Astrophysical Fluid Dynamics

Assignment #2: due October 30

1 Rotating liquid

Consider a constant density, incompressible fluid rotating inside a container at a constant angular speed ω in a constant gravitational field g (see figure). The gravitational field is $\Phi = gz$. Find the shape of the liquid surface.

Can you think of any application of this result?

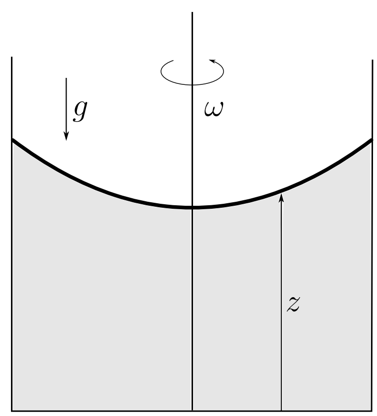

Figure 1: Rotating fluid in a container.

2 Bulging of the Earth

The radius of the Earth is slightly bigger at the equator than it is at the poles due to the centrifugal force arising from the rotation of the Earth around its axis. The goal of this problem is to find the shape of the Earth, first incorrectly, and then correctly. We assume that the Earth is made of an incompressible fluid of constant density.

1. A common *incorrect* method assumes that the gravitational potential of the Earth can be approximated by that of a sphere. One can then find the shape of the Earth by finding the equipotential surfaces in Earth's rotating frame (including the centrifugal potential). Show that this method leads to a surface whose height is given by

$$R = R_0 \left[1 - \left(\frac{R_0 \omega^2}{3g} \right) P_2(\cos\theta) \right] \tag{1}$$

where

$$P_2(\cos\theta) = \frac{1}{2} \left(3\cos^2\theta - 1 \right) \tag{2}$$

is the second Legendre polynomial, R_0 is the radius of the Earth if it were spherical, θ is the angle measured from the poles ($\theta = 90°$ is the equator), $g = GM/R_0^2$ is the gravitational potential at R_0 and ω is the angular velocity of the Earth.

2. The above method is incorrect, because the distortion of the Earth slightly changes its gravitational potential, which changes the equipotential surfaces. Turns out that this effect is of the same order of the one found in the previous item and cannot be neglected. Assuming that answer is of the form

$$R = R_0\left[1 - \beta\left(\frac{R_0\omega^2}{3g}\right)P_2(\cos\theta)\right] \tag{3}$$

Calculate the correct value of β. What is the difference between the radius of the Earth at the equator and at the poles?
Hint: the gravitational potential of a thin spherical shell of radius R_0 and surface density

$$\sigma(\theta) = \sigma_0 P_2(\cos\theta) \tag{4}$$

where σ_0 is a constant, is given by (can you prove this?)

$$\Phi_2(r,\theta) = 4\pi G\,\sigma(\theta) \times \begin{cases} -\frac{1}{5}R_0^{-1}r^2 & \text{if } r \le R_0 \\ -\frac{1}{5}R_0^4 r^{-3} & \text{if } r \ge R_0 \end{cases}$$

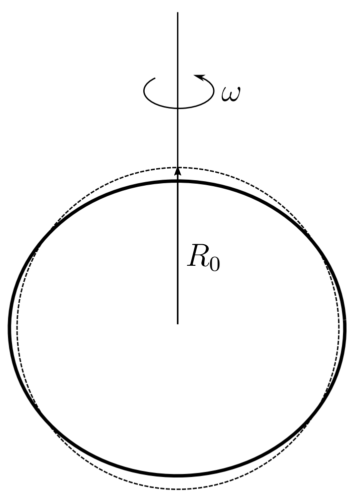

Figure 2: Schematic illustration of the bulging of the Earth. The dashed circle is the shape of the Earth if it were not spinning. The solid line represents the actual shape.