4.3 Hydrodynamical shocks

4.3.1 Steepening of sound waves

- Up to this point, we have assumed that the perturbations that we are dealing with are small, justifying our use of perturbation theory. What happens when this is not the case?

- For simplicity, we will start by considering the 1D case. In one dimension, the continuity equation can be written as

$$\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}\right) + \frac{\partial v}{\partial x} = 0, \tag{349}$$

and the Euler equation becomes

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{c_{\rm s}^2}{\rho}\frac{\partial \rho}{\partial x}. \tag{350}$$

- We can eliminate ρ from these equations by using the fact that

$$\frac{{\rm d}\rho}{\rho} = \frac{2}{\gamma - 1}\frac{{\rm d}c_{\rm s}}{c_{\rm s}}, \tag{351}$$

which follows directly from the definition of the adiabatic sound speed. We find that

$$\frac{\partial}{\partial t}\left(\frac{2}{\gamma-1}c_{\rm s}\right) + v\frac{\partial}{\partial x}\left(\frac{2}{\gamma-1}c_{\rm s}\right) + c_{\rm s}\frac{\partial v}{\partial t} \;=\; 0, \tag{352}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + c_{\rm s}\frac{\partial}{\partial x}\left(\frac{2}{\gamma-1}c_{\rm s}\right) \;=\; 0. \tag{353}$$

Adding these two equations together then yields

$$\left[\frac{\partial}{\partial t} + (v + c_{\rm s})\frac{\partial}{\partial x}\right]\left(u + \frac{2}{\gamma-1}c_{\rm s}\right) = 0, \tag{354}$$

while subtracting them yields

$$\left[\frac{\partial}{\partial t} + (v - c_{\rm s})\frac{\partial}{\partial x}\right]\left(u - \frac{2}{\gamma-1}c_{\rm s}\right) = 0. \tag{355}$$

- Using the method of characteristics, one can show that the first of these equations implies that the quantity

$$Q \equiv v + \frac{2}{\gamma-1}c_{\rm s} \tag{356}$$

is conserved along trajectories that satisfy

$$\frac{{\rm d}x}{{\rm d}t} = v + c_{\rm s}, \tag{357}$$

while the second implies that the quantity

$$R \equiv v - \frac{2}{\gamma - 1} c_{\rm s} \tag{358}$$

is conserved along trajectories that satisfy

$$\frac{{\rm d}x}{{\rm d}t} = v - c_{\rm s}. \tag{359}$$

Lines in the (x, t) plane along which Equation 357 or Equation 359 hold are known as **characteristics**. They represent the trajectories along which our perturbations move in the (x, t) plane – we simply have the bulk velocity v, plus or minus the speed of sound.

- We have therefore recovered a similar result to the one which we derived using perturbation theory: perturbations to a flow of gas propagate at the speed of sound in the local rest frame of the flow. Note, however, that in the derivation above, we have made no assumption regarding the size of the perturbations or the nature of the background flow.

- Why does this matter? In our perturbation theory analysis, we found that our small perturbations propagated at a velocity

$$c_{\rm s} = \left(\frac{\partial p}{\partial \rho}\right)^{1/2} \simeq \frac{\gamma p_0}{\rho_0}, \tag{360}$$

where p_0 and ρ_0 are the pressure and density of the unperturbed fluid, and where the second (approximate) equality follows from the fact that we are considering only small perturbations and expanding to first order. Therefore, in the perturbation theory analysis, all sound waves propagate at the same velocity.

- In the analysis that we have carried out in this section, however, we have not assumed that the perturbations are small, and hence the second equality above does not apply. Instead, we know that for an adiabatic gas, $p = K\rho^{\gamma}$ and hence

$$c_{\rm s} = \left(\gamma K \rho^{\gamma - 1}\right)^{1/2}. \tag{361}$$

Therefore, in general, the speed of sound depends on the density: sound waves propagate faster in denser gas.

