Theoretical Astrophysics

Priv.-Doz. Dr. Matteo Maturi

Winter Semester 2022/23

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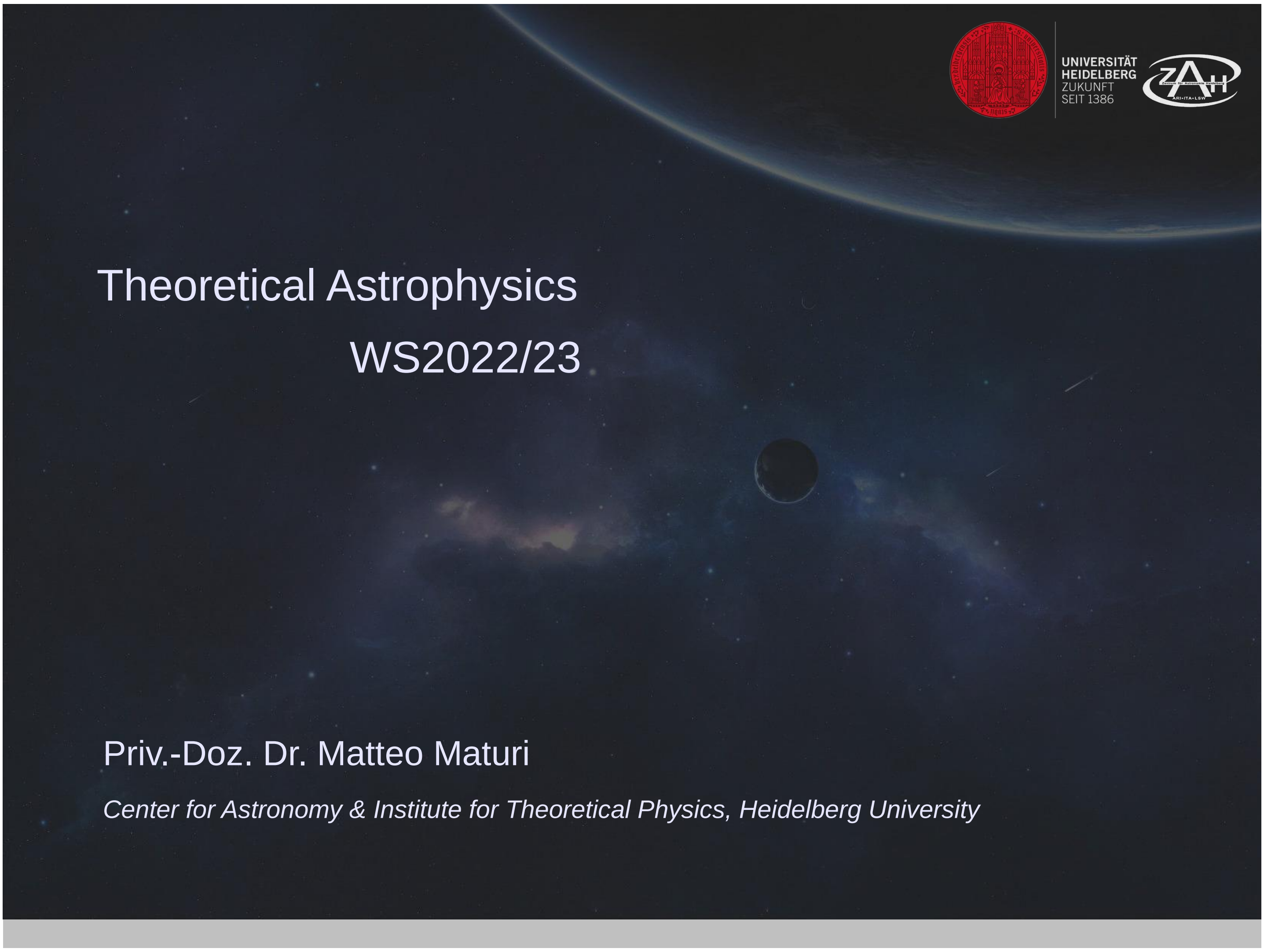
Part I

Introduction and special relativity

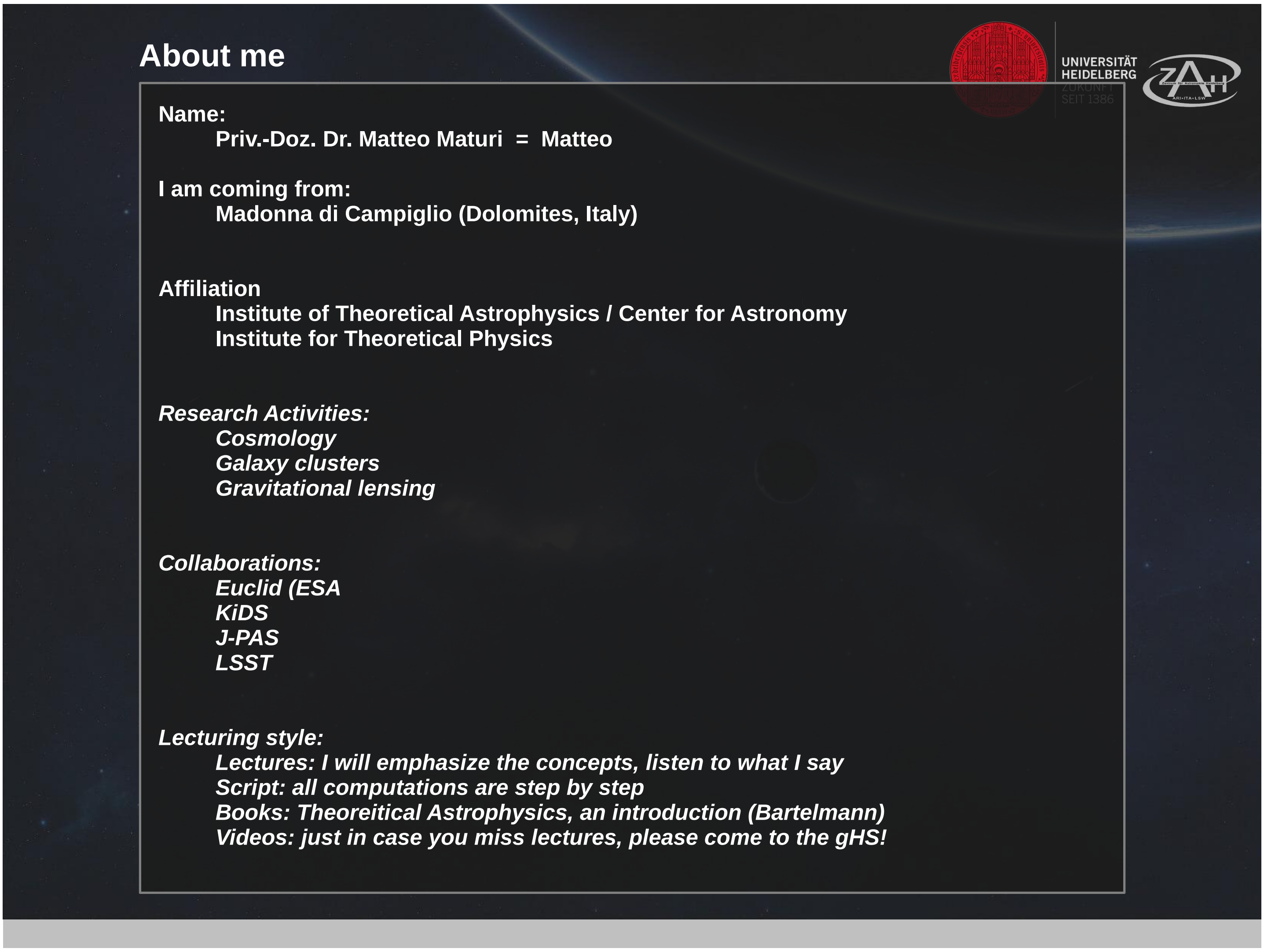


Theoretical Astrophysics

WS2022/23

Priv.-Doz. Dr. Matteo Maturi

Center for Astronomy & Institute for Theoretical Physics, Heidelberg University



About me

Name:
Priv.-Doz. Dr. Matteo Maturi = Matteo

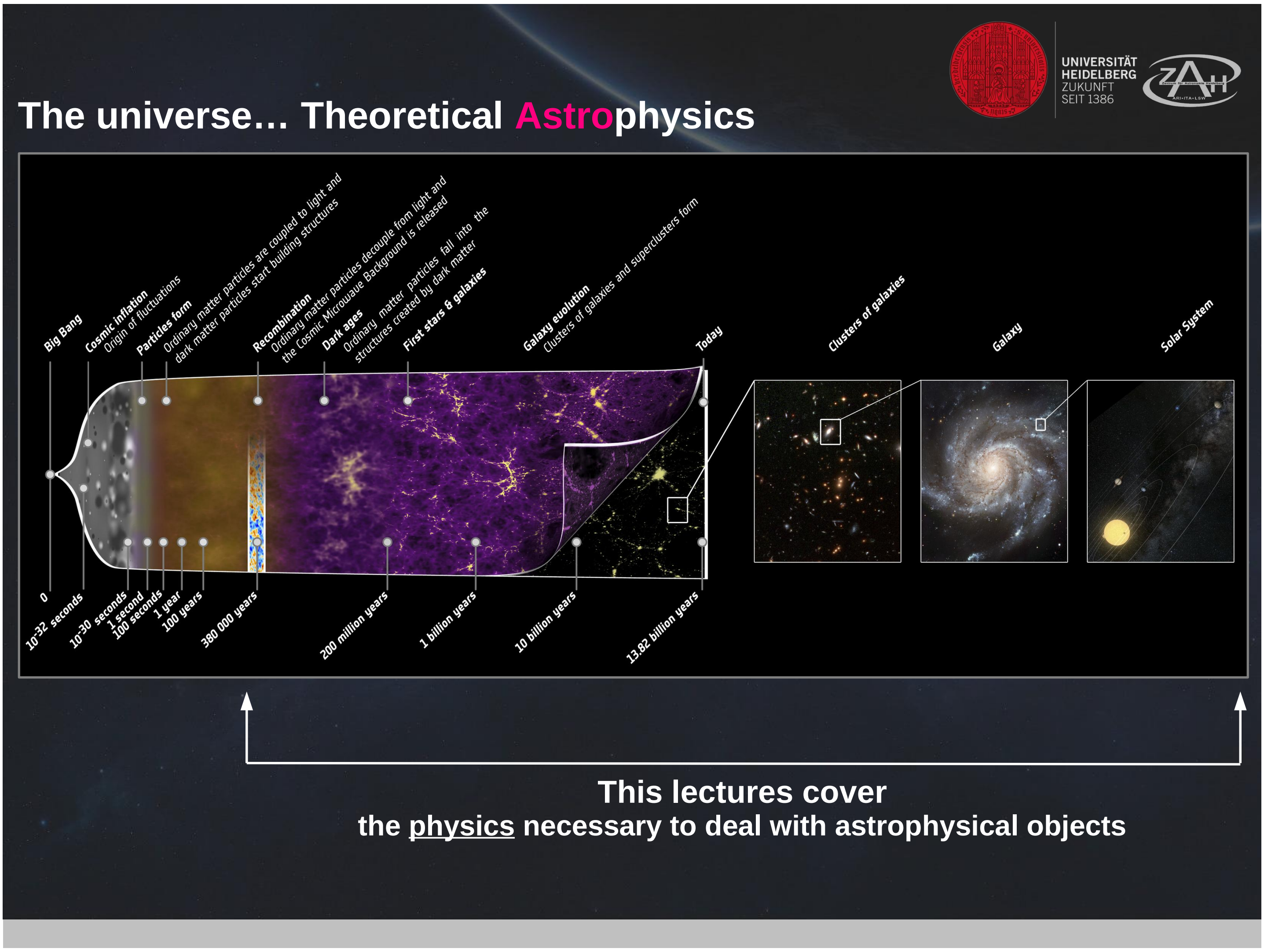
I am coming from:
Madonna di Campiglio (Dolomites, Italy)

Affiliation
Institute of Theoretical Astrophysics / Center for Astronomy
Institute for Theoretical Physics

Research Activities:
Cosmology
Galaxy clusters
Gravitational lensing

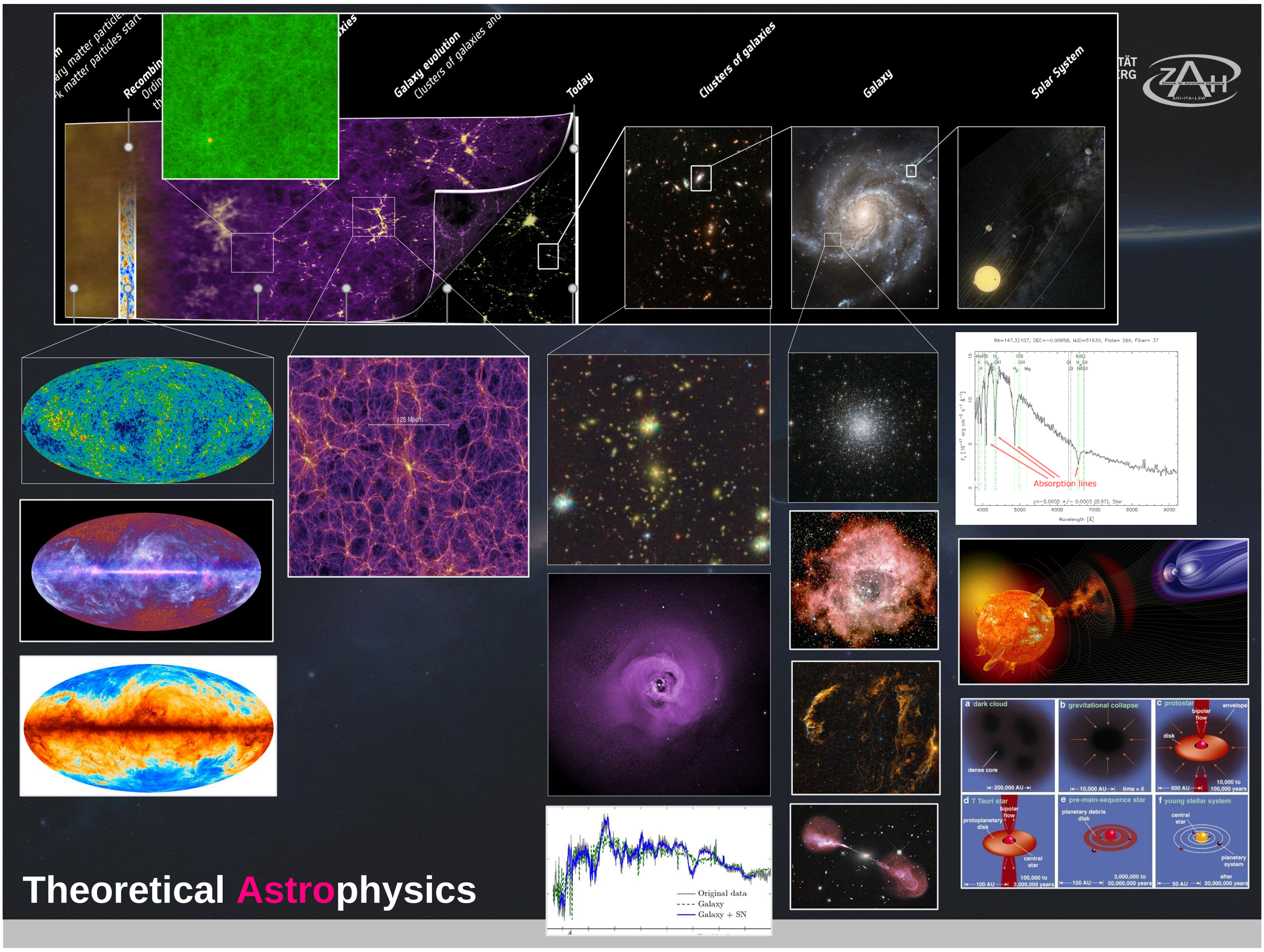
Collaborations:
Euclid (ESA
KiDS
J-PAS
LSST

Lecturing style:
Lectures: I will emphasize the concepts, listen to what I say
Script: all computations are step by step
Books: Theoreitical Astrophysics, an introduction (Bartelmann)
Videos: just in case you miss lectures, please come to the gHS!

