm{d}k && (350)\\
&= \int_{-\infty}^{\infty} \tilde{\rho}_a(k) e^{ik(x - c_0 t)} \mathrm{d}k + \int_{-\infty}^{\infty} \tilde{\rho}_b(k) e^{ik(x + c_0 t)} \mathrm{d}k && (351)\\
&\equiv F_a(x - c_0 t) + F_b(x + c_0 t) && (352)
\end{aligned}$$

Thus, the general solution is simply the sum of two waves travelling in opposite directions but conserving their shapes. However, this is only true if the dispersion relation is of the type (332), because it has the special property that all wavelengths travel at the same speed $c_0 = \omega/k$. For a generic dispersion relation, different wavelengths propagate at different speed, so this is not true anymore (see Section 5.4).

Three dimensional case What about the three dimensional case? Now, the most general solution of plane waves is not only a sum of sound waves, but a superposition of sound waves and vortex waves:

$$\rho_1(\mathbf{x},t) = \int_{-\infty}^{\infty} \tilde{\rho}_a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \mathrm{d}^3 k + \int_{-\infty}^{\infty} \tilde{\rho}_b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}+i\omega t} \mathrm{d}^3 k \tag{353}$$

$$\begin{aligned}\mathbf{v}_1(\mathbf{x},t) = & \int_{-\infty}^{\infty} \tilde{\mathbf{v}}_a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \mathrm{d}^3 k + \int_{-\infty}^{\infty} \tilde{\mathbf{v}}_b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}+i\omega t} \mathrm{d}^3 k \\ & + \int_{-\infty}^{\infty} \tilde{\mathbf{v}}_c(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3 k \end{aligned}\tag{354}$$

where

$$\omega = c_0 k. \tag{355}$$

and

$$\tilde{\mathbf{v}}_a = \frac{c_0^2}{\omega\rho_0}\mathbf{k}\tilde{\rho}_a \quad ; \quad \tilde{\mathbf{v}}_b = -\frac{c_0^2}{\omega\rho_0}\mathbf{k}\tilde{\rho}_b \quad ; \quad \tilde{\mathbf{v}}_c(\mathbf{k})\cdot\mathbf{k} = 0 \tag{356}$$

$\tilde{\mathbf{v}}_c$ is the part related to vortex waves. These waves have no density perturbation, so $\tilde{\rho}_c = 0$. We have also used that for vortex waves $\omega = 0$. How do we determine $\tilde{\rho}_a$, $\tilde{\rho}_b$ and $\tilde{\mathbf{v}}_c$ given the initial conditions (335), (336)? Note that for sound waves, $\tilde{\mathbf{v}}_a(\mathbf{k})$ and $\tilde{\mathbf{v}}_b(\mathbf{k})$ are parallel to $\mathbf{k}$, while for vortex waves $\tilde{\mathbf{v}}_c(\mathbf{k})$ is perpendicular to $\mathbf{k}$. In real space this means that for sound waves $\boldsymbol{\nabla}\cdot\mathbf{v} \neq 0$, $\boldsymbol{\nabla}\times\mathbf{v} = 0$, while for vortex waves $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, $\boldsymbol{\nabla}\times\mathbf{v} \neq 0$. Any linear superposition will preserve these characteristics. How do we separate the initial velocity field into a curl-free and a divergence-free component? At this point we need a little digression.

Helmholtz decomposition An arbitrary vector field $\mathbf{F}(\mathbf{x})$ can be *uniquely* decomposed into the sum of a curl-free and a divergence-free component:

Helmholtz Decomposition.

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_\parallel(\mathbf{x}) + \mathbf{F}_\perp(\mathbf{x}) \tag{357}$$

such that

$$\boldsymbol{\nabla}\times\mathbf{F}_\parallel = 0 \quad \text{and} \quad \boldsymbol{\nabla}\cdot\mathbf{F}_\perp = 0. \tag{358}$$

This is called **Helmholtz decomposition**. $\mathbf{F}_\parallel$ is called the **longitudinal** field and is curl-free, $\mathbf{F}_\perp$ is called the **transverse** field and is divergence-free. To see why these names, consider the Fourier transform of $\mathbf{F}$:

$$\mathbf{F}(\mathbf{x}) = \int_{-\infty}^{\infty} \tilde{\mathbf{F}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k \tag{359}$$

We can split this into two parts by separating the parts of $\tilde{\mathbf{F}}(\mathbf{k})$ that are parallel and perpendicular to $\mathbf{k}$:

$$\mathbf{F}(\mathbf{x}) = \int_{-\infty}^{\infty} \tilde{\mathbf{F}}_{\parallel}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k + \int_{-\infty}^{\infty} \tilde{\mathbf{F}}_{\perp}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k \tag{360}$$

where

$$\tilde{\mathbf{F}}_{\parallel}(\mathbf{k}) = \frac{1}{k^2}\mathbf{k}\left[\mathbf{k}\cdot\tilde{\mathbf{F}}(\mathbf{k})\right] \tag{361}$$

$$\tilde{\mathbf{F}}_{\perp}(\mathbf{k}) = \tilde{\mathbf{F}}(\mathbf{k}) - \tilde{\mathbf{F}}_{\parallel}(\mathbf{k}) \tag{362}$$

so that

$$\mathbf{k}\times\tilde{\mathbf{F}}_{\parallel}(\mathbf{k}) = 0 \quad ; \quad \mathbf{k}\cdot\tilde{\mathbf{F}}_{\perp}(\mathbf{k}) = 0 \tag{363}$$

Thus the first term in (360) is curl-free, while the second term is divergence-free. Going back to real space we can identify the longitudinal and transverse fields as follows:

$$\mathbf{F}_{\parallel}(\mathbf{x}) = \int_{-\infty}^{\infty} \tilde{\mathbf{F}}_{\parallel}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k \tag{364}$$

$$\mathbf{F}_{\perp}(\mathbf{x}) = \int_{-\infty}^{\infty} \tilde{\mathbf{F}}_{\perp}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k \tag{365}$$

It is now clear how we should proceed to solve our problem. Start with taking the Fourier transform of the given initial conditions:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k \tag{366}$$

$$\mathbf{g}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \tilde{\mathbf{g}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k \tag{367}$$

Then find the longitudinal and transverse component of $\mathbf{g}$:

$$\tilde{\mathbf{g}}_{\parallel}(\mathbf{k}) = \frac{1}{k^2}\mathbf{k}\left[\mathbf{k}\cdot\tilde{\mathbf{g}}(\mathbf{k})\right] \tag{368}$$

$$\tilde{\mathbf{g}}_{\perp}(\mathbf{k}) = \tilde{\mathbf{g}}(\mathbf{k}) - \tilde{\mathbf{g}}_{\parallel}(\mathbf{k}) \tag{369}$$

Then we need to match these with the general solution (353) and (354) at $t=0$:

$$\rho_1(\mathbf{x}, t=0) = \int_{-\infty}^{\infty} \tilde{\rho}_a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3 k + \int_{-\infty}^{\infty} \tilde{\rho}_b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3 k \tag{370}$$

$$\begin{aligned}\mathbf{v}_1(\mathbf{x}, t=0) = &\int_{-\infty}^{\infty} \frac{c_0^2}{\omega \rho_0} \mathbf{k} \tilde{\rho}_a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3 k - \int_{-\infty}^{\infty} \frac{c_0^2}{\omega \rho_0} \mathbf{k} \tilde{\rho}_b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3 k \\ &+ \int_{-\infty}^{\infty} \tilde{\mathbf{v}}_c(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3 k \end{aligned}\tag{371}$$

It is clear that since $\tilde{\mathbf{v}}_c(\mathbf{k})$ is perpendicular to $\mathbf{k}$, it will correspond to the transverse part $\tilde{\mathbf{g}}_\perp(\mathbf{k})$, while $\tilde{\mathbf{v}}_a(\mathbf{k})$ and $\tilde{\mathbf{v}}_b(\mathbf{k})$ are parallel to $\mathbf{k}$ and will correspond to the longitudinal part $\tilde{\mathbf{g}}_\parallel(\mathbf{k})$. Equating (370) and (371) to the Fourier transforms of the given initial conditions (366) and (367) we find:

$$\frac{\tilde{f}(\mathbf{x})}{(2\pi)^3} = \tilde{\rho}_a + \tilde{\rho}_b \tag{372}$$

$$\frac{\tilde{\mathbf{g}}_\parallel}{(2\pi)^3} = \frac{c_0^2}{\omega \rho_0} \mathbf{k} \tilde{\rho}_a - \frac{c_0^2}{\omega \rho_0} \mathbf{k} \tilde{\rho}_b \tag{373}$$

$$\frac{\tilde{\mathbf{g}}_\perp}{(2\pi)^3} = \tilde{\mathbf{v}}_c \tag{374}$$

which can be solved to find $\tilde{\rho}_a$, $\tilde{\rho}_b$ and $\tilde{\mathbf{v}}_c$.

