Astrophysical Fluid Dynamics

Assignment #1: due October 23

1 Validity of the fluid approximation

Astrophysical fluids, such as the gas in the interstellar medium or the plasma in stars, are ultimately made of particles. When does it make sense to approximate this collection of particles as a continuous fluid? We have seen in the lectures that the fluid approximation is valid if the mean free path of a particle is small compared to the typical length over which macroscopic quantities, such as the density, vary. In this problem, we check the validity of the fluid approximation by estimating the mean free path in some astrophysical situations.

The mean free path in elementary kinetic theory is given by:

$$\lambda = \frac{1}{n\sigma} \tag{1}$$

where n is the number of particles per unit volume and σ is the average cross section per particle. Use this formula to estimate the mean free path of

(a) The Warm Neutral Medium (WNM), one of the possible phases of the Interstellar Medium (ISM), for which $n \sim 0.1\,\mathrm{cm}^{-3}$.

Hint: the main constituents of the WNM are atomic hydrogen and atomic helium. Since collisions between *neutral* atoms do not involve a strong long-range interaction, the collision cross-section is given approximately by $\sigma \simeq \pi(r_1^2 + r_2^2)$, where r_1 and r_2 are the radius of the two atoms involved in a collision.

(b) The Cold Neutral Medium, (CNM), another possible phase of the ISM, for which $n \sim 100\,\mathrm{cm}^{-3}$.

Hint: the main constituent of the CNM is molecular hydrogen, H_2.

(c) A protostellar accretion disk, for which $n \sim 10^{10}\,\mathrm{cm}^{-3}$.

(d) Particles in the solar corona. In the solar corona, $n \sim 10^7\,\mathrm{cm}^{-3}$, $T \sim 10^6\,\mathrm{K}$.

Hint: the solar corona is almost completely ionised. Charged particles interact via the Coulomb force over distances much larger than atomic radii, which enhances the cross section as compared to hard sphere collisions. Thus the formula that we used in previous items to calculate cross sections for neutral particles is not valid, and you need to come up (or find in some book) an appropriate formula.

(e) Particles in the solar wind at a distance of 1 AU from the Sun (i.e., at around the location of the Earth). At this location, typical values are $n \sim 10\,\mathrm{cm}^{-3}$, $T \sim 10^5\,\mathrm{K}$.

Note: remember this result when we will discuss the Parker wind!

(f) Gas in a galaxy cluster. Plausible values for the density and temperature of gas within a galaxy cluster are $n \sim 10^{-3}\,\mathrm{cm}^{-3}$ and $T \sim 10^8\,\mathrm{K}$.

Hint: it follows from the high value of the temperature that the gas in a galaxy cluster is highly ionised.

2 Estimating Reynolds numbers

(a) Suppose we have a gas with temperature T and particle number density n that is composed of particles of mass m. This gas is flowing in the y direction with a bulk velocity

$$\mathbf{v} = v(x)\,\hat{\mathbf{e}}_y \tag{2}$$

that is a function of x only. Now consider a plane S located at $x = x_0$ (see figure). Since the gas is made of particles moving in random directions and colliding with each other, some particles will cross this plane, transporting momentum, even if the bulk velocity is completely in the y direction. We can assume that the particles that cross the plane are coming from within a layer whose thickness is roughly the mean free path λ.

Show that the amount of y component of momentum transported from left to right per unit area and per unit time, apart from numerical coefficients of order unity, can be estimated as

$$\dot{P}_y \simeq \frac{1}{6} n v_{\mathrm{th}} m \left[v(x_0) - \lambda \frac{\partial v}{\partial x}(x_0) \right], \tag{3}$$

where

$$v_{\mathrm{th}} = \sqrt{\frac{3kT}{m}} \tag{4}$$

is the thermal velocity of the gas particles. Show that the net flux, which can be obtained by also considering the corresponding expression for the momentum transport from right to left, is:

$$\Delta \dot{P}_y = \frac{1}{3} n v_{\mathrm{th}} m \lambda \frac{\partial v}{\partial x} \tag{5}$$