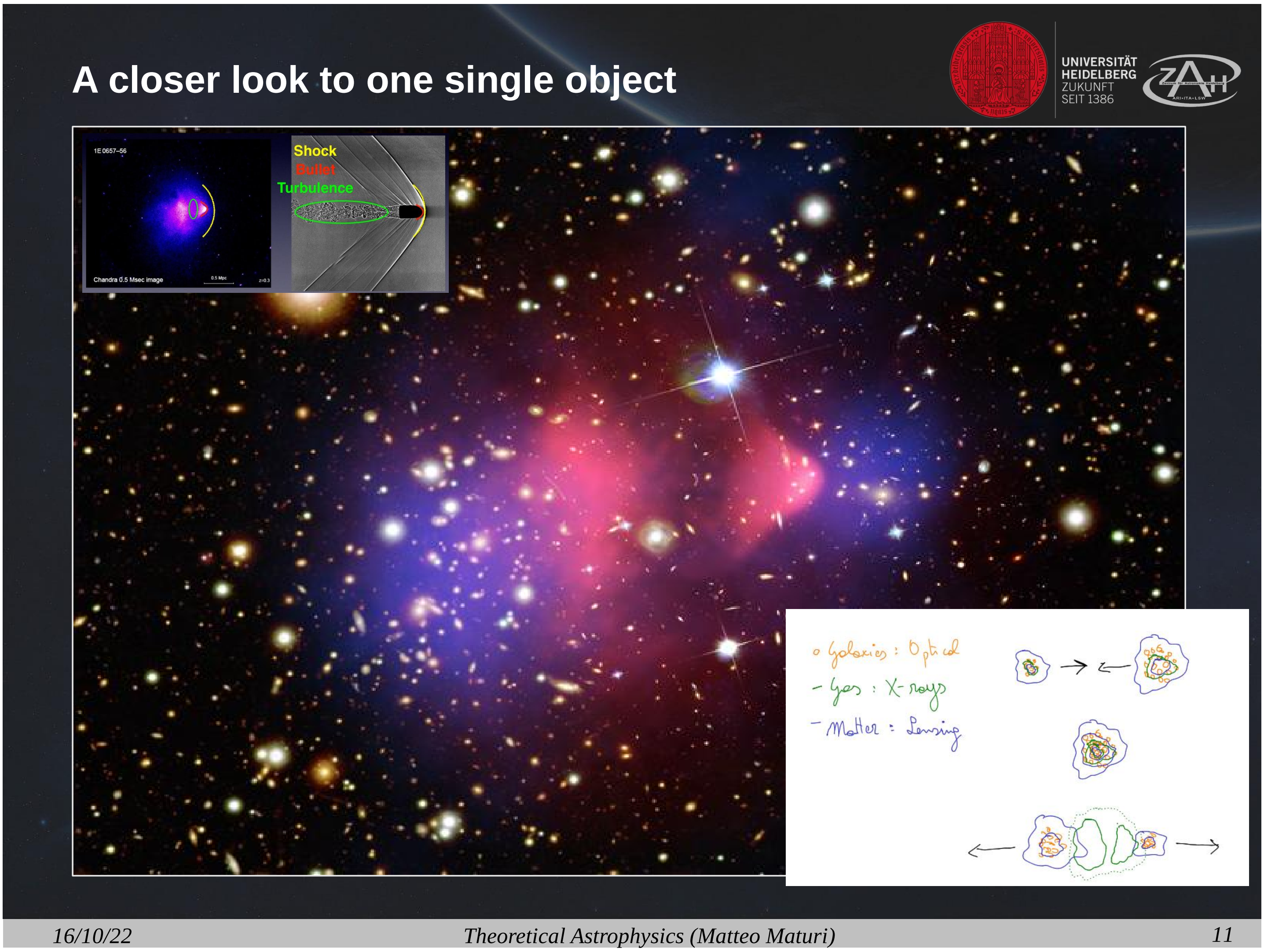


The universe… Theoretical Astrophysics

This lectures cover
the physics necessary to deal with astrophysical objects



Theoretical Astrophysics

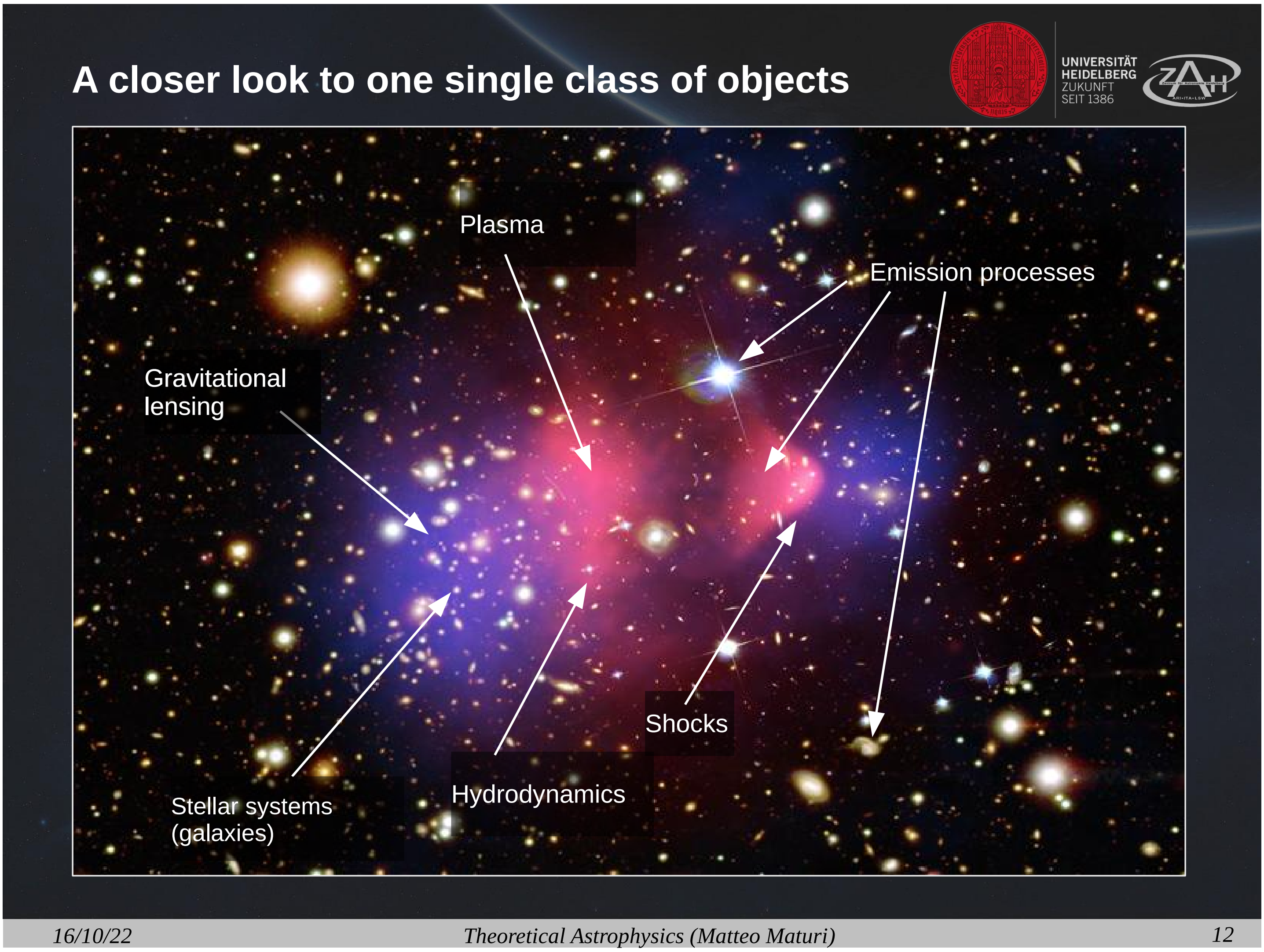


A closer look to one single object

o Galaxies : Optical

– Gas : X-rays

– Matter : Lensing



A closer look to one single class of objects

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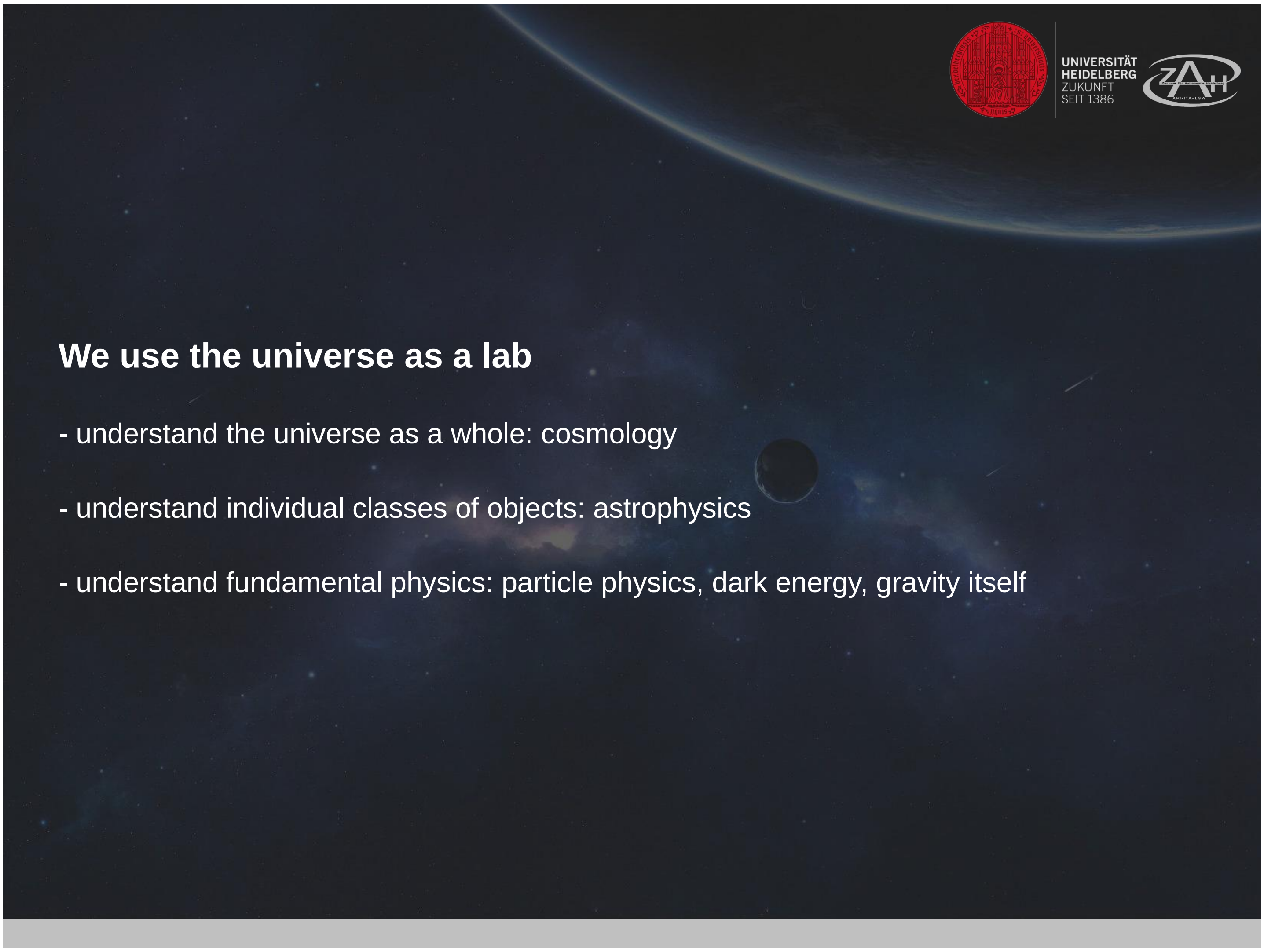

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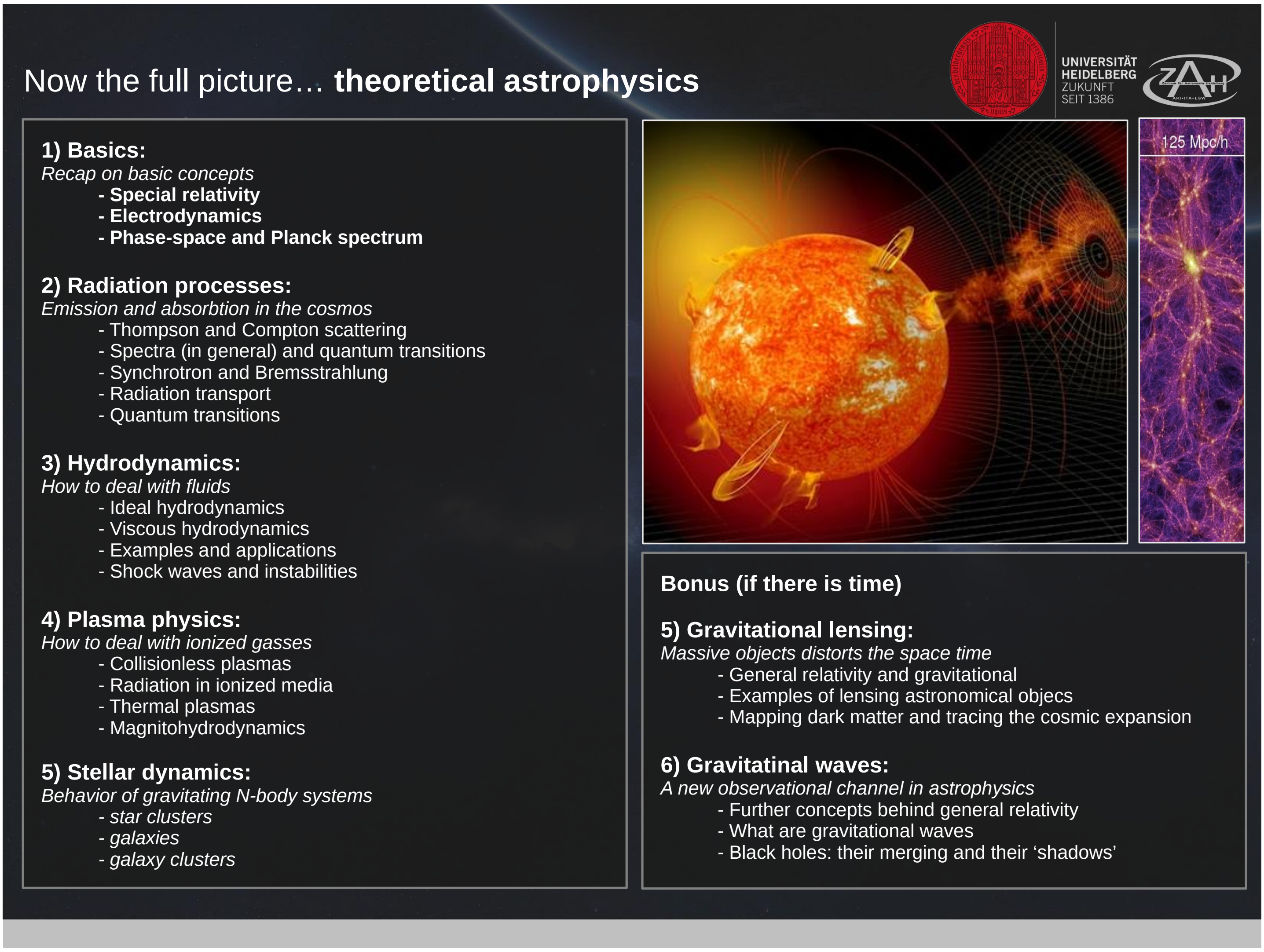
We use the universe as a lab

- understand the universe as a whole: cosmology

- understand individual classes of objects: astrophysics

- understand fundamental physics: particle physics, dark energy, gravity itself



Now the full picture… **theoretical astrophysics**

1) Basics:
Recap on basic concepts
- **Special relativity**
- **Electrodynamics**
- **Phase-space and Planck spectrum**

2) Radiation processes:
Emission and absorbtion in the cosmos
- Thompson and Compton scattering
- Spectra (in general) and quantum transitions
- Synchrotron and Bremsstrahlung
- Radiation transport
- Quantum transitions

3) Hydrodynamics:
How to deal with fluids
- Ideal hydrodynamics
- Viscous hydrodynamics
- Examples and applications
- Shock waves and instabilities