Remarks

- Sound waves are purely *longitudinal* waves. They are made by compression of layers of fluid elements. Indeed, $\boldsymbol{\nabla} \cdot \mathbf{v} \neq 0$ for sound waves, which is associated to change in volumes. Sound waves carry no vorticity, $\boldsymbol{\nabla} \times \mathbf{v} = 0$.

- In the three dimensional case, the curl-free part travels in space as sound waves, while the divergence-free part remains stationary because vortex waves do not evolve in time! A purely divergence-free velocity field does not evolve in time, simply keeps flowing forever. This is somewhat intuitive: if we create a vortex without any overdensity ($\rho_1 = 0$), in absence of viscosity it just keeps circulating.

- What is the exact meaning of *small amplitude* waves? We have stated above that the linear approximation is valid when disturbances are very small compared to the corresponding equilibrium values of the steady state solution. However, the background velocity was $\mathbf{v}_0 = 0$ so it cannot be small compared to anything. Let us see more precisely when the approximations made in this section are valid. When we linearised the equations of motion, we have neglected the term $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ compared to $\partial_t \mathbf{v}$ in the Euler equation. For a plane wave of the form (324) we have:

$$\partial_t \mathbf{v} = -i\omega \mathbf{v} \tag{375}$$
$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = (\mathbf{v}\cdot i\mathbf{k})\mathbf{v} \tag{376}$$

For the RHS of the second to be much smaller than the RHS of the first we need

$$v \ll \frac{\omega}{k} \tag{377}$$

using the dispersion relation for sound waves this means

$$v \ll c_0 \tag{378}$$

Thus our approximations are valid for velocities much smaller than the sound speed.

- Taking the ∂_t derivative of equation (321) we get:

$$\partial_t^2 \rho_1 + \boldsymbol{\nabla}\cdot(\rho_0\,[\partial_t \mathbf{v}_1]) = 0 \tag{379}$$

now substitute (319) into (322) and then the result into (379) to obtain:

$$\boxed{\partial_t^2 \rho_1 - a_0^2 \boldsymbol{\nabla}^2 \rho_1 = 0} \tag{380}$$

Wave equation

Thus, ρ_1 obeys the classical **wave equation**.

5.3 Water waves

Let us examine two-dimensional water waves. Consider an infinite sea of depth H in a constant downward gravitational field g (see figure 18). The equations of motion are:

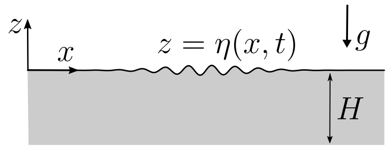

Figure 18: A group of water waves.

$$\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0 \tag{381}$$
$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} - g\hat{\mathbf{e}}_z \tag{382}$$
$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0 \tag{383}$$

where we have assumed (as is appropriate for water) that the fluid is incompressible. In equilibrium the surface of the water is at $z = 0$, and the equilibrium state for $z < 0$ is

$$\rho = \rho_0 = \text{constant} \tag{384}$$
$$P_0 = -\rho_0 g z + P_{\text{atm}} \tag{385}$$
$$\mathbf{v}_0 = 0 \tag{386}$$

We have assumed that at $z = 0$ the pressure is equal to the atmospheric pressure P_{atm}.

We want to study the propagation of *linear* waves. Since the fluid is incompressible, the fluid motion will necessarily be accompanied by a deformation of the water surface. We denote the surface as

$$z = \eta(x, t). \tag{387}$$

Linearising the equations of motion around the equilibrium state we get

$$\partial_t \mathbf{v}_1 = -\frac{\boldsymbol{\nabla} P_1}{\rho_0} \tag{388}$$
$$\boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0 \tag{389}$$

We have not written the continuity equation, which simply tells us that the density does not change with time as we would expect for an incompressible fluid. Note also that g disappears from the linearised equations. Taking the divergence of equation (388) and using (389) we get

$$\boldsymbol{\nabla}^2 P_1 = 0 \tag{390}$$

The pressure satisfies the Laplace equation! We want solutions that in the x direction look like a wave. Hence we look for solutions of the form:

$$P_1 = f(z) e^{ikx - i\omega t} \tag{391}$$
$$\mathbf{v}_1 = \mathbf{g}(z) e^{ikx - i\omega t} \tag{392}$$

Substituting (391) into Laplace equation (390) we get

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} = k^2 f(z) \tag{393}$$

The general solution to this equation is

$$f(z) = A e^{kz} + B e^{-kz} \tag{394}$$

where A and B are constants. We can find the relation between A and B imposing that at the bottom of the ocean, $z = -H$, the *vertical* velocity vanishes. Substituting (391) and (392) where $f(z)$ is given by (394) into (388) we obtain

$$\mathbf{v}_1 = \frac{1}{i\omega\rho_0}\left[ikf(z)\hat{\mathbf{e}}_x + f'(z)\hat{\mathbf{e}}_z\right]e^{ikx-i\omega t} \tag{395}$$

Requiring that $v_{1z}(x, z = -H) = 0$ implies $f'(z = -H) = 0$. We find

$$B = e^{-2kH}A \tag{396}$$

we can therefore rewrite f as

$$\begin{aligned}
f(z) &= Ae^{kz} + Be^{-kz} && (397)\\
&= A\left[e^{kz} + e^{-2kH}e^{-kz}\right] && (398)\\
&= Ae^{-kH}\left[e^{k(z+H)} + e^{-k(z+H)}\right] && (399)\\
&= A2e^{-KH}\cosh\left[k(z+H)\right] && (400)\\
&\equiv \tilde{P}\cosh\left[k(z+H)\right] && (401)
\end{aligned}$$

Thus pressure perturbation has the form

$$P_1 = \tilde{P}\cosh\left[k(z+H)\right]e^{ikx-i\omega t} \tag{402}$$

and the associated velocity is

$$\mathbf{v}_1 = \frac{\tilde{P}k}{i\omega\rho_0}\left[i\cosh\left[k(z+H)\right]\ \hat{\mathbf{e}}_x + \sinh\left[k(z+H)\right]\hat{\mathbf{e}}_z\right]e^{ikx-i\omega t} \tag{403}$$

To derive the dispersion relation, we need one more boundary condition. This condition is that at the surface the pressure always equals the atmospheric pressure $P_{\rm atm}$. We also assume, and we will check later that is actually the case, that points of the surface remain on the surface. Hence if we follow a fluid element on the surface, its pressure is always equal to the atmospheric pressure. This condition is written

$$\frac{DP}{Dt} = 0 \qquad \text{at } z = 0 \tag{404}$$

i.e.

$$\partial_t P + \mathbf{v}\cdot\boldsymbol{\nabla}P = 0 \qquad \text{at } z = 0 \tag{405}$$