- Now consider a sinusoidal density perturbation propagating with respect to a uniform background. At $t = 0$, there are three points in this profile that have a density that is the same as the background medium, $\rho = \rho_0$. For a wave propagating to the right, these points subsequently propagate along plus characteristics defined by

$$\frac{{\rm d}x}{{\rm d}t} = v + c_{\rm s,0}, \tag{362}$$

where v is the initial bulk velocity of the flow and $c_{\rm s,0} \equiv \sqrt{\gamma p_0/\rho_0}$. However, points in the wave that have densities $\rho > \rho_0$ propagate at velocities $v + c_{\rm s} > v + c_{\rm s,0}$. Similarly, points on the wave that have initial densities $\rho < \rho_0$ propagate at velocities $v + c_{\rm s} < v + c_{\rm s,0}$.

- The result of this difference in velocities is that our sinusoidal perturbation **steepens** as it propagates. This steepening continues until we reach a point at which this analysis predicts that the density, velocity etc. become double-valued. Of course, physically, this is impossible. Instead, the flow becomes discontinuous and a **shock** forms.

4.3.2 Non-radiative shocks – jump conditions

- A shock wave is a region of small thickness over which the properties of the flow change rapidly. For the time being, we will ignore what is going on within the shock wave itself, and treat it simply as a true mathematical discontinuity in the flow.

- Consider a stationary shock separating an upstream flow with density ρ_1, pressure p_1 and velocity v_1 and a downstream flow with density ρ_2, pressure p_2 and velocity v_2. If we treat the shock as a discontinuity, then no mass or momentum can accumulate within the shock itself, and hence the flow of mass into the shock must equal the flow out of the shock. Similarly, if the shock is non-radiative[6], then the flow of energy into the shock must balance the flow of energy out of the shock.

- These considerations allow us to derive three equations linking the upstream and downstream properties of the flow. In the simple case in which the upstream velocity is perpendicular to the shock, we have

$$\begin{aligned} \rho_1 v_1 &= \rho_2 v_2, && (363) \\ p_1 + \rho_1 v_1^2 &= p_2 + \rho_2 v_2^2, && (364) \\ \frac{1}{2} v_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} &= \frac{1}{2} v_2^2 + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2}. && (365) \\ & && (366) \end{aligned}$$

  This set of equations are known as the **shock jump conditions** or the **Rankine-Hugoniot conditions**.

- As an example of the power of these equations, let us suppose that we want to know the density ratio ρ_2/ρ_1. We can rearrange the third jump condition to give us the following expression for p_2:

$$p_2 = \frac{(\gamma - 1)}{2\gamma} \rho_2 \left(v_1^2 - v_2^2\right) + p_1 \frac{\rho_2}{\rho_1}. \tag{367}$$

  Substituting this into the second of the jump conditions then yields

$$p_1 + \rho_1 v_1^2 = \frac{(\gamma - 1)}{2\gamma} \rho_2 \left(v_1^2 - v_2^2\right) + p_1 \frac{\rho_2}{\rho_1} + \rho_2 v_2^2. \tag{368}$$

  To eliminate p_1 from this equation, we use the fact that

$$p_1 = \frac{1}{\gamma} \rho_1 c_{\mathrm{s},1}^2, \tag{369}$$

[6]Such shocks are often referred to as "adiabatic" shocks, although this name is actually rather misleading, as we will see later.

where $c_{s,1}$ is the sound speed in the upstream gas, giving us

$$\rho_1 c_{s,1}^2 + \gamma \rho_1 v_1^2 = \frac{(\gamma - 1)}{2} \rho_2 \left(v_1^2 - v_2^2\right) + c_{s,1}^2 \rho_2 + \gamma \rho_2 v_2^2. \tag{370}$$