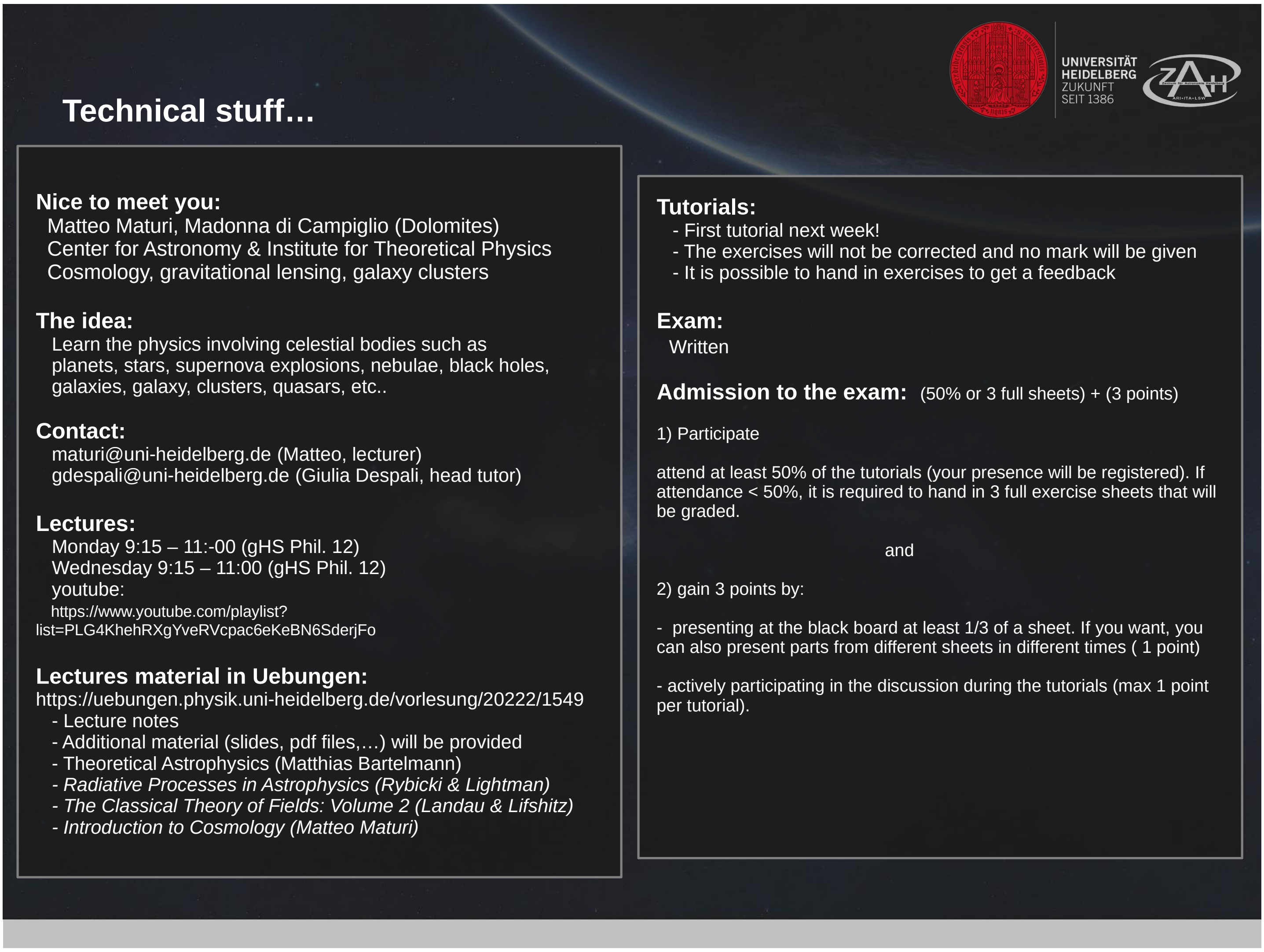
4) Plasma physics:
How to deal with ionized gasses
- Collisionless plasmas
- Radiation in ionized media
- Thermal plasmas
- Magnitohydrodynamics

5) Stellar dynamics:
Behavior of gravitating N-body systems
- *star clusters*
- *galaxies*
- *galaxy clusters*

Bonus (if there is time)

5) Gravitational lensing:
Massive objects distorts the space time
- General relativity and gravitational
- Examples of lensing astronomical objecs
- Mapping dark matter and tracing the cosmic expansion

6) Gravitatinal waves:
A new observational channel in astrophysics
- Further concepts behind general relativity
- What are gravitational waves
- Black holes: their merging and their ‘shadows’



Technical stuff…

Nice to meet you:
Matteo Maturi, Madonna di Campiglio (Dolomites)
Center for Astronomy & Institute for Theoretical Physics
Cosmology, gravitational lensing, galaxy clusters

The idea:
Learn the physics involving celestial bodies such as planets, stars, supernova explosions, nebulae, black holes, galaxies, galaxy, clusters, quasars, etc..

Contact:
maturi@uni-heidelberg.de (Matteo, lecturer)
gdespali@uni-heidelberg.de (Giulia Despali, head tutor)

Lectures:
Monday 9:15 – 11:-00 (gHS Phil. 12)
Wednesday 9:15 – 11:00 (gHS Phil. 12)
youtube:
https://www.youtube.com/playlist?list=PLG4KhehRXgYveRVcpac6eKeBN6SderjFo

Lectures material in Uebungen:
https://uebungen.physik.uni-heidelberg.de/vorlesung/20222/1549
- Lecture notes
- Additional material (slides, pdf files,…) will be provided
- Theoretical Astrophysics (Matthias Bartelmann)
- *Radiative Processes in Astrophysics (Rybicki & Lightman)*
- *The Classical Theory of Fields: Volume 2 (Landau & Lifshitz)*
- *Introduction to Cosmology (Matteo Maturi)*

Tutorials:
- First tutorial next week!
- The exercises will not be corrected and no mark will be given
- It is possible to hand in exercises to get a feedback

Exam:
Written

Admission to the exam: (50% or 3 full sheets) + (3 points)

1) Participate

attend at least 50% of the tutorials (your presence will be registered). If attendance < 50%, it is required to hand in 3 full exercise sheets that will be graded.

and

2) gain 3 points by:

- presenting at the black board at least 1/3 of a sheet. If you want, you can also present parts from different sheets in different times (1 point)

- actively participating in the discussion during the tutorials (max 1 point per tutorial).

Special Relativity: the concept

- $P=(t,x,y,z)$ <u>Event</u> : something happening in P
- S, S' <u>Coordinate frames</u> (in standard configuration)
- <u>Generic linear transformation</u> (preserve homogeneity!)

$$x' = At + Bx \qquad y' = y$$
$$t' = Dt + Ex \qquad z' = z$$

Find coefficients:

$$\begin{cases} x = 0 \Rightarrow \begin{matrix} x' = At \\ t' = Dt \end{matrix} \quad \frac{x'}{t'} \equiv v' = \frac{A}{D} \quad \boxed{A = Dv'} \\ x' = 0 \Rightarrow 0 = At + B\underset{\substack{\uparrow \\ x = vt}}{x} = At + Bvt \quad A = -Bv \end{cases} \Rightarrow D\overset{\substack{-v \\ \downarrow}}{v'} = -Bv \Rightarrow \boxed{D = B}$$

$$\boxed{x'} = Dv't + Dx = D(v't + x) = \boxed{D(x - vt)} \qquad y' = y \qquad \circledast \qquad D = ? \quad E = ?$$
$$\boxed{t' = Dt + Ex} \qquad z' = z$$

- <u>Postulate : $t \overset{!}{=} t'$</u> (Absolute time , Galilean frame)

$$t' = Dt + Ex \quad \forall t, x \iff D = 1 \quad E = 0 \quad \Rightarrow \quad \boxed{\begin{matrix} x' = x - vt \\ t' = t \end{matrix}}$$

- <u>Postulate : $c \overset{!}{=}$ const $\forall$ frame</u> (observations)

- c = const => it make sense to investigate light
- Consider a photon in 2 ≠ frames

$$\begin{matrix} c^2t^2 = x^2 + y^2 + z^2 \\ c^2t'^2 = x'^2 + y'^2 + z'^2 \end{matrix} \Rightarrow \begin{matrix} 0 = -c^2t^2 + x^2 + y^2 + z^2 \\ 0 = -c^2t'^2 + x'^2 + y'^2 + z'^2 \end{matrix} \Rightarrow \boxed{-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2}$$

Plug $\circledast$

$$\Rightarrow \boxed{\begin{matrix} ct' = \gamma(ct - \beta x) & y' = y \\ x' = \gamma(x - \beta ct) & z' = z \end{matrix}} \qquad \gamma = (1 - \beta^2)^{-1/2} \qquad \beta = \frac{v}{c}$$

<- <u>Lorentz transformations</u>

↳ Space and time are connected "Not like the frames of a movie"

↳ $\boxed{\text{The Space-Time}}$! => event $P \equiv (x^\mu) \equiv (ct, x, y, z)$

ct to have units of lengths
x^μ elements of a 4-vector

Some abmiguity is left

For photons $ds^2 = 0 = ds'^2$ also $\phi(v)\, ds^2 = 0$ $\phi \in \mathbb{R}$

$$ds'^2 = \phi(v)\, ds^2$$

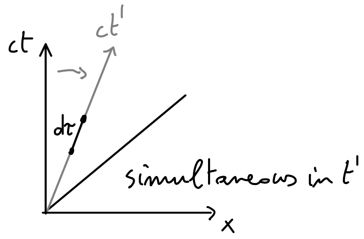
$$ds^2 = \phi(v)\, ds'^2 = \phi^2(v)\, ds^2$$

(e.g. $-c^2dt^2 + d\vec{r}^2 = 0$ or $c^2dt^2 - d\vec{r}^2 = 0$) $\Rightarrow$ $\boxed{\phi(v) = \pm 1}$ signature one can choose

Here we will use $(-,+,+,+)$ signature

The proper time

Time as measured with a clock at rest in a frame



$$-c^2dt^2+dx^2+dy^2+dz^2 = -c^2dt'^2+\underbrace{dx'^2+dy'^2+dz'^2}_{\text{center such that } \overset{!}{=}0}$$

$$\overset{!}{=} -c^2dt'^2$$

we call $dt' \equiv d\tau$ proper time

$$-c^2d\tau^2 = -c^2dt^2+dx^2+dy^2+dz^2 = ds^2 \Rightarrow \boxed{d\tau^2 = -\frac{ds^2}{c^2}}$$

$$\overset{!}{=} -c^2dt^2\underbrace{\left(1-\frac{dx^2+dy^2+dz^2}{c^2dt}\right)}_{\left(1-\frac{\bar{v}^2}{c^2}\right)=\left(1-\bar{\beta}^2\right)=\gamma^{-2}} \Rightarrow \boxed{d\tau = \gamma^{-1}dt}$$

$$dt = \gamma\, d\tau$$

↑

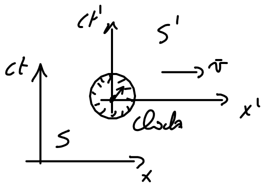
time dilation

Time interval

$$\Delta\tau = \int_w d\tau = \int_w \gamma^{-1}dt$$

$$= \int_w \frac{1}{c}\sqrt{-ds^2}$$

$$= \int_w \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} \leftarrow \text{the metric is "hidden" there!}$$



Implication of the Lorentz transforms

- $ct' = \gamma(ct - \frac{v}{c}x)$ $\quad x' = \gamma(x - vt)$ $\quad y' = y$ $\quad z' = z$ (Standard configuration of S, S')

- Time dilation

$t_0 \equiv t_B - t_A$ (proper time)

$t = t_0(1-\beta^2)^{-1/2} = t_0\gamma \quad \Rightarrow \quad v\uparrow \Rightarrow t\uparrow$

- Length contraction

$l_0 \equiv x'_0 - x'_A$ (proper length)

$l = l_0(1-\beta^2)^{1/2} = l_0\gamma^{-1} \quad (v\uparrow \Rightarrow l\downarrow)$

what if $v = c$? $\Rightarrow l \to 0$

$\Rightarrow \rho = \frac{m}{l^3} = \infty!$

$\Leftrightarrow v = c$ not possible for massive particles

- Velocity transformation

$$\boxed{u'_x} \equiv \frac{\overset{\downarrow}{dx'}}{\underset{\uparrow}{dt'}} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \boxed{\frac{u_x - v}{1 - \frac{u_x v}{c^2}}}$$

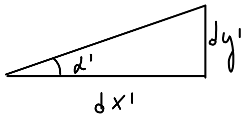
(in lab frame: $u_x = \frac{u'_x + v}{1 + \frac{u_x v}{c^2}}$)

$$\boxed{u'_y \equiv \frac{dy'}{\underset{\nwarrow}{dt'}} = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}} \; (!) \qquad \boxed{u'_z \equiv \frac{dz'}{\underset{\nwarrow}{dt'}} = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}} \; (!)$$

because the time is effected

- Accelerations $\quad a'_x \equiv \frac{\overset{\searrow}{du'_x}}{\underset{\nwarrow}{dt'}}$ ....

- Angles

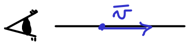


$dy' = dx' \,tg\,\alpha \Rightarrow \boxed{tg\,\alpha'} = \frac{\overset{\downarrow}{dy'}}{\underset{\nwarrow}{dx'}}$

$$= \frac{dy}{\gamma(dx - v\,dt)} = \frac{dy}{\gamma\,dt\left(\frac{dx}{dt} - v\right)} = \boxed{\frac{u_y}{\gamma(u_x - v)}}$$

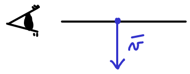
- Doppler effect

↳ longitudinal :



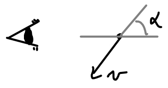
$\frac{\nu}{\nu'} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$

↳ transverse :



$\nu = \nu'\gamma^{-1} \quad (t = t'\gamma)$ purely relativistic effect

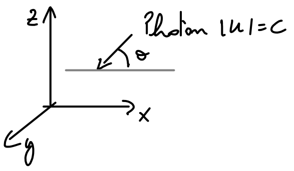
↳ Arbitrary direction :



$\frac{\nu}{\nu'} = \left[\gamma(1 + \beta\cos\theta)\right]^{-1}$ $\qquad \frac{\nu}{\nu'} = \gamma(1 - \beta\cos\theta')$

↑ angle seen by the observer $\qquad$ ↑ angle in rest frame of moving source

- Aberration of light



Polar coordinates:

$u'_x = c\cos\theta'$ $u'_y = 0$ $u'_z = c\sin\theta'$

$u_x = c\cos\theta$ $u_y = 0$ $u_z = c\sin\theta$

Lorentz transf. of velocities →
$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{c\cos\theta - v}{1 - \beta\cos\theta} \Rightarrow \boxed{\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}}$$

Photon

in lab frame: $\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$

- Beaming of light : (solid angles)

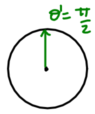
$d\Omega' = d\phi' d\cos\theta'$

$d\Omega = d\phi\, d\cos\theta$: $d\phi = d\phi'$ no change because ⊥ to velocity

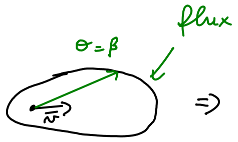
$$d\cos\theta = d\left(\frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}\right) = \frac{d\cos\theta'}{\gamma^2(1 + \beta\cos\theta')^2}$$

$$= \frac{d\phi\, d\cos\theta'}{\gamma^2(1 + \beta\cos\theta')}$$

Super relevant for emission processes :



isotropic emission in source rest frame



anisotropic emission for observer

⇒ priviledge direction of emission

eg. $\theta' = \frac{\pi}{2}$: $\cos\theta' = 0 \Rightarrow \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} = \beta$

$\beta \ll 1 \Rightarrow \cos\theta \sim 0$ $\theta \sim \pi/2$

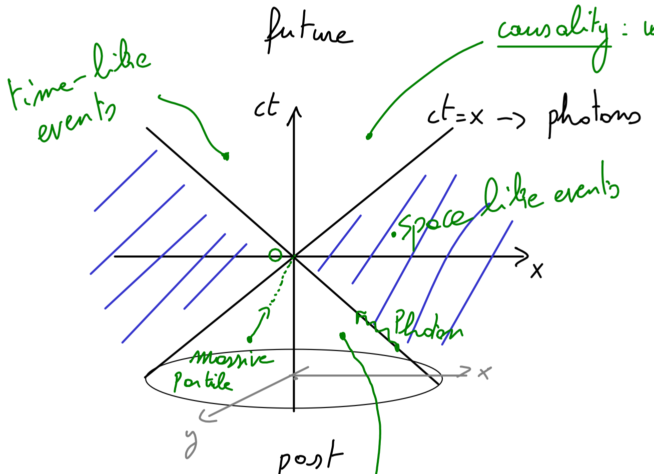
$\beta \sim 1 \Rightarrow \cos\theta \sim 1$ $\theta \sim 0$ (beamed forward)