Linearising this condition we get

$$\partial_t P_1 + \mathbf{v}_1\cdot\boldsymbol{\nabla}P_0 = 0 \qquad \text{at } z = 0 \tag{406}$$

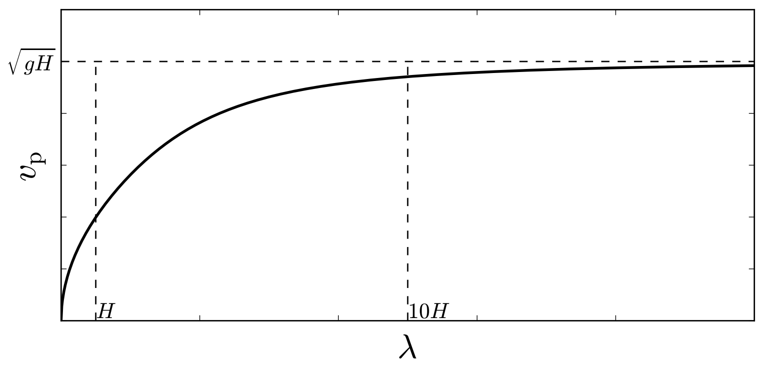

Figure 19: The phase speed $c = \omega/k$ for water waves given by (408) as a function of wavelength $\lambda = 2\pi/k$.

Substituting (402), (403) and (385) into (406) we get

$$\boxed{\omega^2 = gk\tanh(kH)} \tag{407}$$

This is the dispersion relation for small gravity waves in water. The pattern of a wave proportional to $e^{ikx-i\omega t}$ moves with a phase velocity $v_p = \omega/k$ given by

$$v_p^2(k) = gH\left(\frac{\tanh(kH)}{kH}\right) \tag{408}$$

v_p depends on k so these waves are *dispersive*, unlike sound waves studied in the previous section. Waves with different wavelengths travel with different speed. Figure 19 plots the phase speed as a function of wavelength. Longer waves travel *faster*. Moreover, as $\lambda \to \infty$ ($k \to 0$) the phase velocity tends to a constant, $v_p \to \sqrt{gH}$, so very long waves have a speed independent of water, like sound waves studied in the previous section. Long water waves are not dispersive.

An actual wave in the sea will never be an infinitely extended plane wave in all directions. A real "wave packet" is composed of many Fourier components. At small wavelengths, a wave packet composed of many different wavelengths spreads out over time, because the different waves that compose the packet have different speeds. At long wavelengths, a wave packet will retain its shape. This has important and sometimes catastrophic consequences.

Tsunami In the sea, earthquakes can create "wave packets" with a typical wavelength of 200 km (while wind-generated waves have a typical length of 100 metres). These are called *tsunamis*. The wavelength of the tsunami is far greater than the depth of the ocean (which is typically 2-3 km). Initially, tsunamis have relatively small amplitude (one meter is typical), which would seem to render them harmless. In fact, tsunamis often pass by ships in deep ocean without anyone on board even noticing. However, since it is made by long waves, and the phase speed is independent of wavelength at long wavelengths, a tsunami can retain its power while travelling without spreading its amplitude. When the tsunami approaches the coast, the depth H of the water decreases, while the frequency ω is approximately conserved. Since $\omega^2 \simeq gHk^2$ for long waves, k must become larger, so the distance between successive wave crests decreases. The wave "piles up" and inevitably grows in amplitude. The velocity of the waves greatly decreases, typically from $\sim$ 500km/h when the depth is $\sim$ 4 km to $\sim$ 50km/h when the depth is $\sim$ 20 m. Tsunami waves can grow up to 30 meters in height as they hit the shoreline, causing disasters as in the 2011 tsunami in Japan.

Particle paths What is the path followed by a fluid element in a water wave? To find out, let us define the **lagrangian displacement $\boldsymbol{\xi}$**. This is defined as the displacement of a fluid element from its position at $t = 0$. In other words, assume that at $t = 0$ a fluid element is at the position $\mathbf{x}$. At the time t its position will be

$$\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t) \tag{409}$$

and by definition $\boldsymbol{\xi}(\mathbf{x}, 0) = 0$. It follows from its definition that $\boldsymbol{\xi}$ satisfies the following equation:

$$\frac{D\boldsymbol{\xi}}{Dt} = \mathbf{v}(\mathbf{x} + \boldsymbol{\xi}, t) \tag{410}$$

In the water waves that we have studied, $\mathbf{v} = \mathbf{v}_1$ is small, and $\boldsymbol{\xi}$ is also small. Hence neglecting second order terms we can approximate this equation as

$$\partial_t \boldsymbol{\xi} \simeq \mathbf{v}_1(\mathbf{x}, t) \tag{411}$$

Expanding this in components and using the expression (403) for the velocity, we have:

$$\partial_t \xi_x = \frac{\tilde{P}k}{\omega\rho_0} \cosh\left[k(z + H)\right] e^{ikx - i\omega t} \tag{412}$$

$$\partial_t \xi_z = \frac{\tilde{P}k}{i\omega\rho_0} \sinh\left[k(z + H)\right] e^{ikx - i\omega t} \tag{413}$$

Solving this we find (apart from an unimportant constant that would be needed to satisfy the boundary condition $\mathrm{Re}[\boldsymbol{\xi}(\mathbf{x},0)]=0$):

$$\xi_x(x,z,t) = -\frac{\tilde{P}k}{i\omega^2\rho_0}\cosh\left[k(z+H)\right]e^{ikx-i\omega t} \tag{414}$$

$$\xi_z(x,z,t) = \frac{\tilde{P}k}{\omega^2\rho_0}\sinh\left[k(z+H)\right]e^{ikx-i\omega t} \tag{415}$$

Taking the real part to get the actual paths we find:

$$\xi_x(x,z,t) = -\frac{\tilde{P}k}{\omega^2\rho_0}\cosh\left[k(z+H)\right]\sin(kx-\omega t) \tag{416}$$

$$\xi_z(x,z,t) = \frac{\tilde{P}k}{\omega^2\rho_0}\sinh\left[k(z+H)\right]\cos(kx-\omega t) \tag{417}$$

This means that particles path are ellipses of axis ratio $\tanh[k(z+H)]$. The ellipses become flatter with depth, and at $z=-H$ particles oscillate along horizontal segments. Hence particles are not moving very far from their original positions, despite the wave being travelling and transporting energy.

Equation of the surface The surface $\eta(x,t)$ is defined as the surface at constant pressure $P=P_{\mathrm{atm}}$. We have from (385) and (402) that total pressure in a water wave is given by:

$$P = P_0 + P_1 = -\rho_0 g z + P_{\mathrm{atm}} + \tilde{P}\cosh\left[k(z+H)\right]e^{ikx-i\omega t} \tag{418}$$

To find the equation of the surface we use that for points on the surface $z=\eta(x,t)$ and $P=P_{\mathrm{atm}}$. Hence

$$P_{\mathrm{atm}} = -\rho_0 g\eta + P_{\mathrm{atm}} + \tilde{P}\cosh\left[k(\eta+H)\right]e^{ikx-i\omega t} \tag{419}$$

Since η is small and $\tilde{P}$ is also small, we can approximate $\cosh\left[k(\eta+H)\right]\simeq\cosh\left[kH\right]$ in the second term on the RHS. Solving for η we then obtain

$$\eta(x,t) = \frac{\tilde{P}}{\rho_0 g}\cosh\left[kH\right]e^{ikx-i\omega t} \tag{420}$$

Thus, as expected, the shape of the surface oscillates with the usual factor $e^{ikx-i\omega t}$.