Dividing both sides by $\rho_1 c_{s,1}^2$ and using the first jump condition to write $v_2 = \rho_1 v_1 / \rho_2$, we find that

$$1 + \gamma \mathcal{M}_1^2 = \frac{(\gamma - 1)}{2} \frac{\rho_2}{\rho_1} \left[\mathcal{M}_1^2 - \left(\frac{\rho_1}{\rho_2} \right)^2 \mathcal{M}_1^2 \right] + \frac{\rho_2}{\rho_1} + \gamma \frac{\rho_1}{\rho_2} \mathcal{M}_1^2, \tag{371}$$

where $\mathcal{M} \equiv v_1 / c_{s,1}$ is the **Mach number** of the upstream flow. Some rearrangement of terms then gives

$$\left[\frac{(\gamma - 1)}{2} \mathcal{M}_1^2 + 1 \right] \left(\frac{\rho_2}{\rho_1} \right)^2 + \frac{(\gamma + 1)}{2} \mathcal{M}_1^2 = 1 + \gamma \mathcal{M}_1^2 \frac{\rho_2}{\rho_1}. \tag{372}$$

Finally, this quadratic equation can be solved to yield

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) \mathcal{M}_1^2}{2 + (\gamma - 1) \mathcal{M}_1^2}. \tag{373}$$

- If the upstream velocity is subsonic, so that $\mathcal{M}_1^2 < 1$, then this equation implies that $\rho_2 < \rho_1$, i.e. the gas becomes rarefied. However, we know from the jump conditions that

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2}, \tag{374}$$

so this would also imply that the gas would accelerate at the shock. However, this is consistent with energy conservation only if we convert energy from heat into ordered kinetic energy, but as we will see later, doing so would violate the second law of thermodynamics. We can therefore conclude that this is not a physical solution – so-called **rarefaction shocks** do not, in reality, exist, and we will recover physically meaningful values for the density ratio only when $\mathcal{M}_1^2 \geq 1$.

- In the case where $\mathcal{M}_1 = 1$, we see immediately that $\rho_1 = \rho_2$, from which it follows that $v_1 = v_2$ and $p_1 = p_2$; i.e. this is a trivial solution in which there is no discontinuity in any of the flow variables.

- In the most common case, where $\mathcal{M}_1 > 1$, we have $\rho_2 > \rho_1$, which immediately implies that $v_2 < v_1$. Therefore, one effect of the shock is to compress the gas, and to slow it down.

- We can also use the shock jump conditions to determine the pressure ratio p_2/p_1. For brevity, I do not give the derivation here, but instead simply quote the result:

$$\frac{p_2}{p_1} = \frac{2\gamma \mathcal{M}_1^2 - (\gamma - 1)}{\gamma + 1}. \tag{375}$$

If $M_1 > 1$, then we see that $p_2 > p_1$, and so the other important effect of the shock is to raise the gas pressure.

- By combining the expressions for the pressure jump and the density jump, we can derive an expression for the change in the temperature of the gas. If the chemical composition of the gas is unaltered by its passage through the shock, then

$$\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} \frac{T_2}{T_1}. \tag{376}$$

It follows from this that

$$\frac{T_2}{T_1} = \frac{\left[2\gamma \mathcal{M}_1^2 - (\gamma - 1)\right] \left[2 + (\gamma - 1)\mathcal{M}_1^2\right]}{(\gamma + 1)^2 \mathcal{M}_1^2}, \tag{377}$$

and hence if $M_1 > 1$, the post-shock temperature is higher than the pre-shock temperature, $T_2 > T_1$.

- It is also informative to look at how the expressions for the density, pressure and temperature jumps behave in the high $\mathcal{M}_1$ limit, i.e. when we have a very strong shock. As $\mathcal{M}_\infty \rightarrow \infty$, we have

$$\frac{\rho_2}{\rho_1} \rightarrow \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2} = \frac{\gamma + 1}{\gamma - 1} \tag{378}$$

for the density jump,

$$\frac{p_2}{p_1} \rightarrow \frac{2\gamma \mathcal{M}_1^2}{\gamma + 1} \tag{379}$$

for the pressure jump and

$$\frac{T_2}{T_1} \rightarrow \frac{2\gamma(\gamma - 1)}{(\gamma + 1)^2} \mathcal{M}_1^2 \tag{380}$$

for the temperature jump.