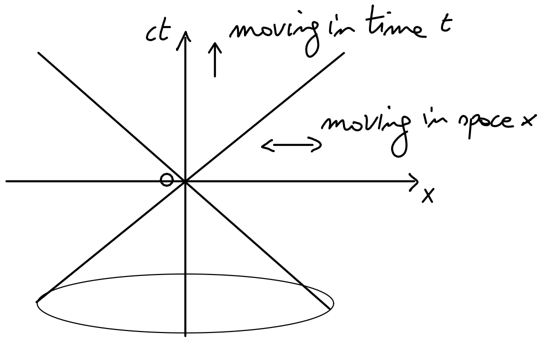
- Cross section

Solid angles are affected ⇒ cross sections transform as well

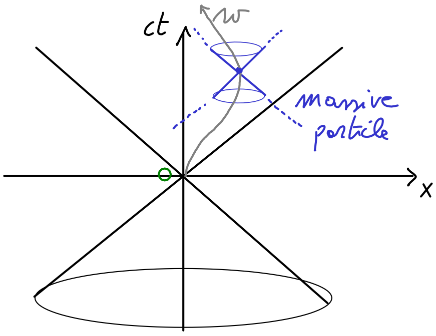
Space-time diagrams

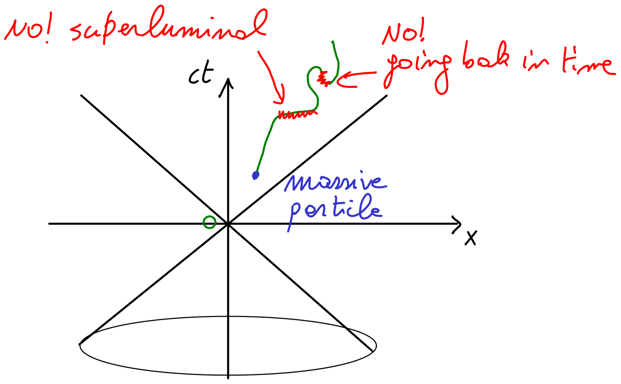
causality: what O can get in contact to



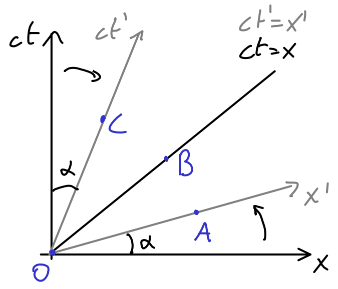


Causality: what can get in contact with O at t=0





Note: ds^2 is not positive definite



OA: Space like event $ds>0$ (frame in which $\Delta t'=0$)

OB: null $ds=0$

OC: Time like event $ds<0$ (frame in which $\Delta x'=0$)

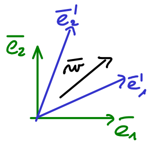
Lorentz transf. are "rotations" (boosts) $\frac{dx}{c\,dt}=\frac{v}{c} \Rightarrow \alpha=\mathrm{atan}\left(\frac{v}{c}\right)$

Because of postulate: null line do not change under Lorentz transf.

ds^2 is invariant under Lorentz transf. (i.e. is the same ∀ frame)

↓

"The modulus of a vector does not depend on the basis"



Vectors, 1-forms, tensors

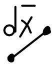
(Korper)
$K=$ field, e.g. $K=\mathbb{R}$

- Vector: $\bar{v} \in V$ "it has no components" $V=\{\bar{v}\in K^n\}$ vector space
- Basis set: $\{\bar{e}_i\}$ $\quad \bar{e}_i \in V$ linearly independent vectors defining the frame
  label
- Components of a vector: $\bar{v} = v^i \bar{e}_i$ as a linear combination of basis vectors
- Transformation of vectors: $x^{i'} = x^{i'}(x^j(\tau))$

$$\frac{dx^{i'}}{d\tau} = \frac{\delta x^{i'}}{\delta x^j}\frac{dx^j}{d\tau} \Rightarrow \boxed{dx^{i'} = \frac{\delta x^{i'}}{\delta x^j} dx^j} \qquad \frac{\delta x^{i'}}{\delta x^j} \equiv J^{i'}{}_j \text{ Jacobian of the transformation}$$

infinitesimal change of $x^{i'}$ with respect to x^j

↳ $d\bar{x}$ (displacement) prototype of a vector



$$\boxed{v^{i'} = \frac{\delta x^{i'}}{\delta x^j} v^j}$$

later I call $\Lambda^{i'}{}_j \equiv \frac{\delta x^{i'}}{\delta x^j}$ (NOT the Lorentz transform!)

- Transformation of basis:

frames (S, S') basis sets $(\{\bar{e}_i\}, \{\bar{e}_{i'}\})$ vector components $(v^i, v^{i'})$ $\quad \bar{v} = \begin{cases} v^i \bar{e}_i \\ v^{i'}\bar{e}_{i'} \end{cases}$ "$\bar{v}$ is $\bar{v}$!"

$$\bar{v} = v^i \underline{\bar{e}_i} = v^{i'}\bar{e}_{i'} = \Lambda^{i'}{}_i v^i \bar{e}_{i'} = v^i \underline{\Lambda^{i'}{}_i \bar{e}_{i'}} \Rightarrow \boxed{\bar{e}_i = \Lambda^{i'}{}_i \bar{e}_{i'}} \quad S' \to S$$

linearity of Λ

inverse:

$$(\Lambda^{-1})^{j'}{}_i \equiv \Lambda_{j'}{}^i \Rightarrow \Lambda_{j'}{}^i \bar{e}_i = \underbrace{\Lambda_{j'}{}^i \Lambda^{i'}{}_i}\, \bar{e}_{i'} = \bar{e}_{j'} \Rightarrow \boxed{\bar{e}_{j'} = \Lambda_{j'}{}^i \bar{e}_i} \quad S \to S'$$

↳ $\Lambda_{j'}{}^i \Lambda^{i'}{}_i = \delta^{i'}_{j'}$ $\quad \Lambda_{j'}{}^i \Lambda^{j'}{}_k = \delta^i_k$ $\qquad \delta^{i'}_{j'}$ Kronecker delta

- Linear map :

$$\boxed{T: V \to G \qquad T(\bar{v}) = T\bar{v} = T_{ij}v^i = \bar{w} \qquad v\in V \quad w\in G}$$

$T(\bar{v}+\bar{u}) = T(\bar{v}) + T(\bar{u})$ $\quad u \in V$ distributive

$T(a\bar{v}) = aT(\bar{v})$ $\quad a\in K$ linear

- Bilinear map : $\boxed{T: V\times V \to G \quad T(\bar{v},\bar{u}) = T_{ij}\, v^i u^j}$

linear in both of its 2 arguments: $T(a\bar{v}+b\bar{u}) = aT(\bar{v}) + bT(\bar{u})$ $\; a,b\in\mathbb{R}$

- The metric

– Now: a generic metric g! (not just η)

– It is a bilinear map identifying the scalar product

$$g: V \times V \to \mathbb{R} \qquad (\bar{u},\bar{v}) \to g(\bar{u},\bar{v}) \equiv \langle \bar{u},\bar{v} \rangle = a \qquad \bar{u},\bar{v} \in V \quad a \in \mathbb{R}$$

– It is also the linear map "linking" the 2 spaces V and $\tilde{V}$

$$g: V \to \tilde{V} \qquad (\bar{v}) \to g(\bar{v},-) = \tilde{v} \qquad v_i = g_{ij} v^i \quad (v_i) \in \tilde{V} \quad (v^i)^T \in V$$

Properties:

- $g(\bar{u},\bar{v}) = 0 \quad \forall \bar{u} \in V \Rightarrow \bar{v} = 0 \quad (\Leftrightarrow \det(g) \neq 0)$ — non degenerate
- $g(\bar{v},\bar{v}) = \langle \bar{v},\bar{v} \rangle = \|\bar{v}\|^2$ — it defines the norm of a vector
- $g(\bar{u},\bar{v}) = g(u^i \bar{e}_i, v^j \bar{e}_j) = u^i v^j g(\bar{e}_i,\bar{e}_j) = u^i v^j g_{ij}$ — components of g $\quad g_{ij} \equiv g(\bar{e}_i,\bar{e}_j)$
- $d\bar{s}^2 = g_{ij} dx^i dx^j \overset{\text{relabeling}}{=} g_{ji}\, \overset{\rightleftarrows}{dx^j dx^i} = g_{ji} dx^i dx^j \Rightarrow g_{ij} = g_{ji}$ — g is symmetric because of the quadratic form of $d\bar{s}^2$

  $ds^2 > 0 \Rightarrow$ Riemanian space

  $ds^2 <> = 0 \Rightarrow$ pseudo-Riemanian space
- $g(\bar{u},\bar{v}) = a = g(u^i \bar{e}_i, v^j \bar{e}_j)$ frame S

  $\overset{!}{=} g(u^{i'} \bar{e}'_i, v^{j'} \bar{e}'_j)$ frame S' — $a \in \mathbb{R}$ is invariant

Example

$$\eta = (\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{diag}(-1,1,1,1) \qquad \text{(cartesian frame)}$$

$$g(\bar{u},\bar{v}) = g_{\mu\nu} u^\mu v^\nu = u_\nu v^\nu = -u_0 v^0 + u_1 v^1 + u_2 v^2 + u_3 v^3 \in \mathbb{R}$$

Careful with the components! Those above are the one for a cartesian frame

For instance:

$$\begin{aligned} ds^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &\overset{!}{=} -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &\overset{!}{=} (\underbrace{c dt + dx^2}_{du})(\underbrace{-c dt + dx}_{dv}) + dy^2 + dz^2 \\ &\overset{!}{=} du\, dv + dy^2 + dz^2 \end{aligned} \Rightarrow \eta = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Same object but expressed on a different basis set

- One-forms = dual vectors = covariant vectors = co-vectors

$$\boxed{\underbrace{\bar{v}^*: V \to K}_{\text{linear map}} \quad \bar{u} \to \bar{v}^*(\bar{u}) \equiv \underbrace{g(\bar{v},\bar{u})}_{\bar{v}^* \equiv g(\bar{v},-) \equiv \tilde{v}} = \langle \bar{v}, \bar{u} \rangle \quad \bar{u},\bar{v} \in V \quad \bar{v}^* \in V^*}$$

dual vector space of V
space of 1-forms

- Components, index lowering

$$\tilde{v}(\bar{u}) = \tilde{v}(u^i \bar{e}_i) = u^i \tilde{v}(\bar{e}_i) = u^i v_i \Rightarrow \boxed{\tilde{v}(\bar{e}_i) \equiv v_i}$$

$\tilde{v}$ lower index, $\bar{u}$ upper index

$$= g(\bar{v},\bar{u}) = g_{ij} u^i v^j \Rightarrow \boxed{\begin{aligned} g_{ij} u^i v^j &= u^i v_i \\ &= u_j v^j \end{aligned}}$$

$\longrightarrow$ g lowers the index

- Example:

$$g = \eta = \text{diag}(-1, 0, 0, 0)$$

(!)

$$\bar{v} = (v^0, v^1, v^2, v^3)^T \in V$$

(!)

$$\tilde{v} = g(\bar{v}, -) = (-v_0, v_1, v_2, v_3) \in V^*$$

(!) take care!

- Both vectors and 1-forms are linear maps

$$\tilde{v}: V \to \mathbb{R} \qquad g(\bar{v},\bar{w}) \equiv \tilde{v}(\bar{w}) \qquad \tilde{v} = g(\bar{v},-)$$

$$\bar{v}: \tilde{V} \to \mathbb{R} \qquad g(\bar{v},\bar{w}) \equiv \bar{w}(\tilde{v})$$

- Basis of one-forms $\{\tilde{\omega}^i\}$

$\tilde{v} = v_i \tilde{\omega}^i$ in analogy to vectors: $\bar{v} = v^i \bar{e}_i$ (lower index)

$\omega^i{}_j = \tilde{\omega}^i(\bar{e}_j)$ j-th component of i-th basis

$$\tilde{v}(\bar{u}) = v_i \tilde{\omega}^i(u^j \bar{e}_j) = v_i u^j \underbrace{\tilde{\omega}^i(\bar{e}_j)}_{\omega^i_j} \overset{!}{=} v_i u^i \Rightarrow \boxed{\omega^i{}_j = \tilde{\omega}^i(\bar{e}_j) = \delta^i_j = \frac{\delta x^i}{\delta x^j}}$$

δ^i_j = Kronecker delta

j-th derivative of the i-th coordinate
is how x^i changes along j
$\tilde{\omega}^i$ are the prototype of gradients

i-th one-form basis in cartesian frame $\longrightarrow$ $\tilde{\omega}^i = (\omega^i_j) = (\omega^i{}_0, \omega^i{}_1, \dots) = (\delta_0 x^i, \delta_1 x^i, \dots \delta_m x^i) = \tilde{\delta x^i}$

Careful:

$\tilde{\omega}^0 = (1, 0, \dots, 0)$ $\quad \bar{e}_0 = (1, 0, \dots, 0)^T$

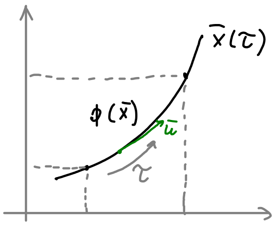
$\tilde{\omega}^1 = (0, 1, \dots, 0)$ $\quad \bar{e}_1 = (0, 1, \dots, 0)^T$

....

$\tilde{\omega}^m = (0, 0, \dots, 1)$ $\quad \bar{e}_m = (0, 0, \dots, 1)^T$

but $\underset{\tilde{V}}{\{\tilde{\omega}^i\}} \neq \underset{V}{\{\bar{e}_i\}}$ they belong to a different spaces!

- They are the prototypes of gradients



$\phi(\bar{x}): \mathbb{R}^m \to \mathbb{R}$ ← some scalar function

$\bar{x}(\tau)$ ← parametric curve

$\bar{u} \equiv \frac{d\bar{x}}{d\tau} = \left(\frac{dx^i}{d\tau}\right)^T$ ← vector tangent to $\bar{x}(\tau)$

$\frac{d\phi}{d\tau} = \frac{\delta\phi}{\delta x^i}\frac{dx^i}{d\tau} = (\delta_i\phi)u^i = a \in \mathbb{R}$ ← Change of ϕ along $x(\tau)$: $\phi(\bar{x}(\tau))$

directional derivative along $\bar{x}(\tau)$

$u^i\delta_i\phi = \nabla_{\bar{u}}\phi$

$= \tilde{\delta}\phi(\bar{u})$ ← in frameless formalism $\tilde{\delta}\phi = \delta_i\phi\,\tilde{\omega}^i$

one-form ↗ ↖ vector

$\hat{\delta}\phi \in$ cotantent space $\bar{u} \in V$ tangent space

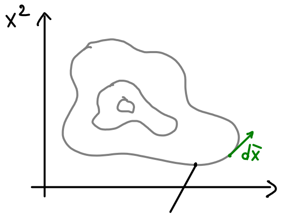
"covectors"

This is the origin of the notation

$\phi_{,\mu} = \frac{\delta\phi}{\delta x^\mu}$ $\phi_{,\mu}$ as a one form

To visualize it:

$\tilde{\delta}\phi(d\bar{x}) = \delta_i\phi\, dx^i = d\phi$ change of ϕ along $d\bar{x}$



$\delta_i\phi \; dx^i$

change ↗ ↖ displacement vector $d\bar{x}$

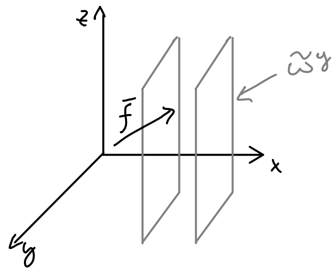
$\phi = const \Rightarrow \frac{d\phi}{d\tau} = \tilde{\delta}\phi\, d\bar{x} = 0$ iso-levels

$d\bar{x}$: prototype of vector (displacement)

$\tilde{\delta}^i$ " " one-form (gradient)

- Useful also to quantify fluxes across a surface

$$\int \bar{f}\cdot d\bar{A} = \int \tilde{dA}(\bar{f})$$



area as a one form

- Transformation of one-forms / basis

$$\boxed{v_{i'}} = \tilde{v}(\bar{e}_{i'}) = \tilde{v}(\Lambda^j{}_{i'}\,\bar{e}_j) = \Lambda^j{}_{i'}\,\tilde{v}(\bar{e}_j) = \boxed{\Lambda^j{}_{i'}\,v_j}$$

$$\tilde{v} = v_i\,\tilde{\omega}^i = v_{i'}\,\tilde{\omega}^{i'} = \Lambda^j{}_{i'}\,v_j\,\omega^{i'} \quad \Rightarrow \quad \boxed{\tilde{\omega}^i = \Lambda^i{}_{i'}\,\omega^{i'}} \quad \boxed{\tilde{\omega}^{i'} = \Lambda_i{}^{i'}\,\tilde{\omega}^i}$$

- This construction guarantees the invariance of 4-intervals

$$\bar{v}(\tilde{w}) = v^i\,\bar{e}_i(w_j\,\tilde{\omega}^j) \overset{!}{=} v^{i'}\bar{e}_{i'}(w_{j'}\,\tilde{\omega}^{j'}) = \Lambda^{i'}{}_i\,v^i\,\bar{e}_{i'}(\Lambda^j{}_{j'}\,w_j\,\tilde{\omega}^{j'}) = \Lambda^{i'}{}_i\,\Lambda^j{}_{j'}\,v^i w_j\,\underbrace{\bar{e}_{i'}(\tilde{\omega}^{j'})}_{\delta^{j'}_{i'}}$$

$$= \underbrace{\Lambda^{i'}{}_i\,\Lambda^j{}_{i'}}_{\delta^j_i}\,v^i w_j = v^i w_i \in \mathbb{R} \text{ invariant}$$

Vectors, 1-forms, tensors

(Korper)
$K=$ field, e.g. $K=\mathbb{R}$

- Vector: $\bar v \in V$ "it has no components" $V=\{\bar v \in K^n\}$ vector space

- Basis set: $\{\bar e_i\}$ $\bar e_i \in V$ linearly independent vectors defining the frame
  label

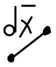
- Components of a vector: $\bar v = v^i \bar e_i$ as a linear combination of basis vectors

- Transformation of vectors: $x^{i'} = x^{i'}(x^j(\tau))$

$$\frac{dx^{i'}}{d\tau} = \frac{\delta x^{i'}}{\delta x^j}\frac{dx^j}{d\tau} \Rightarrow \boxed{dx^{i'} = \frac{\delta x^{i'}}{\delta x^j} dx^j} \qquad \frac{\delta x^{i'}}{\delta x^j} \equiv J^{i'}{}_j$$

Jacobian of the transformation
infinitesimal change of $x^{i'}$ with respect to x^j

↳ $d\bar x$ (displacement) prototype of a vector



$$\boxed{v^{i'} = \frac{\delta x^{i'}}{\delta x^j} v^j}$$

later I call $\Lambda^{i'}{}_j \equiv \frac{\delta x^{i'}}{\delta x^j}$ (NOT the Lorentz transform!)