Something at the surface stays on the surface We can check our assumption that something at the surface stays at the surfaces. A particle on the surface at time t and position x has $z = \eta(x,t)$. In a small interval $\mathrm{d}t$ this particle moves to the point

$$x \to x + v_x \mathrm{d}t, \quad z \to z + v_z \mathrm{d}t \tag{421}$$

If this particle is still on the surfaces, this means that

$$\eta(x + v_x \mathrm{d}t, t + \mathrm{d}t) = z + v_z \mathrm{d}t \tag{422}$$

Expanding we obtain

$$(\partial_x \eta) v_x + \partial_t \eta = v_z \tag{423}$$

Note that this equation is valid also in the case of non linear waves. For linear water waves, velocities are small and η is also small, so we can neglect the first term on the LHS. Thus we can approximate to first order to

$$\partial_t \eta = v_z \tag{424}$$

We can check whether this equation is satisfied by our solutions (420) and (403). Substituting these into (424) we obtain again the dispersion relation (407), and everything is OK.

Exact meaning of "small" waves We have assumed that our waves are "small". Small with respect to what? We can derive a geometrical meaning for "small amplitude" by looking at equation (423). In this equation, we have neglected the term $(\partial_x \eta) v_x$ compared to v_z. Since v_x and v_z are of the same order of magnitude according to (403), for our approximation to be valid we need

$$\partial_x \eta \ll 1 \tag{425}$$

In other words, the *slope* of the free surface must be small.

5.4 Group velocity

A dispersion relation

$$\omega = \omega(\mathbf{k}) \tag{426}$$

relates the frequency ω to the wavenumber $\mathbf{k}$ of plane waves solutions proportional to $e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$. We have seen that for sound waves the dispersion relation (332) is linear in the wavenumber k, while for small water

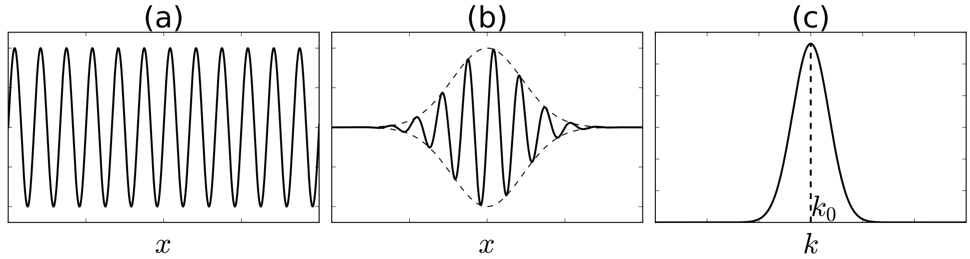

Figure 20: (a) An infinite plane wave with wavenumber k_0. (b) A typical wave packet. The dashed curve is the envelope of the wave packet. (c) The absolute value $|\tilde{f}(k)|$ of the Fourier transform of the packet. Reproduced from [7].

waves ω is related to k through the relation (407). If $\omega(k)$ is linear in k, then the phase velocity

$$v_{\mathrm{p}}(k) = \frac{\omega(k)}{k} \tag{427}$$

is constant. If $\omega(k)$ is not linear in k, waves with different wavelengths travel at different phase velocities, so if they start together after a while they "disperse", and the medium is said to be **dispersive**.

Any real signal is not an infinite plane wave like that shown in Fig. 20 (a), which is unphysical since it extends through all space, but a **wave packet** like that shown in 20 (b), which describes a wave whose amplitude is non-zero over a finite region of space.

How do we find the time evolution of a wave packet in a dispersive medium? Suppose you are given the shape $f(x, t = 0)$ of the packet at $t = 0$. Any packet can be seen as a sum Fourier components:

$$f(x, t = 0) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \tilde{f}(k) e^{ikx} \tag{428}$$

Each component of wavenumber k evolves with its own frequency $\omega(k)$. Hence the packet at time t is:

$$f(x, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \tilde{f}(k) e^{ikx - i\omega(k)t} \tag{429}$$

This general procedure allows us to find the time evolution for any type of linear disturbance once we know the dispersion relation. In the most general case, there isn't any restriction on the shape of the wave packet. This solution is valid for any shape. However, let us analyse the particular case in which the Fourier components that make up the wave packet are all localised around a wavenumber k_0, as in the example of Fig. 20 (b). In this case, we may expand $\omega(k)$ in a Taylor series,

$$\omega(k) \simeq \omega(k_0) + (k - k_0)v_{\rm g} \tag{430}$$

where

$$v_{\rm g} = \left(\frac{{\rm d}\omega(k)}{{\rm d}k}\right)_{k_0} \tag{431}$$

Substituting back into (429) and writing $k = k_0 + u$ we have

$$\begin{aligned} f(x,t) &= \int_{-\infty}^{\infty} \frac{{\rm d}k}{2\pi} \tilde{f}(k) e^{ikx - i\omega(k)t} && (432)\\ &\simeq \int_{-\infty}^{\infty} \frac{{\rm d}k}{2\pi} \tilde{f}(k) e^{i(k_0+u)x - i(\omega(k_0)+uv_{\rm g})t} && (433)\\ &= e^{ik_0x - i\omega(k_0)t} \int_{-\infty}^{\infty} \frac{{\rm d}u}{2\pi} \tilde{f}(k_0+u) e^{iu(x - v_{\rm g}t)} && (434)\\ &\equiv e^{ik_0x - i\omega(k_0)t} F(x - v_{\rm g}t) && (435) \end{aligned}$$

where we have defined

$$F(x) = \int_{-\infty}^{\infty} \frac{{\rm d}u}{2\pi} \tilde{f}(k_0+u) e^{iux} \tag{436}$$

The function $F(x)$ describes the *envelope* of the wave packet, dashed lines in Fig. 20 (b). To see this, note that the expression (435) is composed by a factor $e^{ik_0x - i\omega(k_0)t}$ and by F. $e^{ik_0x - i\omega(k_0)t}$ describes a perfect monochromatic wave with wavenumber k_0. F, on the other hand, is essentially composed of Fourier components of wavenumbers $|u| \ll k_0$, because our original wave packet was made was made by Fourier components localised around $k = k_0$, so $\tilde{f}(k_0+u)$ is non-zero only in a neighbourhood of $u = 0$ in the integral that defines F. Thus $F(x)$ varies much more slowly than e^{ik_0x}. Therefore, (435) at a given time describes a rapidly varying wave proportional to e^{ik_0x} modulated by an envelope given by $F(x)$. It also shows that the envelope propagates with a velocity given by $v_{\rm g}$, which is known as **group velocity**. The group velocity represents the true velocity of a localised physical disturbance. If the original wave packet is not localised in k space, then the envelope of a wave packet will become distorted as it travels.

In the case of a linear dispersion relation $\omega = \pm c_0 k$, such as in the case of sound waves, the phase velocity and the group velocity coincide and are constant. A wave of any shape will travel undistorted at this velocity. In case of dispersive waves, phase and group velocities are generally different. For example, in the case of small water waves the phase velocity is higher than the group velocity (see 407). Therefore, in packet like that shown in 18, wave crests may be seen continually appearing at the back of wave packet and disappearing at the front. A stationary observer would count more crests than one can count in a frozen snapshot like 18 as the whole wave packet passes by.[29]

Although we have not shown it, the group velocity is often thought of as the velocity at which energy is transported by waves of wavelength $2\pi/k$, and at which information is conveyed along a wave. In most cases this is accurate, however if the wave is travelling through a dissipative medium and waves are damped, the group velocity may not be a meaningful quantity or may not even be a well-defined quantity.

5.5 Analogy between shallow water theory and gas dynamics

There is a remarkable analogy between the equations that describe the flow of shallow water in a constant gravitational field and the equations of adiabatic gas dynamics for $\gamma = 2$. This powerful analogy can be used for example to understand what happens in an astrophysical situation by picturing the flow of water on a surface with a given shape that mimics an external gravitational field. You can simulate gas flow in spiral galaxies in a bathtub with a spiral shape.