- If the gas is monatomic, so that $\gamma = 5/3$, we therefore have

$$\frac{\rho_2}{\rho_1} \simeq 4, \tag{381}$$

$$\frac{p_2}{p_1} \simeq \frac{5}{4}\mathcal{M}_1^2, \tag{382}$$

$$\frac{T_2}{T_1} \simeq \frac{5}{16}\mathcal{M}_1^2, \tag{383}$$

for shocks where $\mathcal{M}_1 \gg 1$.

4.3.3 Non-radiative shocks – behaviour inside the shock

- What is responsible for the sudden change in ρ, $\vec{v}$, p and T that occurs within a shock? If we describe shocks only using the Euler equation, then we have no good answer: we are forced to treat them purely as discontinuities in the flow, with zero thickness, and so in this description in makes no sense to talk about the properties *inside* a shock.

- In reality, of course, the behaviour of our gas is governed by the Navier-Stokes equation, rather than the Euler equation, and it is the additional terms corresponding to the effects of viscosity that provide the answer to our question.

- In general, astrophysical fluids have high Reynolds numbers, justifying our neglect of viscous dissipation. In a shock, however, the velocity of the flow changes rapidly over a distance that is comparable to the mean free path of the particles making up the fluid. The Reynolds number of the flow through the shock is therefore

$$\mathrm{Re} \sim \frac{v\lambda}{\nu}, \tag{384}$$

where λ is the mean free path. Since $\nu \sim v_{\mathrm{th}}\lambda$, it follows that

$$\mathrm{Re} \sim \frac{v}{v_{\mathrm{th}}} \sim \mathcal{M}. \tag{385}$$

Therefore, even in highly supersonic flows, the Reynolds number of the flow through the shock is relatively small, demonstrating that on these scales viscous dissipation cannot be neglected.

- The effect of the viscous dissipation occurring within the shock is to convert some fraction of the ordered kinetic energy of the inflowing gas into random kinetic energy, i.e. heat. This explains why the bulk velocity drops and the temperature rises. The density jump then follows as a consequence of the decrease in the velocity of the gas: in a steady shock, gas does not build up within the shock layer and hence must flow out of the downstream side with the same mass flux as is flowing into the upstream side.

- The fact that shocks are fundamentally dissipative events can be seen quite clearly if we compute the difference in the entropy of the pre-shock and post-shock gas. This is given by the expression:

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - \frac{k}{m}\ln\left(\frac{p_2}{p_1}\right), \tag{386}$$

where s_1 and s_2 are the pre-shock and post-shock values of the specific entropy, and c_p is the specific heat at constant pressure.[7]

- Since $c_p = c_V + k/m$, where c_V is the specific heat capacity at constant volume, and since $\gamma = c_p/c_V$, we can rewrite this equation as

$$\frac{s_2 - s_1}{c_V} = \gamma \ln\left(\frac{T_2}{T_1}\right) - (\gamma - 1)\ln\left(\frac{p_2}{p_1}\right). \tag{387}$$

If $\mathcal{M}_1 = 1$, then we already know that $T_1 = T_2$ and $p_1 = p_2$, and so it follows that $s_1 = s_2$; i.e. if there is no shock, then there's no change in entropy.

[7] To see where this expression comes from, note that the first term on the right hand side corresponds to the specific entropy change due to a change in the temperature occurring at constant pressure, while the second term corresponds to a change in the pressure at constant temperature. Since s is a state variable, the change in its value due to changes in both p and T is independent of the path taken from p_1, T_1 to p_2, T_2.