- Transformation of basis:

frames (S, S') basis sets $(\{\bar e_i\}, \{\bar e_{i'}\})$ vector components $(v^i, v^{i'})$ $\bar v = \begin{cases} v^i \bar e_i \\ v^{i'} \bar e_{i'} \end{cases}$ "$\bar v$ is $\bar v$!"

$$\bar v = v^i \underline{\bar e_i} = v^{i'} \bar e_{i'} = \Lambda^{i'}{}_i v^i \bar e_{i'} = v^i \underline{\Lambda^{i'}{}_i \bar e_{i'}} \Rightarrow \boxed{\bar e_i = \Lambda^{i'}{}_i \bar e_{i'}} \quad S' \to S$$

linearity of Λ

inverse:

$$(\Lambda^{-1})^{j'}{}_i \equiv \Lambda_{j'}{}^i \Rightarrow \Lambda_{j'}{}^i \bar e_i = \underbrace{\Lambda_{j'}{}^i \Lambda^{i'}{}_i}_{\delta^{i'}_{j'}} \bar e_{i'} = \bar e_{j'} \Rightarrow \boxed{\bar e_{j'} = \Lambda_{j'}{}^i \bar e_i} \quad S \to S'$$

↳ $\Lambda_{j'}{}^i \Lambda^{i'}{}_i = \delta^{i'}_{j'}$ $\Lambda_{j'}{}^i \Lambda^{j'}{}_k = \delta^i_k$ $\delta^{i'}_{j'}$ Kronecker delta

- Linear map :

$$\boxed{T: V \to G \qquad T(\bar v) = T\bar v = T_{ij} v^i = \bar w \qquad v \in V \quad w \in G}$$

$T(\bar v + \bar u) = T(\bar v) + T(\bar u)$ $u \in V$ distributive

$T(a\bar v) = aT(\bar v)$ $a \in K$ linear

- Bilinear map : $\boxed{T: V \times V \to G \quad T(\bar v, \bar u) = T_{ij} v^i u^j}$

linear in both of its 2 arguments: $T(a\bar v + b\bar u) = aT(\bar v) + bT(\bar u)$ $a, b \in \mathbb{R}$

- The metric

– Now: a generic metric g! (not just η)

– It is a bilinear map identifying the scalar product

$$g: V\times V \to \mathbb{R} \quad (\bar u,\bar v) \to g(\bar u,\bar v) \equiv \langle \bar u,\bar v\rangle = a \qquad \bar u,\bar v\in V \quad a\in\mathbb{R}$$

– It is also the linear map "linking" the 2 spaces V and $\tilde V$

$$g: V\to\tilde V \quad (\bar v)\to g(\bar v,-) = \tilde v \qquad v_i = g_{ij}v^i \quad (v_i)\in\tilde V \quad (v^i)^T\in V$$

Properties:

- $g(\bar u,\bar v)=0 \quad \forall \bar u\in V \Rightarrow \bar v=0$ $(\Leftrightarrow \det(g)\neq 0)$ non degenerate
- $g(\bar v,\bar v)=\langle\bar v,\bar v\rangle = \|\bar v\|^2$ it defines the norm of a vector
- $g(\bar u,\bar v)=g(u^i\bar e_i, v^j\bar e_j) = u^iv^j g(\bar e_i,\bar e_j) = u^iv^jg_{ij}$ components of g $\quad g_{ij}\equiv g(\bar e_i,\bar e_j)$
- $d\bar s^2 = g_{ij}dx^idx^j \overset{\substack{\text{relabeling}\\ \downarrow}}{=} g_{ji}\overset{\rightleftarrows}{dx^jdx^i} = g_{ji}dx^idx^j \Rightarrow g_{ij}=g_{ji}$ g is symmetric because of the quadratic form of $d\bar s^2$

$ds^2>0 \Rightarrow$ Riemanian space

$ds^2<>=0 \Rightarrow$ pseudo-Riemanian space

- $g(\bar u,\bar v)=a = g(u^i\bar e_i, v^j\bar e_j)$ frame S

$\overset{!}{=} g(u^{i'}\bar e'_{i}, v^{j'}\bar e'_{j})$ frame S' $\quad a\in\mathbb{R}$ is invariant

Example

$$\eta = (\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \mathrm{diag}(-1,1,1,1) \qquad \text{(cartesian frame)}$$

$$g(\bar u,\bar v) = g_{\mu\nu}u^\mu v^\nu = u_\nu v^\nu = -u_0v^0 + u_1v^1 + u_2v^2 + u_3v^3 \in\mathbb{R}$$

Careful with the components! Those above are the one for a cartesian frame

For instance:

$$\begin{aligned} ds^2 &= -(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2\\ &\overset{!}{=} -c^2dt^2+dx^2+dy^2+dz^2\\ &\overset{!}{=} (\underbrace{cdt+dx^2}_{du})(-\underbrace{cdt+dx}_{dv}) + dy^2+dz^2\\ &\overset{!}{=} du\,dv + dy^2+dz^2 \quad\Rightarrow\quad \eta = \begin{pmatrix} 0 & 1/2 & 0 & 0\\ 1/2 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\end{aligned}$$

Same object but expressed on a different basis set

- **One-forms = dual vectors = covariant vectors = co-vectors**

$$\boxed{\underbrace{\bar{v}^*: V \to K}_{\text{linear map}} \quad \bar{u} \to \bar{v}^*(\bar{u}) \equiv \underbrace{g(\bar{v},\bar{u})}_{\bar{v}^* \equiv g(\bar{v},-) \equiv \tilde{v}} = \langle \bar{v},\bar{u} \rangle \quad \bar{u},\bar{v} \in V \quad \bar{v}^* \in V^*}$$

dual vector space of V
space of 1-forms

- Components, index lowering

$$\tilde{v}(\bar{u}) = \tilde{v}(u^i \bar{e}_i) = u^i \tilde{v}(\bar{e}_i) = u^i v_i \Rightarrow \boxed{\tilde{v}(\bar{e}_i) \equiv v_i}$$

$\tilde{v}$ lower index, $\bar{u}$ upper index

$$= g(\bar{v},\bar{u}) = g_{ij} u^i v^j \Rightarrow \boxed{g_{ij} u^i v^j = u^i v_i = u_j v^j} \to g \text{ lowers the index}$$

- Example:

$$g = \eta = \mathrm{diag}(\overset{(!)}{-1},1,1,1)$$

$$\bar{v} = (v^0, v^1, v^2, v^3)^T \in V \quad (!)$$

$$\tilde{v} = g(\bar{v},-) = (-v_0, v_1, v_2, v_3) \in V^*$$

(!) take care!

- Both vectors and 1-forms are linear maps

$\tilde{v}: V \to \mathbb{R}$ $\quad g(\bar{v},\bar{w}) \equiv \tilde{v}(\bar{w})$ $\quad \tilde{v} = g(\bar{v},-)$

$\bar{v}: \tilde{V} \to \mathbb{R}$ $\quad g(\bar{v},\bar{w}) \equiv \bar{w}(\tilde{v})$

- Basis of one-forms $\{\tilde{\omega}^i\}$

$\tilde{v} = v_i \tilde{\omega}^i$ in analogy to vectors: $\bar{v} = v^i \bar{e}_i$ (lower index)

$\omega^i{}_j = \tilde{\omega}^i(\bar{e}_j)$ j-th component of i-th basis

$$\tilde{v}(\bar{u}) = v_i \tilde{\omega}^i(u^j \bar{e}_j) = v_i u^j \underbrace{\tilde{\omega}^i(\bar{e}_j)}_{\omega^i_j} \overset{!}{=} v_i u^i \Rightarrow \boxed{\omega^i{}_j = \tilde{\omega}^i(\bar{e}_j) = \delta^i_j = \frac{\delta x^i}{\delta x^j}}$$

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j-th derivative of the i-th coordinate
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$\tilde{\omega}^i$ are the prototype of gradients

i-th one-form basis in cartesian frame $\longrightarrow$ $\tilde{\omega}^i = (\omega^i{}_j) = (\omega^i{}_0, \omega^i{}_1, \ldots) = (\delta_0 x^i, \delta_1 x^i, \ldots \delta_m x^i) = \tilde{\delta} x^i$

Careful:

$\tilde{\omega}^0 = (1,0,\ldots,0)$ $\quad \bar{e}_0 = (1,0,\ldots,0)^T$
$\tilde{\omega}^1 = (0,1,\ldots,0)$ $\quad \bar{e}_1 = (0,1,\ldots,0)^T$
.... $\quad$ ....
$\tilde{\omega}^m = (0,0,\ldots,1)$ $\quad \bar{e}_m = (0,0,\ldots,1)^T$

but $\underset{\tilde{V}}{\{\tilde{\omega}^i\}} \neq \underset{V}{\{\bar{e}_i\}}$ they belong to a different spaces!

– Transformation of one-forms / basis

$$\boxed{v_{i'}} = \tilde{v}(\bar{e}_{i'}) = \tilde{v}(\Lambda^{j}{}_{i'}\,\bar{e}_j) = \Lambda^{j}{}_{i'}\,\tilde{v}(\bar{e}_j) = \boxed{\Lambda^{j}{}_{i'}\,v_j}$$

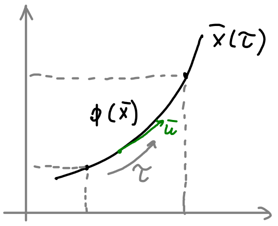
$$\tilde{v} = v_i\,\tilde{\omega}^i = v_{i'}\,\tilde{\omega}^{i'} = \Lambda^{i}{}_{i'}\,v_i\,\omega^{i'} \quad \Rightarrow \quad \boxed{\tilde{\omega}^i = \Lambda^{i}{}_{i'}\,\omega^{i'}} \quad \boxed{\tilde{\omega}^{i'} = \Lambda_i{}^{i'}\,\tilde{\omega}^i}$$

– This construction guarantees the invariance of 4-intervals

$$\bar{v}(\tilde{w}) = v^i\,\bar{e}_i(w_j\,\tilde{\omega}^j) \overset{!}{=} v^{i'}\bar{e}_{i'}(w_{j'}\,\tilde{\omega}^{j'}) = \Lambda^{i'}{}_i\,v^i\,\bar{e}_{i'}(\Lambda^{j}{}_{j'}\,w_j\,\tilde{\omega}^{j'}) = \Lambda^{i'}{}_i\,\Lambda^{j}{}_{j'}\,v^i w_j\,\underbrace{\bar{e}_{i'}(\tilde{\omega}^{j'})}_{\delta^{j'}_{i'}}$$

$$= \underbrace{\Lambda^{i'}{}_i\,\Lambda^{j}{}_{i'}}_{\delta^j_i}\,v^i w_j = v^i w_i \in \mathbb{R} \text{ invariant}$$

– One-forms are the prototypes of gradients



$\phi(\bar{x}): \mathbb{R}^m \to \mathbb{R}$ ← some scalar function

$\bar{x}(\tau)$ ← parametric curve

$\bar{u} \equiv \frac{d\bar{x}}{d\tau} = \left(\frac{dx^i}{d\tau}\right)^T$ ← vector tangent to $\bar{x}(\tau)$

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial x^i}\frac{dx^i}{d\tau} = (\partial_i\phi)\,u^i = a \in \mathbb{R}$$ ← Change of ϕ along $x(\tau)$: $\phi(\bar{x}(\tau))$

directional derivative along $\bar{x}(\tau)$

$u^i\,\partial_i\phi = \bar{\nabla}_{\bar{u}}\,\phi$

$$= \tilde{\partial}\phi(\bar{u})$$ ← in frameless formalism $\tilde{\partial}\phi = \partial_i\phi\,\tilde{\omega}^i$

one-form ↗ ↖ vector

$\tilde{\partial}\phi \in$ cotantent space

"covectors"

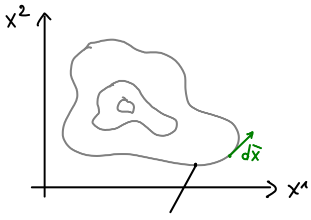
$\bar{u} \in V$ tangent space

this is the origin of the notation

$\phi_{,\mu} = \frac{\partial\phi}{\partial x^\mu}$ $\phi_{,\mu}$ as a one form

To visualize it:

$\tilde{\partial}\phi(d\bar{x}) = \partial_i\phi\,dx^i = d\phi$ change of ϕ along $d\bar{x}$



$\partial_i\phi\ dx^i$

change ↗ ↖ displacement vector $d\bar{x}$

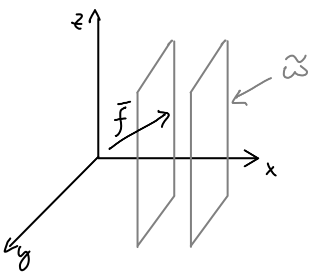
$\phi = \text{const} \Rightarrow \frac{d\phi}{d\tau} = \tilde{\partial}\phi\,d\bar{x} = 0$ iso-levels

$d\bar{x}_i$ prototype of vector (displacement)

$\tilde{\partial}^i$ " " one-form (gradient)

– Useful also to quantify fluxes across a surface

$$\int \bar{f}\cdot d\bar{A} = \int \tilde{dA}(\bar{f})$$



area as a one form

Tensors

Tensors are rulers in physics:

metric tensor (space-time)
elettromagnetic tensor
energy-momentum tensor
....

They are a "generalization":

Scalars: 0 indeces (a), Vector: 1 index (a^i)
Matrix: 2 indeces (a_{ij}), Tensors: N indeces
↑ (tensors are matrixes but not all matrixes are tensors)

General definition:

a matematical object obaying certain transformation roles
i.e components have certain transformation properties
under a change of coordinates

Types of tensors:

a tensor of type $\binom{M}{N}$ is a linear mapping of
M 1-forms and N vectors to scalars (Lorentz invariant)

$$T : \underbrace{\tilde{V} \times \tilde{V} \times \cdots \times \tilde{V}}_{M} \times \underbrace{V \times V \times \cdots \times V}_{N} \rightarrow \mathbb{R} \quad \text{invariant}$$

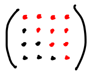
Types of indeces:

type $\binom{M}{N}$ ← # of input 1-forms (index up)
← " " " vectors (index down)

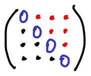
Rank:

total number of indeces
eg. $T_{ij}{}^{k}$: rank = 3 type $\binom{1}{2}$
$i = 1, \ldots, m \quad j = 1, \ldots, m \quad k = 1, \ldots, m$
m^3 components (m-dimensional vector space V eg. $V = \mathbb{R}^m$)

Symmetric tensors

$T(\bar{v}, \bar{w}) = T(\bar{w}, \bar{v}) \iff T_{ij} = T_{ji} \quad \forall\, \bar{v}, \bar{w} \in V \quad \rightarrow$ 

Antisymmetric tensors

$T(\bar{v}, \bar{w}) = -T(\bar{w}, \bar{v}) \iff T_{ij} = -T_{ji} \quad \forall\, \bar{v}, \bar{w} \in V \quad \rightarrow$ 

$T_{ij} \equiv \partial_i A_j - \partial_j A_i$ is guaranteed to be anti-symmetric

Splitting in symmetric and antisymmetric parts:

eg. a type $\binom{0}{2}$ tensor T: $T_{ij} = T_{(i,j)} + T_{[i,j]}$ with $T_{(i,j)} \equiv \frac{1}{2}(T_{ij} + T_{ji})$ Symmetric part
↑
you can always do it
$T_{[i,j]} \equiv \frac{1}{2}(T_{ij} - T_{ji})$ Antisymmetric part

You already meet tensors!