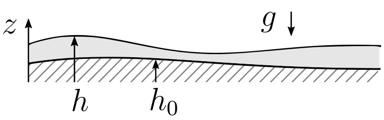

Figure 21: Shallow water waves.

In the **shallow water approximation**, we consider again the equations of motion for water waves (381)-(383). This time we *do not* assume that waves are small compared with the depth, so linear theory does not apply, but we assume that the typical depth $(h - h_0)$ is small compared to typical length-scale L of waves (see Fig. 21):

Shallow water approximation

$$(h - h_0) \ll L \tag{437}$$

where $z = h_0(x, y)$ is a given function describing the bottom of the ocean, and $z = h(x, y, t)$ describes the surface of the water. We also assume that these functions are slowly varying:

$$\partial_x h \sim \frac{h}{L} \ll 1 \tag{438}$$

$$\partial_x h_0 \lesssim \frac{h_0}{L} \ll 1 \tag{439}$$

[29]see for example the nice animation at https://en.wikipedia.org/wiki/Dispersion_(water_waves).

As we shall see, in the shallow water approximation vertical motions can be neglected with respect to horizontal motions, and horizontal motions are considered independent of depth. As a result, the fluid can be considered a two-dimensional medium characterised at each point by a horizontal velocity $\mathbf{v}(x, y)$ and a height $h(x, y)$.

We now derive the equations of shallow water theory. Written out in full, equations (382)-(383) read:

$$\partial_t v_x + (v_x\partial_x + v_y\partial_y + v_z\partial_z)v_x = -\frac{\partial_x P}{\rho} \tag{440}$$

$$\partial_t v_y + (v_x\partial_x + v_y\partial_y + v_z\partial_z)v_y = -\frac{\partial_y P}{\rho} \tag{441}$$

$$\partial_t v_z + (v_x\partial_x + v_y\partial_y + v_z\partial_z)v_z = -\frac{\partial_z P}{\rho} - g \tag{442}$$

$$\partial_x v_x + \partial_y v_y + \partial_z v_z = 0 \tag{443}$$

We do not need to write down the continuity equation which simply tells us that density is constant. First, we neglect all terms on the LHS of (442), that is Dv_z/Dt. We will check the validity of this approximation a posteriori. (442) reduces to:

$$0 = -\frac{\partial_z P}{\rho} - g, \tag{444}$$

which means that the pressure as a function of depth is the same as in hydrostatic equilibrium. We impose that at the surface, defined by $z = h(x, y, t)$, we have $P = P_{\text{atm}}$. Solving (444) with this condition we obtain

$$P = P_{\text{atm}} + \rho g\,[h(x, y, t) - z] \tag{445}$$

Substituting this into (440) and (441) we find

$$\partial_t v_x + (v_x\partial_x + v_y\partial_y + v_z\partial_z)v_x = -g\partial_x h \tag{446}$$

$$\partial_t v_y + (v_x\partial_x + v_y\partial_y + v_z\partial_z)v_y = -g\partial_y h \tag{447}$$

Now we make one further assumption, namely that v_x and v_y are independent of z at the initial time. This assumption is satisfied if for example the fluid is initially at rest. Since h does not depend on z, the right hand side of equations (446) and (447) does not depends on z. Hence, whatever the function h might be, these equations imply that v_x and v_y remain independent of z for all t. Thus we may simplify (446) and (447) as

$$\boxed{\begin{aligned} \partial_t v_x + (v_x\partial_x + v_y\partial_y)v_x &= -g\partial_x h \\ \partial_t v_y + (v_x\partial_x + v_y\partial_y)v_y &= -g\partial_y h \end{aligned}} \qquad \begin{aligned} (448) \\ (449) \end{aligned}$$

Since v_x and v_y are independent of z, we can integrate (443) to give

$$v_z = -\left(\partial_x v_x + \partial_y v_y\right) z + f(x, y, t) \tag{450}$$

where $f(x, y, t)$ is an arbitrary function. We can determine it by imposing that at the bottom of the ocean the component of velocity perpendicular to the surface of the bottom vanishes. To find the vector perpendicular to this surface consider the function

$$F(x, y, z) = h_0(x, y) - z \tag{451}$$

the surface of the bottom is one of the surfaces defined by

$$F(x, y, z) = \text{constant} \tag{452}$$

and $\boldsymbol{\nabla} F = \partial_x h_0 \hat{\mathbf{e}}_x + \partial_x h_0 \hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z$ is perpendicular to these surfaces. Hence the vector

$$\mathbf{N} = \partial_x h_0 + \partial_y h_0 - \hat{\mathbf{e}}_z \tag{453}$$

is perpendicular to the bottom of the ocean. Thus, at the bottom of the ocean we must have

$$\mathbf{v} \cdot \mathbf{N} = 0 \quad \text{at} \quad z = h_0(x, y) \tag{454}$$

Expanding this becomes

$$v_x \partial_x h_0 + v_y \partial_y h_0 - v_z = 0 \quad \text{at} \quad z = h_0(x, y) \tag{455}$$

the expressions for v_z given by (455) and (450) must coincide at $z = h_0(x, y)$. Imposing this we obtain

$$f(x, y, t) = \partial_x(v_x h_0) + \partial_y(v_y h_0) \tag{456}$$

and substituting back into (450) we obtain

$$v_z = -\left(\partial_x v_x + \partial_y v_y\right) z + \partial_x(v_x h_0) + \partial_y(v_y h_0). \tag{457}$$

This equation gives v_z as a function of v_x and v_y. To find an equation for the evolution of the surface, we must impose as we did in Section 5.3 that elements at the surface stay at the surface, so they always feel same pressure, which is the atmospheric pressure P_{atm}. Hence

$$\frac{DP}{Dt} = 0 \quad \text{at} \quad z = h(x, y, t) \tag{458}$$

Using (445) this becomes

$$\partial_t h + (v_x \partial_x + v_y \partial_y) h - v_z = 0 \quad \text{at} \quad z = h(x, y, t). \tag{459}$$

The expressions for v_z given this equation and by (457) must coincide at $z = h(x, y, t)$. Hence we obtain

$$\boxed{\partial_t h + \partial_x \left[v_x \left(h_0 - h \right) \right] + \partial_y \left[v_y \left(h_0 - h \right) \right] = 0} \tag{460}$$

Equations (448), (449) and (460) are the equation of shallow water theory and form a complete system of equations. Given the state of the system at $t = 0$, we can evolve it in time. It remains to check our approximation of neglecting the LHS in (442). Equations (448) and (449) tell us that typical values for the acceleration of fluid elements are $a \sim g \partial_x h \sim gh/L$. Since $a \sim v_{x,y}/T$ and $v_{x,y} \sim L/T$ where T is a typical timescale over which $v_{x,y}$ vary, this implies that typical horizontal velocities are

$$v_{x,y} \sim (gh)^{1/2} \tag{461}$$

Then from (457) we have that

$$v_z \sim \left(\frac{h}{L} \right) v_{x,y} \tag{462}$$

hence, vertical velocities are much smaller than horizontal velocities. Using these results, we can see that all the terms on the LHS of (442) are of order $(h/L)^2 g$. For example $\partial_t v_z \sim (1/T) v_z \sim (1/T)(h/L) v_{x,y}$ and using $T \sim L/(gh)^{1/2}$ we obtain $\partial_t v_z \sim (h/L)^2 g$. Hence the terms on the LHS of (442) can be safely neglected, as assumed.