- If $\mathcal{M}_1 > 1$, then we can substitute in the expressions we derived previously to get an expression for the entropy change. However, the full form of this equation is rather lengthy, so it is more convenient to look at the limiting case when the shock is weak. For a very weak shock, it is convenient to write our expression for the pressure jump as

$$\frac{p_2}{p_1} = \frac{2\gamma\left(\mathcal{M}_1^2 - 1\right) + (\gamma+1)}{\gamma+1} = 1 + \frac{2\gamma}{\gamma+1}\left(\mathcal{M}_1^2 - 1\right), \tag{388}$$

and that for the temperature jump as

$$\frac{T_2}{T_1} = \frac{\left[2\gamma\left(\mathcal{M}_1^2 - 1\right) + (\gamma+1)\right]\left[(\gamma-1)\left(\mathcal{M}_1^2 - 1\right) + (\gamma+1)\right]}{(\gamma+1)^2\left(\mathcal{M}_1^2 - 1\right) + (\gamma+1)^2}. \tag{389}$$

Since $\mathcal{M}_1 \simeq 1$ for a very weak shock, $\mathcal{M}_1^2 - 1 \ll 1$, and so we can approximate T_2/T_1 as

$$\frac{T_2}{T_1} \simeq 2\left[\frac{\gamma-1}{\gamma+1}\left(\mathcal{M}_1^2 - 1\right) + 1\right] \tag{390}$$

where we have dropped all terms of order $(\mathcal{M}_1^2 - 1)^2$ and higher. Our expression for the entropy change therefore becomes

$$\frac{s_2 - s_1}{c_V} = \gamma \ln 2 + \gamma \ln\left[\frac{\gamma-1}{\gamma+1}\left(\mathcal{M}_1^2 - 1\right) + 1\right] - (\gamma-1)\ln\left[1 + \frac{2\gamma}{\gamma+1}\left(\mathcal{M}_1^2 - 1\right)\right]. \tag{391}$$

We can then use the fact that $\ln(1+x) \simeq x$ for $x \ll 1$ to eliminate the logarithms, allowing us to write the entropy jump as

$$\frac{s_2 - s_1}{c_V} = \gamma \ln 2 + \left[\frac{\gamma(\gamma-1)}{\gamma+1}\left(\mathcal{M}_1^2 - 1\right)\right] - \left[\frac{2\gamma(\gamma-1)}{\gamma+1}\left(\mathcal{M}_1^2 - 1\right)\right], \tag{392}$$

$$= \gamma \ln 2 - \frac{\gamma(\gamma-1)}{\gamma+1}\left(\mathcal{M}_1^2 - 1\right). \tag{393}$$

The first term in this expression is of order unity, while the second term is much smaller, since $\mathcal{M}_1^2 - 1 \ll 1$. We therefore see that the entropy of the gas is increased by the shock, confirming its dissipative nature. It is also possible to show (although we will not do so here) that $s_2 - s_1$ is a monotonically increasing function of $\mathcal{M}_1$, and so if it is positive in the case of a weak shock, then it is positive for *any* shock.

- The fact that shocks involve jumps in entropy demonstrates why rarefaction shocks are impossible. A rarefaction shock with the same strength as the compressive shock considered above would produce a change in entropy equal to $s_1 - s_2$, which is negative. Therefore, rarefaction shocks involve a decrease in the entropy, and hence are forbidden by the second law of thermodynamics.

4.3.4 Non-radiative shocks – moving and oblique shocks

- So far, we have implicitly been working in a frame in which the shock is at rest. However, it is often convenient to work in a frame in which the shock is moving. In this case, the shock jump conditions take a slightly different form:

$$\rho_1(v_1 - v_s) = \rho_2(v_2 - v_s), \tag{394}$$
$$p_1 + \rho_1(v_1 - v_s)^2 = p_2 + \rho_2(v_2 - v_s)^2, \tag{395}$$
$$\frac{1}{2}(v_1 - v_s)^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}(v_2 - v_s)^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}, \tag{396}$$

where v_s is the velocity of the shock.

- The other important simplification that we have made so far is that the pre-shock gas is flowing in a direction perpendicular to the shock. Although there are many situations in which this is a reasonable approximation, there are also many situations in which it is not, notably whenever we are dealing with strongly curved shock fronts.