Type $\binom{0}{0}$: scalar $a: \mathbb{R} \to \mathbb{R}$ $a(b) = c \in \mathbb{R}$ $a, b, c \in \mathbb{R}$

Type $\binom{0}{1}$: "1-forms" = dual vector = covector = covariant vector

linear map $\tilde{P}: V \to \mathbb{R}$ $\tilde{P}(\bar{v}) = a \in \mathbb{R}$ $\bar{v} \in V$ $\tilde{P} \in V^*$

Type $\binom{1}{0}$: vector = contra-variant vector = tangent vector

linear map $\bar{P}: V \to \mathbb{R}$ $\bar{P}(\tilde{v}) = a \in \mathbb{R}$ $\tilde{v} \in V^*$ $\bar{P} \in V$

Type $\binom{1}{1}$: $P: V^* \times V \to \mathbb{R}$ $P(\tilde{v}, \bar{w}) = a \in \mathbb{R}$ $\tilde{v} \in V^*$ $\bar{w} \in V$

Type $\binom{0}{2}$: $P: V \times V \to \mathbb{R}$ $P(\bar{v}, \bar{w}) = a \in \mathbb{R}$ $\bar{v}, \bar{w} \in V$

eg. the metric η, g! $\eta: \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ $\eta(\bar{v}, \bar{w}) = \langle \bar{v}, \bar{w} \rangle = \eta_{\alpha\beta} v^\alpha w^\beta$ $\bar{v}, \bar{w} \in \mathbb{M}$

Rising/Lowering indexes

$$P(\overset{\downarrow}{\bar{v}}, \overset{\downarrow}{\tilde{\omega}}) = P(v^i \bar{e}_i, w_j \tilde{\omega}^j) = P(g^{i\alpha} v_\alpha \bar{e}_i, w_j \tilde{\omega}^j) = v_\alpha w_j g^{i\alpha} P(\bar{e}_i, \tilde{\omega}^j) = v_\alpha w_j \underbrace{g^{i\alpha} P_i{}^j}_{P^{\alpha j}}$$

$\binom{M}{N} \to \binom{M-1}{N+1}$ lowering eg. $g_{\alpha\mu} T^\mu{}_{\beta\gamma} = T_{\alpha\beta\gamma}$

$\binom{M}{N} \to \binom{M+1}{N-1}$ rising eg. $g^{\alpha\gamma} T^\mu{}_{\beta\gamma} = T^\mu{}_\beta{}^\alpha$ $\Rightarrow$ $T_{ij}{}^k \neq T_i{}^{jk}$!

Components of a tensor

components are the values of a function (linear mapping) when its arguments are the basis $\{\bar{e}_i\}, \{\tilde{\omega}^i\}$ of the frame. Eg.

$$P: V \times V^* \to \mathbb{R} \quad P(\bar{v}, \tilde{w}) = P(v^i \bar{e}_i, w_j \tilde{\omega}^j) = v^i w_j P(\bar{e}_i, \tilde{\omega}^j) = v^i w_j P_i{}^j$$

$$\boxed{P(\bar{e}_i, \tilde{w}^j) = P_i{}^j}$$

Transformation of the components

$$P^{i'j'} = P(\tilde{\omega}^{i'}, \tilde{\omega}^{j'}) = P(\Lambda^{i'}{}_\alpha \tilde{\omega}^\alpha, \Lambda^{j'}{}_\beta \tilde{\omega}^\beta) = \Lambda^\alpha{}_{i'} \Lambda^\beta{}_{j'} P(\tilde{\omega}^\alpha, \tilde{\omega}^\beta)$$

$$P_{i'}{}^{j'} = P(\bar{e}_{i'}, \tilde{\omega}^{j'}) = P(\Lambda^\alpha{}_{i'} \bar{e}_\alpha, \Lambda^{j'}{}_\beta \tilde{\omega}^\beta) = \Lambda^\alpha{}_{i'} \Lambda^{j'}{}_\beta P(\bar{e}_\alpha, \tilde{\omega}^\beta)$$

$$P_{i'j'} = P(\bar{e}_{i'}, \bar{e}_{j'}) = P(\Lambda^\alpha{}_{i'} \bar{e}_\alpha, \Lambda^\beta{}_{j'} \bar{e}_\beta) = \Lambda^\alpha{}_{i'} \Lambda^{j'}{}_\beta P(\bar{e}_\alpha, \bar{e}_\beta)$$

$$P^{i'}{}_{j'} = P(\tilde{\omega}^{i'}, \bar{e}_{j'}) = P(\Lambda^{i'}{}_\alpha \tilde{\omega}^\alpha, \Lambda^\beta{}_{j'} \bar{e}_\beta) = \Lambda^\alpha{}_{i'} \Lambda^{j'}{}_\beta P(\bar{e}_\alpha, \bar{e}_\beta)$$

$$\boxed{\begin{aligned} P^{i'j'} &= \frac{\delta x^{i'}}{\delta x^\alpha} \frac{\delta x^{j'}}{\delta x^\beta} P^{\alpha\beta} \\ P_{i'}{}^{j'} &= \frac{\delta x^\alpha}{\delta x^{i'}} \frac{\delta x^{j'}}{\delta x^\beta} P_\alpha{}^\beta \\ P_{i'j'} &= \frac{\delta x^\alpha}{\delta x^{i'}} \frac{\delta x^\beta}{\delta x^{j'}} P_{\alpha\beta} \\ P^{i'}{}_{j'} &= \frac{\delta x^{i'}}{\delta x^\alpha} \frac{\delta x^\beta}{\delta x^{j'}} P^\alpha{}_\beta \end{aligned}}$$

Basis for tensors

- We have basis for vectors $\{\bar{e}_i\}$ and 1-forms $\{\tilde{\omega}^i\}$
  $\Rightarrow$ vectors/1-forms as linear combination of basis vectors/1-forms : $\bar{v} = v^i \bar{e}_i$ / $\tilde{w} = w_i \tilde{\omega}^i$
- Do we have basis for tensors ? i.e. tensors expressed as linear combination of something

eg. type $\binom{0}{2}$ $\quad P : V \times V \to \mathbb{R}$ look for something like "$P \equiv P_{ij} \cdot \text{☺}^{ij}$"

$$P_{ij} := P(\bar{e}_i, \bar{e}_j) \equiv P_{k\ell}\, \underbrace{\text{☺}^{k\ell}(\bar{e}_i, \bar{e}_j)}_{\overset{!}{=}\, \delta^k_i \delta^\ell_j} \qquad \Rightarrow \qquad \text{☺}^{k\ell}(\bar{e}_i, \bar{e}_j) \overset{!}{=} \delta^k_i \delta^\ell_j$$

$$\delta^k_i = \frac{\delta x^k}{\delta x^i} = \omega^k_i$$

$$= \omega^k{}_i\, \omega^\ell{}_j$$

$$= \tilde{\omega}^k(\bar{e}_i)\, \tilde{\omega}^\ell(\bar{e}_j)$$

outer product: $\tilde{\omega}^k \otimes \tilde{\omega}^\ell \equiv \tilde{\omega}^k(-)\,\tilde{\omega}^\ell(-) \longrightarrow$

$$\equiv \underbrace{\tilde{\omega}^k \otimes \tilde{\omega}^\ell}_{\text{Basis}}(\bar{e}_i, \bar{e}_j)$$

$$\boxed{\begin{aligned} P(\bar{v}, \bar{w}) &= P_{ij}(\tilde{\omega}^i \otimes \tilde{\omega}^j)(\bar{v}, \bar{w}) \\ P &= P_{ij}(\tilde{\omega}^i \otimes \tilde{\omega}^j) \\ &= P^i{}_j\, \bar{e}_i \otimes \tilde{\omega}^j \\ &= P^{ij}\, \bar{e}_i \otimes \bar{e}_j \\ &= P_i{}^j\, \tilde{\omega}^i \otimes \bar{e}_j \end{aligned}}$$

You have various ways to construct a tensor

Every tensor can be expressed as a combination of outer products

i.e. given $\tilde{p}, \tilde{q} \in V^*$ (for example) you can always build a tensor $T = \tilde{p} \otimes \tilde{q} \equiv \tilde{p}(-)\,\tilde{q}(-)$ $\quad T : V \times V \to \mathbb{R}$

$$T(\bar{v}, \bar{w}) \equiv \underbrace{\tilde{p}(\bar{v})\, \tilde{q}(\bar{w})}_{a \,=\, p_i v^i q_j w^j} \neq \underbrace{\tilde{p}(\bar{w})\, \tilde{q}(\bar{v})}_{p_i w^i q_j v^j \,=\, b} \qquad \bar{v}, \bar{w} \in V$$

in fact in general $a \neq b$

in general $\otimes$ is not commutative $\quad \tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$

Example:

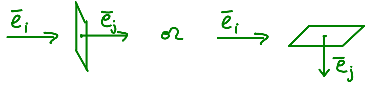
type $\binom{0}{1}$ 1-form : $\tilde{v} : V \to \mathbb{R}$ $\qquad \tilde{v} = v_i \tilde{\omega}^i$ $\quad \{\tilde{\omega}^i\}$ basis of 1-forms

type $\binom{1}{0}$ vector : $\bar{v} : V^* \to \mathbb{R}$ $\qquad \bar{v} = v^i \bar{e}_i$ $\quad \{\bar{e}_i\}$ basis of vectors

type $\binom{0}{2}$ $\quad P : V \times V \to \mathbb{R}$ $\qquad P = P_{ij}\, \tilde{\omega}^i \otimes \tilde{\omega}^j$ $\quad$ eg. metric $g = g_{ij}\, \tilde{\omega}^i \otimes \tilde{\omega}^j$

type $\binom{3}{1}$ $\quad P : V^* \times V^* \times V \times V^* \to \mathbb{R}$ $\qquad P = P^{ij}{}_k{}^\ell\, \bar{e}_i \otimes \bar{e}_j \otimes \tilde{\omega}^k \otimes \bar{e}_\ell$

Intuitive understanding of tensors : is an object that "points" in multiple directions



Tensors operations

3 tensors of the same type, based on the same vector space ↙

- Sum/Subtraction : $S = T + V \qquad T_{ij} + V_{ij} = T(\bar{e}_i, \bar{e}_j) + V(\bar{e}_i, \bar{e}_j) = S(\bar{e}_i, \bar{e}_j) = S_{ij}$

- Multiplication by a scalar : $\alpha \in \mathbb{R} \qquad S = \alpha T \qquad S_{ij} = \alpha T_{ij}$

- Outer ptoduct : $(\tilde{q} \otimes \tilde{p})(\bar{v}, \bar{u}) \equiv \tilde{q}(\bar{v})\tilde{p}(\bar{u}) \qquad \bar{v}, \bar{u} \in V \quad \tilde{q}, \tilde{p} \in V^* \quad (\tilde{q} \otimes \tilde{p})$ Rank 2

$$(\tilde{q} \otimes \tilde{p} \otimes \tilde{r})(\bar{v}, \bar{u}, \bar{s}) = \underbrace{\tilde{q}(\bar{v})\,\tilde{p}(\bar{u})}_{t(\bar{v},\bar{u})}\tilde{r}(\bar{s}) = t(\bar{v}, \bar{u})\tilde{r}(\bar{s})$$
$$\equiv (t \otimes \tilde{r})(\bar{v}, \bar{u}, \bar{s})$$

- Inner product :

eg. $T(\tilde{\omega}^i, \bar{e}_j)\, S(\bar{e}_i) = \boxed{T^i{}_j S_i = Q_j} = Q(\bar{e}_j)$ inner product between 2 tensors T and S gives a new tensor Q

$TS \neq ST : \quad T^{ij} S_i \neq T^{ij} S$ not commutative

- Contraction :

fixed, same index

eg. Rank-3 tensor $T(-, -, -) = (T^\alpha{}_{\beta\gamma}) \qquad T(\tilde{\omega}^\alpha, -, \bar{e}_\alpha) \equiv S(-)$ Rank-1 tensor

components : $\boxed{S_\beta} = T(\tilde{\omega}^\alpha, \bar{e}_\beta, \bar{e}_\alpha) = T^\alpha{}_{\beta\alpha} = \boxed{g^{\alpha\nu}\, T_{\nu\beta\alpha}}$ (repeated index up, down = contraction)

We obtain a lower rank tensor S

- Derivative :

eg. $\partial_\alpha T^{\mu\nu} = R_\alpha{}^{\mu\nu}$ (in flat S-T!) $\begin{pmatrix}2\\0\end{pmatrix} \to \begin{pmatrix}2\\1\end{pmatrix}$ tensor with an higher rank

only $\partial_\alpha \Phi$ is a valid tensor in all S-T (1-form)

Mapping tensors on tensors

$T(-, -, \bar{v}) \qquad \bar{v} \in V$ fixed

If we provide other two "inputs" we get a real number

$\Rightarrow T(-, -, \bar{v}) \equiv S(-, -)$ is a rank-2 tensor

T maps a vector $\bar{v}$ into a rank-2 tensor S

Components $\Rightarrow \boxed{S_{\alpha\beta}} \equiv S(\bar{e}_\alpha, \bar{e}_\beta) \equiv T(\bar{e}_\alpha, \bar{e}_\beta, \bar{v}) = \boxed{T_{\alpha\beta\gamma} v^\gamma}$

another eg. $W(-) \equiv T(\bar{u}, -, \bar{v}) \qquad \bar{u}, \bar{v} \in V$ fixed $\Rightarrow W$ rank-1 tensor (vector)

$W(\bar{e}_\alpha) = T(\bar{u}, \bar{e}_\alpha, \bar{v}) = T_{\gamma\alpha\delta} u^\gamma v^\delta$

Special tensors:

- The Levi-CIvita symbol

$$\varepsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{even permutations of } \; 0,1,2,3 \\ -1 & \text{odd } \quad '' \qquad\qquad '' \qquad '' \\ 0 & \text{otherwise } \quad \text{(i.e. same index)} \end{cases} \qquad \text{starting with } \varepsilon_{0123} = 1$$

$$\text{eg. } \varepsilon_{0123} = 1 \qquad \varepsilon_{0\underset{\smile}{21}3} = -1 \qquad \varepsilon_{0\underset{\smile}{321}} = -1 \qquad \varepsilon_{\underset{\smile}{10}23} = -1 \; \ldots \qquad \varepsilon_{\underset{\smile}{10}\underset{\smile}{32}} = 1 \; \ldots\ldots$$

$$\varepsilon_{\nu 12\nu} = -\varepsilon_{\underset{\curvearrowleft}{\nu 12 \nu}} \Rightarrow \varepsilon_{\nu 12 \nu} = 0 \qquad \nu = 0,1,2,3$$

$$\text{eg. of application: } \quad \bar{\nabla} \times \bar{B} = \varepsilon^{ijk} \partial_j B_k = (\partial_2 B_3 - \partial_3 B_2)\, \bar{e}_1 + (\partial_3 B_1 - \partial_1 B_3)\, \bar{e}_2 + (\partial_1 B_2 - \partial_2 B_1)\, \bar{e}_3$$

is invariant under Lorentz transformations in flat space-time (not in general)

- Kroneker delta

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The same in all coordinate systems and $\forall$ space-time

Tensors are maps $\Rightarrow (\delta^i_j)$ is the identity map from vectors to vectors

$\delta : V \to V \quad \delta(\bar{v}) = \bar{v} \quad \delta^j_i v^i = v^j$ it "selects" a component without changing it

Isotropic tensors
e.g. e^{ikl} completely anti-symmetric unit tensor Landau pg. 34 (in flat)
pg 3-4 curvilinear

Space of all smooth scalar functions on M

$F = \{\phi \mid \phi : V \to K\}$

$\phi : V \to K \qquad \phi(\bar{v}) = a \quad \bar{v} \in V \quad a \in K \qquad \phi$ linear map

<u>Derivative</u> : $\quad d : F \to \mathbb{R}$ linear map

$$(I) \quad d(af + bg) = a\,d(f) + b\,d(g) \qquad f, g \in F \quad a, b \in K$$

$$(II) \quad d(fg) = d(f)\,g + f\,d(g)$$

e.g.