Now introduce the following quantities:

$$\bar{\rho} = \rho (h - h_0) \tag{463}$$

$$\bar{\mathbf{v}} = v_x \hat{\mathbf{e}}_x + v_y \hat{\mathbf{e}}_y \tag{464}$$

$\bar{\rho}$ is simply the surface density of water. Rewriting (460), (448) and (449) in terms of these yields

$$\partial_t \bar{\rho} + \boldsymbol{\nabla} \cdot (\bar{\rho} \bar{\mathbf{v}}) = 0 \tag{465}$$

$$\partial_t \bar{\mathbf{v}} + (\bar{\mathbf{v}} \cdot \boldsymbol{\nabla}) \bar{\mathbf{v}} = -g \frac{\boldsymbol{\nabla} \bar{\rho}}{\rho} - \boldsymbol{\nabla} h_0 \tag{466}$$

These are just the equation of a gas with the following equation of state

$$P = \left(\frac{g}{2\rho} \right) \bar{\rho}^2 \tag{467}$$

flowing in an external gravitational potential $\Phi = h_0$ i.e., the equations of an adiabatic and isentropic gas with $\gamma = 2$.

5.6 MHD waves

Consider waves propagating in a uniform magnetised medium at rest. The only difference between this setup and the sound waves studied in Section 5.2 is the presence of a uniform magnetic field. The equations of motions are the ideal MHD equations (205), (206), (207) and (212). The background steady state is

$$\rho = \rho_0, \qquad P = P_0, \qquad \mathbf{v} = 0, \qquad \mathbf{B} = \mathbf{B}_0 = B_0 \hat{\mathbf{e}}_z \tag{468}$$

where ρ_0, P_0 and B_0 are constants, and without loss of generality we have taken the magnetic field parallel to the z axis. Now as usual we linearise the equations of motions around the background state by writing

$$\rho = \rho_0 + \rho_1, \qquad P = P_0 + P_1, \qquad \mathbf{v} = \mathbf{v}_1, \qquad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 \tag{469}$$

We consider *adiabatic* perturbations, i.e., perturbations in which the entropy of the background state is the same as the entropy of the perturbed state (so they are also called isentropic). Following the same steps as in Section 5.2 (cf equation 319) we then find that (212) implies

$$\boxed{P_1 = c_0^2 \rho_1} \tag{470}$$

where

$$c_0 = \left(\frac{\gamma P_0}{\rho_0} \right)^{1/2}. \tag{471}$$

Linearising the equations (205), (206), (207) and using (470) we find:

$$\partial_t \rho_1 = -\rho_0 \boldsymbol{\nabla} \cdot \mathbf{v}_1 \tag{472}$$

$$\partial_t \mathbf{v}_1 = -c_0^2 \frac{\boldsymbol{\nabla} \rho_1}{\rho_0} + \frac{1}{4\pi\rho_0} (\boldsymbol{\nabla} \times \mathbf{B}_1) \times \mathbf{B}_0 \tag{473}$$

$$\partial_t \mathbf{B}_1 = \boldsymbol{\nabla} \times (\mathbf{v}_1 \times \mathbf{B}_0) \tag{474}$$

Now we look for solutions of the form

$$\rho_1 = \tilde{\rho}_1 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{475}$$

$$\mathbf{v}_1 = \tilde{\mathbf{v}}_1 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{476}$$

$$\mathbf{B}_1 = \tilde{\mathbf{B}}_1 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{477}$$

We need to do quite some algebra to get to the dispersion relation, so in the following we drop the ˜ to make notation shorter. Substituting

(475)-(477) into (472)-(474) we obtain

$$-\omega\rho_1 = -\rho_0 \mathbf{k}\cdot\mathbf{v}_1 \tag{478}$$

$$-\omega\mathbf{v}_1 = -c_0^2\frac{\mathbf{k}\rho_1}{\rho_0} + \frac{1}{4\pi\rho_0}(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0 \tag{479}$$

$$-\omega\mathbf{B}_1 = \mathbf{k}\times(\mathbf{v}_1\times\mathbf{B}_0) \tag{480}$$

Isolating ρ_1 and $\mathbf{B}_1$ from the first and last equation we have[30]

$$\rho_1 = \rho_0\frac{(\mathbf{k}\cdot\mathbf{v}_1)}{\omega} \tag{481}$$

$$\mathbf{B}_1 = -\frac{1}{\omega}\mathbf{k}\times(\mathbf{v}_1\times\mathbf{B}_0) \tag{482}$$

$$= -\frac{1}{\omega}\left[(\mathbf{k}\cdot\mathbf{B}_0)\mathbf{v}_1 - (\mathbf{k}\cdot\mathbf{v}_1)\mathbf{B}_0\right] \tag{483}$$

where in the second step we have used the identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}\,. \tag{484}$$

Substituting (481) and (483) into (479) (so that we get an equation in which the only unknown is $\mathbf{v}_1$) we obtain

$$-\omega^2\mathbf{v}_1 = -c_0^2(\mathbf{k}\cdot\mathbf{v}_1)\mathbf{k} - \frac{1}{4\pi\rho_0}\left\{\mathbf{k}\times\left[(\mathbf{k}\cdot\mathbf{B}_0)\mathbf{v}_1 - (\mathbf{k}\cdot\mathbf{v}_1)\mathbf{B}_0\right]\right\}\times\mathbf{B}_0\,. \tag{485}$$

Now using the following identities

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\,\mathbf{A} \tag{486}$$

$$(\mathbf{k}\times\mathbf{v}_1)\times\mathbf{B}_0 = (\mathbf{k}\cdot\mathbf{B}_0)\mathbf{v}_1 - (\mathbf{v}_1\cdot\mathbf{B}_0)\mathbf{k} \tag{487}$$

$$(\mathbf{k}\times\mathbf{B}_0)\times\mathbf{B}_0 = (\mathbf{k}\cdot\mathbf{B}_0)\mathbf{B}_0 - B_0^2\mathbf{k} \tag{488}$$

and rearranging a bit, we can rewrite (485) as

$$\left[\omega^2 - \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{4\pi\rho_0}\right]\mathbf{v}_1 + (\mathbf{k}\cdot\mathbf{v}_1)\left[\frac{(\mathbf{k}\cdot\mathbf{B}_0)}{4\pi\rho_0}\mathbf{B}_0 - \left(c_0^2 + \frac{B_0^2}{4\pi\rho_0}\right)\mathbf{k}\right] + \frac{(\mathbf{k}\cdot\mathbf{B}_0)(\mathbf{v}_1\cdot\mathbf{B}_0)}{4\pi\rho_0}\mathbf{k} = 0 \tag{489}$$

We now assume without loss of generality that $\mathbf{k} = k_x\hat{\mathbf{e}}_x + k_z\hat{\mathbf{e}}_z$ (because the x and y direction are physically identical). Introducing the **Alfvén speed**

$$\boxed{v_{\mathrm{A}}^2 = \frac{B_0^2}{4\pi\rho_0}} \tag{490}$$

Alfvén speed

[30]Note that we are dividing by ω, so we are assuming $\omega \neq 0$. Thus we are losing possible vortex modes such as those discussed in Section 5.2.

and recalling that we have also assumed $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_z$, we have

$$(\mathbf{k}\cdot\mathbf{v}_1) = k_x v_{1x} + k_z v_{1z} \tag{491}$$

$$\frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{4\pi\rho_0} = k_z^2 v_\mathrm{A}^2 \tag{492}$$

$$(\mathbf{v}_1\cdot\mathbf{B}_0) = v_{1z}B_0 \tag{493}$$

Hence we can rewrite (489) as

$$\left[\omega^2 - k_z^2 v_\mathrm{A}^2\right]\mathbf{v}_1 + (k_x v_{1x} + k_z v_{1z})\left[k_z v_\mathrm{A}^2\hat{\mathbf{e}}_z - \left(c_0^2 + v_\mathrm{A}^2\right)\mathbf{k}\right] + k_z v_{1z} v_\mathrm{A}^2\mathbf{k} = 0 \tag{494}$$

Or equivalently as

$$\begin{pmatrix} \omega^2 - k^2 v_\mathrm{A}^2 - k_x^2 c_0^2 & 0 & -k_x k_z c_0^2 \\ 0 & \omega^2 - k_z^2 v_\mathrm{A}^2 & 0 \\ -k_x k_z c_0^2 & 0 & \omega^2 - k_z^2 c_0^2 \end{pmatrix}\begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z}\end{pmatrix} = 0 \tag{495}$$

This system admits a non-trivial solution for $\mathbf{v}_1$ (i.e., a solution different from $\mathbf{v}\equiv 0$) if and only if the determinant of the big square matrix is zero. Imposing this we obtain the dispersion relation

$$\left(\omega^2 - k_z^2 v_\mathrm{A}^2\right)\left[\omega^4 - \omega^2 k^2\left(c_0^2 + v_\mathrm{A}^2\right) + c_0^2 k^2 k_z^2 v_\mathrm{A}^2\right] = 0 \tag{496}$$

There are three different types of solutions of this equation, corresponding to three different types of waves.