- If the flow of gas into the shock front is not perpendicular to the shock front, then we refer to it as an **oblique shock**. As with a normal shock, the jump conditions for an oblique shock follow directly from the conservation of mass, momentum and energy. If the pre-shock gas has velocity v_1, and is flowing at an angle ϕ relative to the plane of the shock front, then we can write the velocity component that is perpendicular to the shock front as $v_{1,\perp} = v_1 \sin\phi$, and the velocity component that is parallel to the shock front as $v_{1,\parallel} = v_1 \cos\phi$.[8]

- The velocity component parallel to the shock is unaffected by the passage through the shock front: $v_{1,\parallel} = v_{2,\parallel}$, and so
$$v_{2,\parallel} = v_1 \cos\phi. \tag{397}$$
For the component perpendicular to the shock, we simply have the standard jump conditions, only now instead of using v_1 as the pre-shock velocity and v_2 as the post-shock velocity, we instead us $v_{1,\perp}$ and $v_{2,\perp}$. Therefore,
$$\rho_1 v_1 \sin\phi = \rho_2 v_{2,\perp}, \tag{398}$$
$$p_1 + \rho_1(v_1 \sin\phi)^2 = p_2 + \rho_2 v_{2,\perp}^2, \tag{399}$$
$$\frac{1}{2}(v_1 \sin\phi)^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}v_{2,\perp}^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}. \tag{400}$$

- Note that in order for a shock to be possible at all, $v_{1,\perp}$ must be supersonic. This implies that
$$v_1 \sin\phi > c_{s,1}, \tag{401}$$
$$\sin\phi > \mathcal{M}_1^{-1}, \tag{402}$$
$$\phi > \arcsin\left(\mathcal{M}_1^{-1}\right). \tag{403}$$

[8]Note that there are two different conventions in the literature for which direction is the "perpendicular" direction and which the "parallel" direction. The convention that we adopt here is the same as in Shu's book on *Gas Dynamics*. However, Draine & McKee, in their review article on the *Theory of Interstellar Shocks* (1993, ARA&A, 31, 373) adopt the other convention – flow which is parallel to the normal vector of the shock is "parallel" and flow perpendicular to the normal (i.e. parallel to the shock front) is "perpendicular". Personally, I think Shu's convention makes more sense, but you should be prepared to encounter both...

There is therefore a minimum degree of obliqueness for which a shock solution is possible, which depends on the Mach number of the upstream flow.

- One important consequence of these jump conditions is that the post-shock gas flows in a different direction from the pre-shock gas: the gas is **refracted** by the oblique shock. To see this, note that the angle that the flow makes with the plane of the shock front is given by

$$\chi = \arctan\left(\frac{v_\perp}{v_\parallel}\right). \tag{404}$$

For the pre-shock flow, we have

$$\chi = \arctan\left(\frac{v_1 \sin\phi}{v_1 \cos\phi}\right) = \phi. \tag{405}$$

For the post-shock flow, we have instead

$$\chi = \arctan\left(\frac{(\rho_1/\rho_2) v_1 \sin\phi}{v_1 \cos\phi}\right) = \arctan\left(\frac{\rho_1}{\rho_2}\tan\phi\right). \tag{406}$$

For a compressive shock, $\rho_1 < \rho_2$, and so $\chi < \phi$, demonstrating that the flow turns towards the plane of the shock.