$f \in F \quad \underline{f(\bar{v}) \equiv c} \quad \forall \bar{v}$ (constant function)

$(I) \Rightarrow \quad d(c \cdot f) = c\,d(f)$

$(II) \Rightarrow \quad \underline{d(f^2)} = d(ff) = d(f) \cdot f + f\,d(f) = 2f\,d(f) = 2c\,d(f) = \underline{2\,d(cf)}$

but $d(f^2) = d(cf) \Rightarrow \quad 2c\,d(f) = c\,d(f)$ true only if $d(f) = 0$

$\Rightarrow$ derivatives of const. functions are zero

$\rightarrow$ derivatives have the structure of a vector space Mathias 17

Lorentz geometry

Geometric interpretation of the ct,x plane

- Define a vector space
- $\bar{x} = (x^\mu) = (x^0, x^1, x^2, x^3)$ 4-vector $x^0 = ct$ $\bar{x} \in \mathbb{M}$ $\mathbb{M}$ Minkowski space
- Separations in the space-time are expressed by the 4-interval $d\bar{s}$

  ⇒ Distances are measured in termes of a metric ⇒ metric vector space

$$d\bar{s}^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

4-interval ⇒ vector space $d\bar{s} \in \mathbb{M}$ vector in Minkowski space

$$= \underbrace{\sum_{\nu=0}^{3} \sum_{\mu=0}^{3}}\, \eta_{\mu\nu}\, dx^\mu dx^\nu$$

Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$ $\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$= \eta_{\mu\nu}\, dx^\mu dx^\nu \longleftarrow$$

$$= dx_\nu\, dx^\nu \longleftarrow$$

Einstein notation and index lowering

Based on a frame ⇒ components

Frameless representation

$$= d\bar{x}^T \eta\, d\bar{x}$$

matrix notation

$$= \boxed{\eta(d\bar{x},} d\bar{y})$$

$\eta : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ $\eta(\bar{u}, \bar{v}) = a \in \mathbb{R}$ $\bar{u}, \bar{v} \in \mathbb{M}$ bilinear map identifying the scalar product

$$= \langle d\bar{x}, d\bar{x} \rangle$$

scalar product

$$= d\tilde{x}(d\bar{x})$$

$\eta(\bar{v}, -) \equiv \tilde{v}$ $\tilde{v} : \mathbb{M} \to \mathbb{R}$ $\tilde{v}(\bar{u}) = a \in \mathbb{R}$ $(v_\nu) \equiv \bar{v}^* \equiv \tilde{v} \in \mathbb{M}^*$ $\mathbb{M}^*$ Dual space of $\mathbb{M}$

1-form, dual vector, tangent vector....

$$= \|d\bar{s}\|^2 \in \mathbb{R}$$

norm of $d\bar{s} \in \mathbb{M}$ ⇒ invariant, it can be negative!

Here we used η but this formalism is valid for any arbitrary metric g! in a grav. field $g \neq \eta$!

The metric is "hidden" in many places!

- Proper time $\tau = \frac{1}{c}\sqrt{-ds^2} = \frac{1}{c}\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \frac{1}{c}\sqrt{-\eta(d\bar{x}, d\bar{x})}$

  $\uparrow$

- Do you see?! The metric is hidden in crucial places!

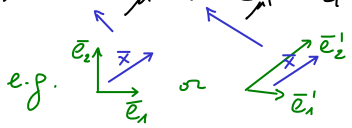
- Another example? It is even in the dear old Kinetic energy...

$$T = \frac{1}{2} m \bar{v}^2 = \frac{1}{2} m \bar{v} \bar{v} = \frac{1}{2} m\, \delta_{ij} v^i v^j \qquad \bar{v} \in \mathbb{R}^3$$

$\uparrow$ $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$ (Euclidean 3D space + cartesian system)

Note: above we have choosen a specific basis set (cartesian)

$\bar{x} = x^{\mu}\bar{e}_{\mu} = x^{\mu'}\bar{e}_{\mu'}$ 4-vectors as linear combination of basis



different $\{\bar{e}_i\}$ but $\bar{x}$ is $\bar{x}$

$\{\bar{e}_{\mu}\}$ basis set $\eta_{\mu\nu} = \eta(\bar{e}_{\mu}, \bar{e}_{\nu}) = \bar{e}_{\mu}\bar{e}_{\nu}$ η defines the scalar product

↑
some linearly independent vector

$\bar{e}_0\bar{e}_0 = -1$ $\bar{e}_i\bar{e}_i = 1$ $\bar{e}_i\bar{e}_j = 0$ $i \neq j$ ← Cartesian frame

$\bar{e}_0 = (1,0,0,0)^T$, $\bar{e}_1 = (0,1,0,0)^T$....

$d\bar{s}^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$\bar{e}_{\mu}$ $\bar{e}_{\mu'}$ basis of S and S' frame

Minkowski metric η can also be not diagonal. Example with ≠ basis:

$$ds^2 = \underbrace{-c^2dt^2 + dx^2 + dy^2 + dz^2}_{\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} = (\underbrace{-c\,dt + dx}_{\equiv du})(\underbrace{c\,dt + dx}_{\equiv dv}) + dy^2 + dz^2 \equiv \underbrace{du\,dv + dy^2 + dz^2}_{\eta = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Other coordinate frames: polar, Kruskal, Eddington-Finkenstein, ...

Frame transformations

- We are dealing with inertial frames

$$\frac{dx^{\mu}}{d\tau} = \text{const.} \qquad \frac{dx^{\mu'}}{d\tau} = \text{const.}$$ (shift of coordinates through a point)

$$\frac{d^2x^{\mu}}{d\tau^2} = 0 \qquad \frac{d^2x^{\mu'}}{d\tau^2} = 0$$ (no acceleration)

here, we parameterize $x^{\mu}(\tau)$ with proper time τ, but any affine parameter can be used $\lambda = a+\tau b$

- Relation between frames S, S'

$$x^{\mu'} = x^{\mu'}(x^{\mu}(\tau))$$

transformations law

transformed component

$$\frac{dx^{\mu'}}{d\tau} = \frac{\delta x^{\mu'}}{\delta x^{\mu}} \frac{dx^{\mu}}{d\tau}$$

$\frac{\delta x^{\mu'}}{\delta x^{\mu}}$ = Jacobian of the transformation

$$\underset{\overset{!}{=}0}{\frac{d^2x^{\mu'}}{d\tau^2}} = \frac{d}{d\tau}\left(\frac{dx^{\mu'}}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\delta x^{\mu'}}{\delta x^{\mu}}\frac{dx^{\mu}}{d\tau}\right) = \frac{\delta^2 x^{\mu'}}{\delta x^{\nu}\delta x^{\mu}}\underbrace{\frac{dx^{\nu}}{d\tau}\frac{dx^{\mu}}{d\tau}}_{\text{constant}} + \frac{\delta x^{\mu'}}{\delta x^{\mu}}\underset{\overset{!}{=}0 \text{ (inertial frame)}}{\frac{d^2x^{\mu}}{d\tau^2}}$$

$$\Downarrow$$

$$\frac{\delta^2 x^{\mu'}}{\delta x^{\nu}\delta x^{\mu}} = 0 \qquad 0 = \int \frac{\delta^2 x^{\mu'}}{\delta x^{\nu}\delta x^{\mu}} x^{\mu} dx^{\nu}$$ integrate by part (see box next page)

- This implies a linear transfrmation of vectors between frames

$$\boxed{x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} + \cancel{a^{\mu'}}}$$

not relevant (origin shift)

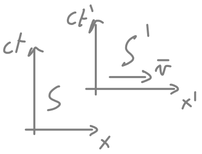
→ Preserves homogeneity ($\bar{x}' = \Lambda \bar{x}$ frameless)

→ " metric

→ i.e. Preserves the scalar product $\langle \bar{u}, \bar{v} \rangle = \langle \bar{u}', \bar{v}' \rangle$

- Here we want 'c' invariant ⇒ Lorentz transforms as we have seen in previous lectures

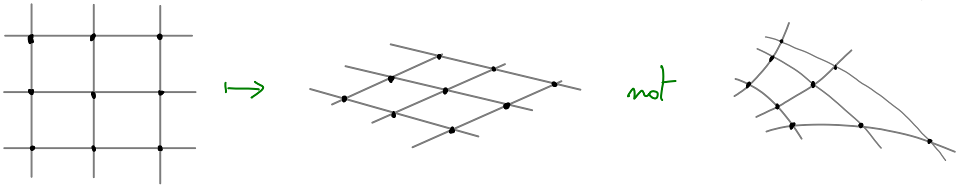
$$\left(\Lambda^{\mu'}{}_{\nu}\right) = \left(\frac{\delta x^{\mu'}}{\delta x^{\nu}}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ for "normal configuration"



$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix}$$

Another way to define the needed transformation

1) The 4-interval must be invariant (to preserve homogeneity)



$\Rightarrow$ Need a linear transformation and it actually is the Lorentz transf. which leaves the scalar product unchanged $\langle \bar{u}, \bar{v} \rangle = \langle \bar{u}', \bar{v}' \rangle$

$$\boxed{x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} + a^{\mu'}} \qquad \text{with } \Lambda^{\mu'}{}_{\nu} = \frac{\delta x^{\mu'}}{\delta x^{\nu}} \text{ transformation}$$

2) Λ must preserve the metric (η is the same in all frames)

(same η on booth sides as we used cartesian frames on booth sides

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2$$
)

in fact $d\bar{s}^2 = d\bar{s}'^2$ with $\eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'}$

$$d\bar{s}^2 = d\bar{x}^T \eta \, d\bar{x}$$
$$(!) \; = d\bar{x}'^T \eta \, d\bar{x}'$$
$$= d\bar{x}^T \Lambda^T \eta \Lambda d\bar{x}$$

$$\overset{(1)}{\boxed{\eta = \Lambda^T \eta \Lambda}} \leftrightarrow \overset{(1)}{\boxed{\eta_{\mu\nu} = \Lambda^{\alpha'}{}_{\mu} \Lambda^{\beta'}{}_{\nu} \eta_{\alpha'\beta'}}} \quad \text{or} \quad \overset{(1)}{\eta_{\alpha'\beta'} = \Lambda^{\mu}{}_{\alpha'} \Lambda^{\nu}{}_{\beta'} \eta_{\mu\nu}}$$

! the same $\Rightarrow$ Λ such that η invariant under such transf.

(1) It is the necessary and sufficient condition that a transformation $\Lambda^{\mu}{}_{\nu'} = \frac{\delta x^{\mu}}{\delta x^{\nu'}}$ (being $\eta \overset{!}{=} \eta'$) is a Lorentz transform

$$\textcircled{*}$$

$$0 = \int \frac{\delta^2 x^{\mu'}}{\delta x^{\nu} \delta x^{\mu}} x^{\mu} dx^{\nu} \overset{\text{int. by part}}{=} \frac{\delta x^{\mu'}}{\delta x^{\mu}} x^{\mu} - \int \overbrace{\underbrace{\frac{\delta x^{\mu'}}{\delta x^{\mu}}}_{\Lambda^{\mu'}{}_{\mu}} \underbrace{\frac{\delta x^{\mu}}{\delta x^{\nu}}}_{\Lambda^{\mu}{}_{\nu}}}^{\Lambda^{\mu'}{}_{\nu}} dx^{\nu} + a^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} - \overbrace{\int dx^{\mu'}}^{x^{\mu'}} + a^{\mu'}$$

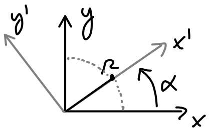
$$\Rightarrow x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} + a^{\mu'}$$

Lorentz invariant quantities

Why are there quantities that do not change under coordinate transformations?

A familiar example: rotation in 3D

i.e. same values ∀ frame



e.g. in 2D $r^2 = x^2 + y^2 = \delta_{ij} x^i x^j$ Invariant under rotations

↳ corresponds to the statement $\cos^2\alpha + \sin^2\alpha = 1 \quad \forall \alpha \in \mathbb{R}$

$= (r\cos\alpha)^2 + (r\sin\alpha)^2 \qquad r^2 \geq 0$

Likewise, for Lorentz transforms: define a "rotation angle" $\alpha \to \psi$

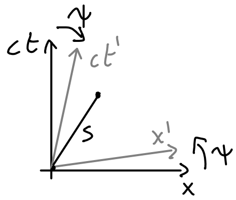
ψ = rapidity parameter

$\cosh\psi \equiv \gamma \qquad \in [1,\infty]$

$\sinh\psi \equiv \beta\gamma \qquad \in [-\infty,\infty]$

$\tanh\psi = \frac{\beta\gamma}{\gamma} = \beta$

$$\underline{\cosh^2\psi - \sinh^2\psi} = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = \underline{1}$$



$$s^2 = -c^2dt^2 + dx^2$$
$$= \eta_{\mu\nu} x^\mu x^\nu$$
$$= -(s\sinh\psi)^2 + (s\cosh\psi)^2 = s^2 \underbrace{(\cosh^2\psi - \sinh^2\psi)}_{=1}$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Lorentz transformations ~ hyperbolic rotation : boost

- Λ form the so called Lorentz group (see next page)
  the invariance is an intrinsic property of this group
  not just a property of sinh, cosh

The Lorentz group

- What is a groups?

Set of elements $\{G_i\}$ with a connection $*$ between elements $(G,*)$ with these properties:

$*$	connection between elements
$a*b=c$	addition, $\forall a,b\in G \Rightarrow c\in G$
$(a*b)*c=a*(b*c)$	associativity, $\forall a,b,c\in G$
$e*a=a*e=a$	$\exists$ of identiy element e, $\forall a\in G$
$a*b=b*a=e$	$\exists$ of inverse, $\forall a,b\in G$
extra $a*b=b*a$	commutativity, $\forall a,b\in G$ $\Rightarrow$ Abelian group, e.g. $GL(n)$ is not abelian Rotations

: e.g. $(a\in\mathbb{R},\cdot)$ $1_G=1$
$(a\in\mathbb{R},+)$ $1_G=0$
$(a\in\mathbb{R}^n\times\mathbb{R}^n,\cdot)$ $1_G=I^n$

- What is a Lie group?

Continuous group $\{G_i(\alpha_1,\dots,\alpha_m)\}$ i.e Elements are "functions" of α_j continuous parameters
Have finite dimensional differentiable (i.e. smooth) manifold

- What is the Lorentz group?

A non Abelian Lie group which elements are Lorentz transformations

$$SO(3,1,\mathbb{R})=\{\Lambda\in\mathcal{M}(4,\mathbb{R})\,|\,<\bar{u},\bar{v}>=<\overset{\bar{u}'}{\Lambda\bar{u}},\overset{\bar{v}'}{\Lambda\bar{v}}>\ \forall\bar{u},\bar{v}\in\mathbb{M}\}$$

↑ orthogonal $\Lambda^T\Lambda=I_4$ ↑ 4x4 real matrixes

↑ Special : $\det(\Lambda)=1$

- Continuous group $\{\Lambda\}$
- Elements (Λ) are "functions" of continuous parameters : v or ψ
- Finite dimensional differentiable (i.e. smooth) manifold

} Lie group

- $\dim O(n)=\frac{n(n-1)}{2}$ $n=4$
  - 6 free parameters (1) : 3 boosts $(x^i, x^0$ planes$)$, 3 rotations $(x^i, x^j$ $i\neq j$ plans$)$
  - 4 translations (2) : shift in time and time

$$\overset{(1)}{\downarrow}\qquad\overset{(2)}{\downarrow}$$
$$x^{\mu'}=\Lambda^{\mu'}{}_{\nu}x^{\nu}+a^{\mu'}$$

$\Rightarrow$ 10 parameters (Poincaré group)

- Non abelian group (Λ along different directions do not commute)

Inverse transformation

$(v \to -v)$ $x^{\mu'} \to x^{\mu}$

$\bar{x} \to \bar{x}'$: $x^{\mu'} = \Lambda^{\mu'}{}_{\alpha} x^{\alpha}$

$\bar{x}' \to \bar{x}$: ? look for it: $x^{\mu} = \Lambda^{\mu}{}_{\alpha'} \Lambda^{\alpha'}{}_{\nu} x^{\nu} \Rightarrow \Lambda^{\mu}{}_{\alpha'} \Lambda^{\alpha'}{}_{\nu} = \delta^{\mu}{}_{\nu}$

$$\Lambda_{\mu'}{}^{\nu} = \frac{\delta x^{\nu}}{\delta x^{\mu'}}$$

$\Rightarrow (\Lambda^{\mu}{}_{\alpha'})$ is the inverse of $(\Lambda^{\alpha'}{}_{\nu})$

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$x^{\mu'} = \Lambda^{\mu'}{}_{\alpha} x^{\alpha}$ $\quad \underbrace{\Lambda^{\beta}{}_{\mu'} x^{\mu'}}_{= x^{\alpha}} = \Lambda^{\beta}{}_{\mu'} \Lambda^{\mu'}{}_{\alpha} x^{\alpha} \Rightarrow \Lambda^{\beta}{}_{\mu'} \Lambda^{\mu'}{}_{\alpha} = \delta^{\beta}_{\alpha}$?