(b) Use your results from part (a) to show that we can estimate the coefficient of dynamic viscosity as

$$\eta = \frac{1}{3} n v_{\mathrm{th}} m \lambda. \tag{6}$$

which implies that

$$\eta \sim \frac{m v_{\mathrm{th}}}{\sigma}. \tag{7}$$

(c) Estimate the Reynolds number of the gas in:

(i) A protostellar accretion disk

(ii) A giant molecular cloud

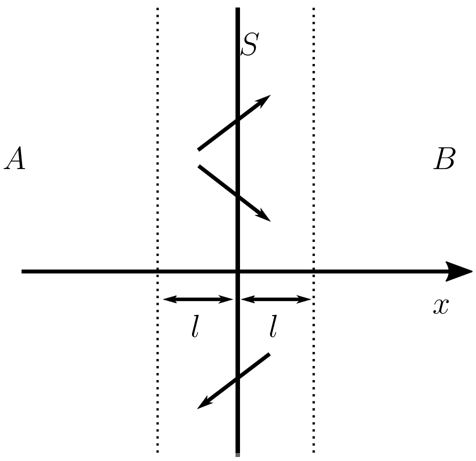

Figure 1: The microscopic origin of viscosity.

(iii) Gas in a galaxy cluster

Hint: you can use the results of the previous exercise, but you will also need to search (e.g. in books, or the internet) for typical values of some quantities.

3 Conservation of energy in an external gravitational potential.

We have seen that the following is a statement energy conservation for an adiabatic fluid subject only to pressure forces:

$$\partial_t \left[\frac{\rho \mathbf{v}^2}{2} + \frac{P}{\gamma - 1} \right] + \boldsymbol{\nabla} \cdot \left[\left(\frac{\rho \mathbf{v}^2}{2} + P + \frac{P}{\gamma - 1} \right) \mathbf{v} \right] = 0. \tag{8}$$

Show that the analogous statement in the presence of a given static external gravitational field $\Phi(\mathbf{x})$ is

$$\partial_t \left[\frac{\rho \mathbf{v}^2}{2} + \frac{P}{\gamma - 1} + \rho\Phi \right] + \boldsymbol{\nabla} \cdot \left[\left(\frac{\rho \mathbf{v}^2}{2} + P + \frac{P}{\gamma - 1} + \rho\Phi \right) \mathbf{v} \right] = 0. \tag{9}$$

Hint: start from the Euler equation

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla}\Phi. \tag{10}$$

and follow the derivation in the lecture notes with the appropriate differences.

4 Lagrangian derivative of line and volume elements.

(a) Show that the Lagrangian derivative of a short line element is

$$\frac{D(\mathrm{d}\mathbf{l})}{Dt} = (\mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla})\mathbf{v} \tag{11}$$

which means that in a time $\mathrm{d}t$ it changes as $\mathrm{d}\mathbf{l} \to \mathrm{d}\mathbf{l} + (\mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla})\mathbf{v}\mathrm{d}t$.

Hint: consider how the endpoints move from t to $t + \mathrm{d}t$.

(b) Show that the Lagrangian derivative of a volume element is

$$\frac{D(\mathrm{d}V)}{Dt} = (\mathrm{d}V)\boldsymbol{\nabla} \cdot \mathbf{v} \tag{12}$$

which means that in a time $\mathrm{d}t$ the volume of a fluid element changes as $\mathrm{d}V \to \mathrm{d}V(1 + (\boldsymbol{\nabla} \cdot \mathbf{v})\mathrm{d}t)$, and $\boldsymbol{\nabla} \cdot \mathbf{v}$ is its rate of change.

Hint: consider a parallelepiped whose edges are $\mathrm{d}\mathbf{x} = \mathrm{d}x\hat{\mathbf{e}}_x$, $\mathrm{d}\mathbf{y} = \mathrm{d}y\hat{\mathbf{e}}_y$, $\mathrm{d}\mathbf{z} = \mathrm{d}z\hat{\mathbf{e}}_z$ and use the result of the previous point and that the volume of a parallelepiped is $\mathrm{d}V = |\mathrm{d}\mathbf{x} \cdot (\mathrm{d}\mathbf{y} \times \mathrm{d}\mathbf{z})|$.