- Using the oblique shock jump conditions, one can derive an expression for χ in terms of the upstream Mach number and the inflow angle ϕ:

$$\tan\chi = \tan\phi \frac{(\gamma-1)\mathcal{M}_1^2 \sin^2\phi + 2}{(\gamma+1)\mathcal{M}_1^2 \sin^2\phi} \tag{407}$$

For a weak shock, $\sin\phi \sim 1$, $\mathcal{M}_1 \sim 1$, and hence $\tan\chi \sim \tan\phi$; i.e. there is almost no refraction of the flow by the shock. For a strong shock, on the other hand, $\tan\chi$ tends to a limiting value given by

$$\lim_{\mathcal{M}_1\to\infty} (\tan\chi) = \frac{\gamma-1}{\gamma+1}\tan\phi. \tag{408}$$

- Another useful expression is the relation between the upstream and downstream Mach numbers for an oblique shock. This is given by

$$\mathcal{M}_2^2 \sin^2\chi = \frac{2 + (\gamma-1)\mathcal{M}_1^2 \sin^2\phi}{2\gamma\mathcal{M}_1^2 \sin^2\phi - (\gamma-1)}. \tag{409}$$

In a planar shock, we know already that $\mathcal{M}_2 < 1$, i.e. that the post-shock flow is subsonic. For an oblique shock, this is no longer the case: shocks with $\mathcal{M}_2 > 1$ are possible, provided that ϕ is sufficiently small. Note, however, that even in this case, the Mach number of the component of the downstream flow that is perpendicular to the shock, $\mathcal{M}_{2,\perp} \equiv v_{2,\perp}/c_{\mathrm{s},2}$, satisfies $\mathcal{M}_{2,\perp} < 1$, regardless of the Mach number of the upstream flow.

4.3.5 Radiative shocks

- Up to this point, we have assumed that any radiative heat losses from the gas passing through the shock are negligible, implying that the total thermal plus kinetic energy is conserved across the shock. This is a reasonable assumption if the cooling length of the post-shock gas (i.e. the distance travelled by the flow during the time that it takes to radiate away a significant fraction of its thermal energy) is long compared to any other length scales of interest.

- When dealing with interstellar shocks, however, particularly those in dense environments such as molecular clouds, we often find ourselves in a regime where the cooling length is short compared to other dynamical length scales of interest, while still being much longer than the particle mean free path.[9]

- In this regime, the extremely narrow shock region, with width comparable to the particle mean free path, is immediately followed by a narrow cooling-dominated zone, often referred to as a **radiative relaxation layer**. If we are only interested in the behaviour of the flow on scales that are much larger than the width of this radiative relaxation layer, then it is useful to write down jump conditions that relate the pre-shock conditions to the conditions at the end of this layer (i.e. once the gas temperature has reached equilibrium).

- If we denote the density, pressure and velocity of the cooled gas as ρ_3, p_3, and v_3, respectively, then we can immediately write down two of the necessary jump conditions:

$$\rho_1 v_1 = \rho_3 v_3, \tag{410}$$
$$p_1 + \rho_1 v_1^2 = p_3 + \rho_3 v_3^2. \tag{411}$$

  These have the same form as for a non-radiative shock, because despite the radiative cooling that occurs in the radiative relaxation layer, the flow must still conserve mass and momentum.

- The change in the internal energy in the radiative relaxation layer is governed (in 1D) by the equation

$$\rho u \frac{\mathrm{d}\epsilon}{\mathrm{d}x} = -p \frac{\mathrm{d}v}{\mathrm{d}x} - \rho \Lambda(\rho, T), \tag{412}$$

  where ϵ is the internal energy and Λ is the radiative cooling rate per unit mass, which, if the gas is optically thin, will depend only on local gas properties such as the density, temperature and chemical composition.

- For an ideal, chemically inert gas, ϵ and p are related by

$$\epsilon = \frac{1}{\gamma - 1} \frac{p}{\rho}, \tag{413}$$

[9] The extreme case where the cooling length is comparable to the mean free path cannot be treated using a fluid description, but instead requires a full kinetic treatment, as in this case one cannot, for instance, assume a thermal distribution of velocities.

and so we can rewrite the energy equation as

$$\frac{1}{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}x}(pv) = -p\frac{\mathrm{d}v}{\mathrm{d}x} - \rho\Lambda(\rho, T), \tag{414}$$

where we have used the fact that $\rho v =$ constant within the radiative relaxation layer to take it inside the differential. This equation can then be further rearranged to give

$$\frac{v}{\gamma-1}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\gamma}{\gamma-1}p\frac{\mathrm{d}v}{\mathrm{d}x} = -\rho\Lambda(\rho, T). \tag{415}$$