Alfvén waves The first solution has

$$\omega^2 = k_z^2 v_\mathrm{A}^2 \tag{497}$$

let us first consider the case $k_x = 0$, $k_z \neq 0$, i.e. a wave propagating along z, parallel to the magnetic field. Then any vector of the following form is an eigenvector of the square matrix in (495)

$$\mathbf{v}_1 = v_{1x}\hat{\mathbf{e}}_x + v_{1y}\hat{\mathbf{e}}_y \tag{498}$$

From (481) and (483) we see that $\rho_1 = 0$, so there are no density and pressure perturbations associated with this wave, and that $\mathbf{B}_1 \propto \mathbf{v}_1$, so the directions of velocity and magnetic perturbation coincide. Thus we have waves that propagate with velocity v_A parallel to the unperturbed magnetic field, fluid elements that oscillate perpendicular to the direction of propagation, and the magnetic field that bends like a string under tension. These waves are in fact very similar to waves on a string, such as a violin string. The restoring force is not pressure (the pressure perturbation vanishes!), but the magnetic tension that we studied in Section 2.2. Since $\boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0$ there is no associated compression, and this wave bends magnetic field-lines without squeezing them together.

If we now allow for $k_x \neq 0$, eigenvectors of the square matrix in (495) are of the following form

$$\mathbf{v}_1 = v_{1y}\hat{\mathbf{e}}_y \tag{499}$$

Now the wave can propagate along x too, and the magnetic field lines bend in the y direction. The bending has a phase which depends on x, resulting in a pattern travelling in the x direction. Physically, this case is similar to the previous one. The velocity of propagation along z is v_{A} as before, while the velocity of propagation along x is arbitrary and does not carry information, as one would expect since it is not associated with any restoring force but is just the result of a "phase tuning".

Fast and slow waves the other two solutions of the dispersion relation can be written

$$\omega^2 = k^2 v_{\pm}^2 \tag{500}$$

where

$$v_{\pm}^2 = \frac{1}{2}\left[v_{\mathrm{A}}^2 + c_0^2 \pm \sqrt{\left(v_{\mathrm{A}}^2 + c_0^2\right)^2 - 4v_{\mathrm{A}}^2 c_0^2 (k_z/k)^2}\right] \tag{501}$$

The eigenvector of the square matrix in (495) for these waves are of the form

$$\mathbf{v}_1 = v_{1x}\hat{\mathbf{e}}_x + v_{1z}\hat{\mathbf{e}}_z \tag{502}$$

From (481) and (483) we then see that $\rho_1 \neq 0$ (since $(\mathbf{k} \cdot \mathbf{v}_1) \neq 0$), so these waves are associated with density and pressure perturbations. To interpret these waves, let us introduce the angle θ between $\mathbf{k}$ and $\mathbf{B}_0$ and consider the particular case $\cos\theta = k_z/k \ll 1$. Then (501) simplifies to

$$v_+^2 \simeq v_{\mathrm{A}}^2 + c_0^2; \qquad v_-^2 \simeq \frac{v_{\mathrm{A}}^2 c_0^2}{v_{\mathrm{A}}^2 + c_0^2}\cos\theta; \tag{503}$$

The first is called **fast wave**, the second **slow wave**. These are pressure waves, analogous to the sound waves studied in section (5.2). In the fast wave, also called a **magnetosonic** wave, the pressure and magnetic pressure work together reinforcing each other, while in the slow wave, pressure and magnetic forces work in opposition. This can be also seen by comparing the signs of the perturbations to the standard pressure P_1 and magnetic pressure $\mathbf{B}_0 \cdot \mathbf{B}_1/4\pi$ that are associated with the wave.

5.7 Steepening of sounds waves and the formation of shocks

In Section 5.2 we have studied sound waves propagating in a uniform medium, and we have seen that in the linear approximation a small perturbation travels without changing its shape. However, this result rests on neglecting second order terms in the perturbed quantities. In reality, when the equations are solved exactly, any perturbation in an inviscid fluid, no matter how small, will eventually distort and become a shock. This process is called non-linear "steepening" of the wave, and is what we are going to study in this section.

Consider a one-dimensional inviscid isentropic fluid (see Section 1.5 for the definition of isentropic). The equations of motions are

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{504}$$

$$\partial_t v + v\partial_x v = -\frac{\partial_x P}{\rho} \tag{505}$$

$$P\rho^{-\gamma} = P_0 \rho_0^{-\gamma} \tag{506}$$

where P_0 and ρ_0 are constants. We are going to analyse these equations *exactly*, i.e. without making any approximation. This is possible thanks to a brilliant mathematical transformation due to Riemann, one of the greatest mathematicians of the nineteenth century (the same person of the Riemann hypothesis, which many consider to be the most important unsolved problem in pure mathematics).

Let us rewrite the continuity equation (504) as:

$$\partial_t \rho + v\partial_x \rho + \rho \partial_x v = 0 \tag{507}$$

Now let us replace ρ with P in this equation. Differentiating Eq. (506) we find

$$\frac{\mathrm{d}P}{P} = \gamma \frac{\mathrm{d}\rho}{\rho} \,. \tag{508}$$

Hence

$$\partial_t \rho = \frac{1}{c^2}\partial_t P \,; \qquad \partial_x \rho = \frac{1}{c^2}\partial_x P \,, \tag{509}$$

where

$$c^2 = \frac{\gamma P}{\rho} \tag{510}$$

is the local sound speed (which now is a function of x and t, not a constant!). Using (509) and multiplying by c/ρ we can rewrite (507) as

$$\frac{1}{\rho c}\partial_t P + \frac{v}{\rho c}\partial_x P + c\,\partial_x v = 0 \,. \tag{511}$$

Adding and subtracting (505) and (511) we find

$$[\partial_t v + (v+c)\partial_x v] + \frac{1}{\rho c}\,[\partial_t P + (v+c)\partial_x P] = 0 \tag{512}$$

$$[\partial_t v + (v-c)\partial_x v] - \frac{1}{\rho c}\,[\partial_t P + (v-c)\partial_x P] = 0 \tag{513}$$

The equations of motion written in this form can be interpreted as follows. Recall that the time derivative of a generic function $f(x,t)$ along an arbitrary curve $x = \varphi(t)$ in the (x,t) plane can be written as:

$$\frac{\mathrm{d}f}{\mathrm{d}t}_{\varphi} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\,. \tag{514}$$

Therefore Eqs. (512) and (513) contain derivatives of the quantities v and P only along the curves $C_{\pm}(t)$ such that $\mathrm{d}x/\mathrm{d}t = (v \pm c)$ respectively. This means that the fluid equations can be interpreted as propagation of two signals that move with velocities $\mathrm{d}x/\mathrm{d}t = (v \pm c)$ from any given point in the fluid. The curves $C_{\pm}$ are called **characteristics** (see for example Chapter 1 in [8] if you want to know more).