- The fact that $p + \rho v^2$ is also conserved throughout this region the allows us to write

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\rho v\frac{\mathrm{d}v}{\mathrm{d}x}, \tag{416}$$

and hence the energy equation can be put into the form

$$\frac{1}{\gamma-1}(c_{\mathrm{s}}^2 - v^2)\frac{\mathrm{d}v}{\mathrm{d}x} = -\Lambda(\rho, T). \tag{417}$$

- In the immediate post-shock gas, at the start of the radiative relaxation layer, we know that $v_2 < c_{\mathrm{s},2}$, and hence $c_{\mathrm{s},2}^2 - v_2^2 > 1$. Since Λ is positive if the gas is cooling, then it follows that within the radiative relaxation layer, the gas velocity will fall: $\mathrm{d}v/\mathrm{d}x < 0$.

- One important consequence of this derives from mass conservation: since $\rho v =$ constant, a fall in the velocity implies an increase in the density. A second important consequence derives from momentum conservation: since $p + (\rho v)v =$ constant and $\rho v =$ constant, a decrease in v implies an increase in the pressure p.

- Therefore, within the radiative relaxation layer, the velocity of the gas decreases, but the pressure and density both increase. The behaviour of the temperature is governed by the equation of state

$$p = \frac{\rho k T}{m}, \tag{418}$$

but since the fractional increase in p is typically much smaller than the fractional increase in ρ, it is usually the case that T decreases substantially.

- The details of these changes depend on the form of the cooling function $\Lambda(\rho, T)$, as do the final density, pressure etc., which correspond to the values in the flow at the point when the gas reaches thermal equilibrium, $\Lambda(\rho, T) = 0$.

- In some circumstances (e.g. within dense molecular clouds), the equilibrium temperature is relatively insensitive to the density of the gas. In this case, an isothermal approximation is often appropriate, i.e.

$$T_1 = T_3. \tag{419}$$

By combining this with the other two jump conditions, it is relatively straightforward to show that

$$v_1 v_3 = c_{\mathrm{I}}^2, \tag{420}$$

where $c_{\mathrm{I}} = p/\rho$ is the isothermal sound speed. This also implies that the density jump for an isothermal shock is

$$\frac{\rho_3}{\rho_1} = \left(\frac{v_1}{c_{\mathrm{I}}}\right)^2. \tag{421}$$

Radiative shocks can therefore produce much larger density contrasts than non-radiative shocks.

- Finally, whenever we have a radiative shock, we need to consider what happens to the photons produced in the radiative relaxation layer. Often, the surrounding gas will be optically thin to these photons, and they will simply escape from the vicinity of the shock without affecting the gas in any way. However, if the shock is very strong, so that the post-shock temperature is very large, then the gas in the radiative relaxation layer will produce a significant number of ionizing photons as it cools. If the pre-shock gas is largely neutral, it will typically have a high optical depth to these ionizing photons, and the post-shock emission will therefore ionize and heat the pre-shock gas, creating a region known as a **radiative precursor**.

- In order to account for the effects of this radiative precursor, one typically needs to make use of iterative methods, as by changing the temperature of the pre-shock gas, the radiation from the radiative relaxation layer changes its sound-speed, and hence the strength of the shock. This in turn effects the temperature of the post-shock gas, and hence the details of the emission. It is therefore necessary to iterate until one finds a consistent solution for both the pre-shock and post-shock temperatures.

- For reference, detailed modeling of very fast shocks show that the minimum shock speed for which enough ionizing photons are produced in order to create a significant radiative precursor is around $100\ \mathrm{km\,s^{-1}}$. Velocities of this order of magnitude are uncommon in much of the ISM, but can be encountered in protostellar outflows or young supernova remnants, for example.