Now to proceed further let us express P and c as a function of ρ (use Eqs. 506 and 510):

$$P = P_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}\,, \tag{515}$$

$$c = c_0\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}\,, \tag{516}$$

where $c_0 = (\gamma P_0/\rho_0)^{1/2}$. Using these expressions we find

$$\frac{\mathrm{d}P}{\rho c} = \frac{2}{(\gamma-1)}\mathrm{d}c\,. \tag{517}$$

Using (517) we can rewrite Eqs. (512) and (513) as

$$[\partial_t J_+ + (v+c)\partial_x J_+] = 0 \tag{518}$$

$$[\partial_t J_- + (v-c)\partial_x J_-] = 0 \tag{519}$$

where

$$J_{\pm} = v \pm \frac{2}{(\gamma-1)}c \tag{520}$$

Or in other words

$$\frac{\mathrm{d}J_+}{\mathrm{d}t} = 0 \qquad \text{along the curve} \quad C_+ : \frac{\mathrm{d}x}{\mathrm{d}t} = v + c\,, \tag{521}$$

$$\frac{\mathrm{d}J_-}{\mathrm{d}t} = 0 \qquad \text{along the curve} \quad C_- : \frac{\mathrm{d}x}{\mathrm{d}t} = v - c\,. \tag{522}$$

This is the clever way in which Riemann recast the equations. The quantities $J_\pm$ are called **Riemann invariants**. The meaning of Eqs. (521) and (522) is as follows. The first equation shows that the quantity J_+ is constant along a trajectory $x(t)$ in the (x,t) plane such that $\mathrm{d}x/\mathrm{d}t = v+c$. The second equation similarly shows that J_- is constant along the curve $\mathrm{d}x/\mathrm{d}t = v - c$.

We are now ready to use the equations to understand the non-linear development of waves. Consider initial conditions $\rho(x,t=0)$ and $v(x,t=0)$ such that $J_-(x,t=0) = J_0 = \text{constant}$. Equation (522) tells us that J_- is invariant along characteristic lines, and since it is initially constant, it will stay constant to the same value at all times. Therefore we have that at all times the following relation holds:

$$v - \frac{2}{(\gamma-1)}c = J_0 \tag{523}$$

Since in isentropic flow c is a function of ρ (Eq. 516), this means that *the velocity is a function of density only*, $v = v(\rho)$ (at all times!). Now let's see what the other equation (Eq. 521) tells us. Since both the velocity (Eq. 523) and the sound speed (Eq. 516) are functions of ρ only, also $J_+ = J_+(\rho)$ is a function of ρ only (Eq. 520). Eq. (521) says that J_+ is constant along the lines $\mathrm{d}x/\mathrm{d}t = v+c$. But since we have just shown that J_+ is a function of ρ only, this means that also the density is constant along the lines $\mathrm{d}x/\mathrm{d}t = v+c$. Moreover, since $v = v(\rho)$, also the velocity is constant along the lines $\mathrm{d}x/\mathrm{d}t = v+c$! Hence the characteristic lines originating from each point are straight lines:

$$x(t) = [v + c(v)]t\,. \tag{524}$$

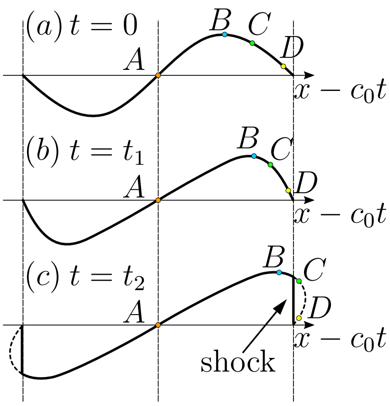

Figure 22: Steepening of an acoustic wave into a shock. The black line represent the velocity profile at successive instant of times.

This also implies that if the velocity is positive everywhere at $t=0$, then it will stay always positive at all times, so we have a wave travelling in one direction only. In general, a solution of the equations of motion in which one of the Riemann invariants is constant (and therefore the characteristics that are straight lines) is called a **simple wave**.

To understand the meaning of Eq. (524), consider a wave whose v profile at $t=0$ is as shown in Fig. 22(a). We want to find the v profile at a later time t_1. Consider point A. This point initially has $v=0$ and $c=c_0$. Therefore, according to Eq. (524) we find that the point $x_A(t) = c_0 t$ will always have $v=0$. Loosely speaking, we can say that at $t=t_1$ the point A has "moved" to $x_A(t_1)$, but note that this is *not* the motion of the actual fluid element that was initially at A. What we really mean is that the point with constant velocity $v=0$ has moved from its initial position to $x = x_A(t_1)$. The fluid element at A follows a different trajectory: indeed, this point initially has $v = 0$ and so initially does not move. What we are doing here is mathematical construction to find the solution at later times, we are not following fluid elements.

Now consider points B, C, D. According to Eq. (524), point B "moves" (in the sense above) the fastest because it has the highest initial v ($c(v)$ is an increasing function of v, see Eq. 523). Point C moves slightly less fast, and point D is the slowest of the three. The points get closer with time and the shape of the wave is distorted. If we continue our mathematical construction long enough, at some time t_2 (Fig. 22c) point C will surpass point B. At t_2 our solution is double valued, which is physically impossible. What actually happens is that instead of becoming double valued, the solution develops a discontinuity (a shock). As can be seen from this argument, this always inevitably happens whenever there is a segment of the wave in which the velocity decreases in the direction of wave propagation. Thus, the exact "simple wave" solution that we derived in this section is generally valid only for a limited amount of time.

How long does it take for a wave to become a shock? We can estimate the time of shock formation as follows. Consider two points, say C and D in Fig. 22 that are initially separated by a distance Δx and whose velocity difference is $\Delta v = v_C - v_D$. The shock will form when point C reaches point D (time t_2 in Fig. 22). Using Eqs. (524) and (523) this happens after a time interval

$$\Delta t = \frac{\Delta x}{[v_A + c(v_A)] - [v_B + c(v_B)]} \tag{525}$$

$$= \left(\frac{2}{1+\gamma}\right)\frac{\Delta x}{\Delta v}\,. \tag{526}$$

Therefore, the time of shock formation is related to the slope of the velocity profile. The steeper the profile, the sooner the shock will form.

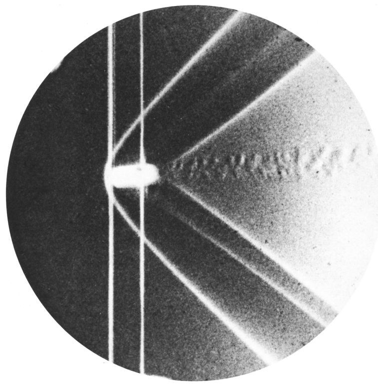

Figure 23: Photograph of a bullet in supersonic flight, published by Ernst Mach in 1887

The idea of *shock waves* is now familiar, but it was not readily accepted at first. In a paper titled "On a difficulty in the theory of sound" published in 1848, Stokes noted that waves of finite amplitude distort, as in Fig. 22. Stokes hesitantly suggested that "*Perhaps the most natural supposition to make for trial is, that a surface of discontinuity is formed, in passing across which there is an abrupt change of density and velocity*". The actual proof that shocks exist in nature was given later by Ernst Mach, which was fascinated by the fact that one usually perceives two "bangs" when a bullet is flying by. Shock waves are normally invisible to the naked eye, but Mach devised a special optical arrangement (called a shadowgraph) that allowed him to photograph shock waves. In 1887, he presented a photograph to the Academy of Sciences in Vienna of a bullet moving at supersonic speeds (see Fig. 23). The bow shock and trailing edge shock are visible in the figure. This historic photograph allowed scientists to actually see shock waves for the first time (see for example https://history.nasa.gov/SP-4219/Chapter3.html for more historical info).