$\Lambda^{T} \Lambda = I_4$

$(\Lambda^{-1})^{\nu'}{}_{\mu} \equiv \Lambda_{\nu'}{}^{\mu} \Rightarrow \Lambda_{\nu'}{}^{\alpha} \Lambda^{\mu'}{}_{\alpha} = \delta^{\mu'}_{\nu'}$ or $\Lambda_{\alpha'}{}^{\mu} \Lambda^{\alpha'}{}_{\nu} = \delta^{\mu}_{\nu}$

comol

Lie-groups

- Continuous groups $\{X(\alpha_1, \dots, \alpha_m)\}$
- Elements are "functions" of α_i continuous parameters
- Have finite dimensional differentiable (i.e. smooth) manifold

e.g. collection of phase factors $e^{i\alpha}$, α = phase $U(1) = \{e^{i\alpha}; \alpha \in \mathbb{R}\}$

" " rotations $R(\alpha)$ angle α (in a plane), $R(\alpha, \beta, \gamma)$ in 3D

" " boosts $\Lambda(\psi)$ rapidity ψ (in a plane)

Lie-algebra

- Define an element of the group with a set of "basis" : generators of the group M_k
- Elements of a Lie group X can be written as

$$X = \exp\left(i \sum \alpha_k M_k\right) \qquad \alpha_k \in \mathbb{R} \text{ parameters} \qquad k = 1, \dots, r$$

$$= e^{i\alpha_1 M_1} \dots e^{i\alpha_r M_r} \qquad M_k = -i\left(\frac{\delta X}{\delta \alpha_k}\right)_{\{\alpha_k = 0\}} \text{ generator}$$

$\uparrow$

X satisfies a Lie algebra if $[M_k, M_\ell] = i f_{k\ell m} M_m$ ($[a,b] \equiv ab - ba$ commutator)

$f_{k\ell m}$ = structure constant of the group (completely anti-symmetric on $k\ell m$)

it defines the connection between the elements X

- if set of generators $\{M_k\}$ $M_k \rightarrow M'_k = S M_k S'$ S $n \times n$ invertible matrix

- We have $X = e^Y \Rightarrow \det X = e^{\mathrm{Tr} Y}$

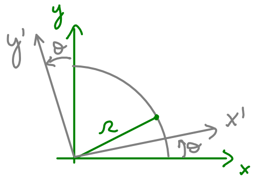
for $\mathrm{Tr} Y = 0 \Rightarrow \det X = 1$ you have conserved quantities

$\uparrow$

$\mathrm{Tr}\left(\sum_k \alpha_k M_k\right) = 0$

eg. $X = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$ Rotations

$\det(X) = \cos^2\vartheta + \sin^2\vartheta = 1$



$r = r'$

eg. Lorentz boosts Λ : $ds^2 = ds'^2$

The Lorentz group is a Lie group

- Here, consider rotation in the x^0, x^1 plane $\Rightarrow$ only 1 parameter $\alpha = \psi$

$$X = \exp\left(i \sum \alpha_K M_K\right) \qquad K=1 \quad \alpha_1 = \psi \quad M_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\sigma^{(3)}$$

as generators we can use a Pauli matrix

$$\Lambda = \exp\left(\psi \sigma^{(3)}\right)$$

Taylor expansion

$$= \sum_n \frac{1}{n!} \left(\psi \sigma^{(3)}\right)^n$$

$$\left.\begin{aligned} \hookrightarrow \left(\sigma^{(3)}\right)^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^{(0)} \\ \left(\sigma^{(3)}\right)^1 &= \sigma^{(3)} \\ \left(\sigma^{(3)}\right)^2 &= \sigma^{(3)}\sigma^{(3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^{(0)} \\ \left(\sigma^{(3)}\right)^3 &= \sigma^{(0)}\sigma^{(3)} = \sigma^{(3)} \\ &\cdots\cdots \end{aligned}\right\} \Rightarrow \begin{array}{lll} n=0 & \text{even} : & \sigma^{(0)} \\ n=1 & \text{odd} : & \sigma^{(3)} \\ n=2 & \text{even} : & \sigma^{(0)} \\ n=3 & \text{odd} : & \sigma^{(3)} \end{array}$$

$$= \sigma^{(0)} \sum_n \frac{1}{(2n)!} \psi^{2n} + \sigma^{(3)} \sum_n \frac{1}{(2n+1)!} \psi^{2n+1}$$

Taylor expansion of sinh, cosh

$$= \sigma^{(0)} \cosh(\psi) + \sigma^{(3)} \sinh(\psi)$$

$$= \begin{pmatrix} \cosh(\psi) & \sinh(\psi) \\ \sinh(\psi) & \cosh(\psi) \end{pmatrix}$$

for $x^{\mu'} \to x^{\mu}$; inverse transf. $x^{\mu} \to x^{\mu'}$ $\Rightarrow \psi \to -\psi$ $\Lambda = \begin{pmatrix} \cosh(\psi) & -\sinh(\psi) \\ -\sinh(\psi) & \cosh(\psi) \end{pmatrix}$

- we could have been using $\sigma^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as well (i.e. there are different possibilities for M_K)

Invariants under boosts (Lorentz invariance)

- $\Lambda = e^{(\psi\sigma^{(3)})}$: $\det\Lambda = e^{Tr(\psi\sigma^{(3)})} = e^0 = 1 \Rightarrow \boxed{\det\Lambda = 1}$

because $\sigma^{(3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow$ It provides invariant quantities $ds^2 = ds'^2$!

$\Rightarrow$ This is a property of the group rather then of sinh, cosh

$\det(\Lambda) = \cosh^2\psi - \sinh^2\psi = 1$ as we have seen before

Combination of boosts in one plane

- here the structure constant of the group $f = \varepsilon$ Levi-Civita symbol

$$\Lambda(\phi)\Lambda(\psi) = \exp(\phi\sigma^{(3)})\exp(\psi\sigma^{(3)}) = \exp[(\phi+\psi)\sigma^{(3)}] = \Lambda(\phi+\psi) \quad (1)\ (2)$$

$$= \Lambda(\psi+\phi) = \Lambda(\psi)\Lambda(\phi) \quad (3)$$

$\Rightarrow$

(1) Two boosts in the same plane is given by one boost with $\psi+\phi=\psi_{tot}$ (additive in ψ not v!)
(as in a Galilean transformation with velocities $v_1+v_2=v_{Tot}$)

(2) $\exists$ of inverse Λ^{-1}: $\Lambda(\psi)\Lambda(-\psi)=\Lambda(0)=I$ $\quad \Lambda(\psi)^{-1}=\Lambda(-\psi)$

(3) In one plane, boosts Λ are commutative $[\Lambda(\phi),\Lambda(\psi)]=0$

Λ is orthogonal with respect to η not δ!

Conbination of boosts in different planes

- A,B basis generators elements generating boosts around 2 ≠ axis

1° apply A then apply B, i.e. $\exp(A)\exp(B) \neq \exp(A+B)$ because

→ Final boost given by Baker-Hausdorf-Campbell formula

$\exp(A)\exp(B) = \exp(A+B)\exp\left(-\frac{1}{2}[A,B]\right)$ $\quad [A,B] \equiv AB-BA \neq 0$ As from the f of the group these generators do not commute

Rotation group $O(3, K)$

$K \to K$
$I \to L$

$$\boxed{R_{\bar{m}}(\delta\phi) = I_{d_m} + \delta\phi\, i\, \bar{m}\bar{J} + O((\delta\phi)^2)}$$

infinitesimal rotation around $\bar{m}$ (unity vec)

$\bar{J}$ = infinitesimal generators

$$\boxed{J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \Rightarrow (J_k)_{\ell m} = -i\varepsilon_{k\ell m} \quad \text{Generators}$$

↳ Rotations around the 3 axes

$$[J_k, J_\ell] = i\varepsilon_{k\ell m} J_m$$

algebra for (e.g.) angular momentum $\hbar\bar{J}$

$$= \varepsilon_{k\ell 1} J_1 + \varepsilon_{k\ell 2} J_2 + \varepsilon_{k\ell 3} J_3$$

$$\rightarrow [J_k, J_\ell] = J_k J_\ell - J_\ell J_k$$

$$\Rightarrow R(\phi) = \exp(i\phi\bar{m}\bar{J}) = \sum_{k=0}^{\infty} \frac{1}{k!}(i\phi\bar{m}\bar{J})^k = I_d + i\phi\bar{m}\bar{J} - \frac{(\phi\bar{m}\bar{J})^2}{2} + i\frac{1}{3}\overbrace{(\phi\bar{m}\bar{J})^3}^{\phi\bar{m}\bar{J}} \dots \quad ?$$

$$= I_d + i\sin(\theta)\,\bar{m}\bar{J} + (\cos\phi - 1)(\bar{m}\cdot\bar{J})^2$$

- You could transform J_1, J_2, J_3 as $J_i^s = S J_i S^{-1}$ to have spherical coordinates like in quantum mechanics $J_1^s = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $J_2^s = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $J_3^s = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Lorentz group

$$R_{\bar{e}}(\delta\chi) = I_d + \delta\chi\, i\, \bar{e}\bar{K} + O((\delta\chi)^2)$$

Has the generators of the spatial rotations $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{J_k} & \\ 0 & & & \end{pmatrix}$

mixed t,x terms →

$$K_x = \begin{pmatrix} 0 & \boxed{i} & 0 & 0 \\ \boxed{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Generators for the boost

↑ pure space terms (J_i)

$$[J_k, J_\ell] = i\varepsilon_{k\ell m} J_m \quad [J_k, K_\ell] = i\varepsilon_{k\ell m} K_m \quad [K_k, K_\ell] = -i\varepsilon_{k\ell m} J_m$$

Lie algebra for the Lorentz group

$$\Rightarrow \boxed{L(\chi) = \exp(i\chi\bar{e}\bar{K}) = I_d + i\sinh(\chi)\,\bar{e}\bar{K} + (\cosh(\chi) - 1)(\bar{e}\bar{K})^2}$$

Pure boost along $\bar{e}$

$$\cosh(\chi) = \gamma^{-1} = \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

Part II

Neutral particles

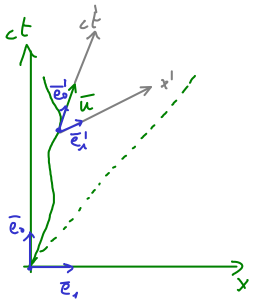
Relativistic mechanics

- Behavejour of paticles in the space-time

- 4-vector : $(x^\mu) = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$ not invariant

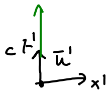
- World-line : $\boxed{x^\mu(\tau)}$ trajectory of particle in space-time τ = proper time

- 4-velocity : $\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt}}$ $(u^\mu) = \gamma\left(c\frac{dt}{dt}, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right) = \gamma(c, \bar v) = c\gamma(1, \bar\beta)$



  - rate at which coordinates pass by an observer
  - not invariant
  - $\langle \bar u, \bar u\rangle = u_\mu u^\mu = \eta_{\mu\nu}u^\nu u^\mu = c^2\gamma^2\overbrace{(-1+\bar\beta^2)}^{-\gamma^{-2}} = -c^2 < 0$ invariant

    (this is because of ds^2: $-c^2d\tau = dx_\mu dx^\mu \Rightarrow -c^2 = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = u_\mu u^\mu$)
  - time-like $\Rightarrow$ particles can only move inside the light cone
  - $\bar u$ is tangent to the world line because it is a multiple of $d\bar x$: $\bar u = \frac{d\bar x}{d\tau}$ ↖ scalar
  - In particle rest frame $\bar u' = (c, 0, 0, 0)^T \Rightarrow \bar u' = c\,\bar e_0$



- 4-acceleration : $\boxed{a^\mu \equiv \frac{du^\mu}{d\tau}}$ $(a^\mu) = \frac{d}{d\tau}[\gamma(c, \bar v)] = \gamma\frac{d}{dt}[\overset{\gamma(v)}{\downarrow}\gamma(c, \bar v)] = \gamma(\dot\gamma c, \dot\gamma\bar v + \gamma\dot{\bar v})$

  - note: $\frac{d}{d\tau}(\underbrace{u^\mu u_\mu}_{-c^2}) = \frac{du^\mu}{d\tau}u_\mu + u^\mu\frac{du_\mu}{d\tau} = \frac{du^\mu}{d\tau}u_\mu + u_\mu\frac{du^\mu}{d\tau} = 2u_\mu a^\mu = 0$

    $\Rightarrow \bar u \perp \bar a$ always! (orthogonal in 4-D)
  - In istantaneous rest frame : $\gamma = 1$ $\dot\gamma = 0$ $(a^\mu) = (0, \bar a)$

- 4-momentum : $\boxed{p^\mu \equiv m_0 u^\mu}$ $(p^\mu) = \gamma(\underset{(2)}{m_0 c}, \underset{(1)}{m_0\bar v})$ $\Rightarrow$ $\bar p = \gamma m_0 \bar v$ $\frac{E}{c} = \gamma m_0 c$ ↳ $\bar p = \frac{E}{c^2}\bar v$ ↲

  m_0 = rest mass

  (1) $\bar p = \gamma m_0 \bar v$

  (2) $\gamma m_0 c = m_0 c(1-\bar\beta^2)^{-1/2} \underset{\beta \ll 1}{\simeq} m_0 c\left(1 + \frac{1}{2}\bar\beta^2 + \dots\right) = m_0 c + \frac{1}{2}m_0\cancel{c}\frac{\bar v^2}{c^{\cancel{2}}} = \frac{1}{c}\left(m_0c^2 + \frac{1}{2}m_0\bar v^2\right) = \frac{E}{c}$

  rest energy ⇐ kinetic energy

$\Rightarrow (p^\mu) = \gamma\left(\frac{E_0}{c}, m_0\bar v\right) = \left(\frac{E}{c}, \bar P\right)$ ← Energy-momentum (one entity!)

↓

- $\langle P, P\rangle = -\frac{E^2}{c^2} + \bar P^2$

  $\langle P, P\rangle = P_\mu P^\mu = m_0^2 u_\mu u^\mu = -m_0^2c^2$

  ↑ ↑ ↓ invariant.

  $\Bigg\}$ $-\frac{E^2}{c^2} + \bar P^2 = -m_0^2c^2$ $\boxed{E^2 = m_0^2c^4 + c^2\bar P^2}$ at rest $E = m_0c^2$

  $\partial_\mu P^\mu = 0$

- Energy and momentum are components of the same entity

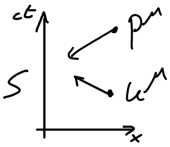
  $\Rightarrow$ not any more energy conservation + momentum conservation BUT energy-momentum conservation!

- Massive particles have 0<v<c

$$P_\mu P^\mu = -P_0^2 + P_1^2 = -m_0^2c^2 \Rightarrow P_1 = (P_0 - m_0^2c^2)^{1/2} \quad \frac{P_1}{P_0} = \left(1 - \frac{m_0^2c^2}{P_0^2}\right)^{1/2} \to 1 \text{ (but not 1!) for } P_0 = \frac{E}{c} \to \infty$$

⇒ No matter how much energy we give to the particle, $v < c$!

- A convenient expression: energy of a particle measured by a moving observer



P^μ of a particle in frame S (eg. a cosmic ray)

u^μ 4-velocity of an observer moving in S

$$\Rightarrow \boxed{E' = -P_\mu u^\mu}$$

$S: u = \gamma(c, \bar{v})^T \quad P = (\frac{E}{c}, \bar{P})^T$ (lab)

$S': u = (c, \bar{0})^T \quad E' = ?$ (moving observer)

$$P_\mu u^\mu \underset{\uparrow}{=} P'_\mu u'^\mu = \left(\frac{E'}{c}, \bar{P}'\right)\begin{pmatrix} c \\ \bar{0} \end{pmatrix} = -\frac{E'}{\cancel{c}} \cdot \cancel{c} + \cancel{P'0_i} = -E' \quad ✓$$

scalarproduct is the same in all frames

- 4-force:

$$f^\mu \equiv \frac{dP^\mu}{d\tau} \quad (f^\mu) = \gamma \frac{d}{dt}\left(\frac{E}{c}, \bar{P}\right) = \gamma\left(\frac{\bar{f}\bar{v}}{c}, \bar{f}\right)$$

① $dE = \bar{f}\,d\bar{x} = \bar{f}\,\bar{v}\,dt \Rightarrow \dot{E} = \bar{f}\bar{v}$ work

② $\dot{\bar{P}} = \bar{f}$ 3-force

remember: a force is a change in momentum not $\bar{f} = m\bar{a}$

$$\bar{f} = \frac{d\bar{P}}{dt} = \frac{d}{dt}(m\bar{v}) \underset{\substack{\uparrow \\ m = \text{const.}}}{=} m\dot{\bar{v}}$$

- Change in rest mass ?!

product rule (let m free)

$$u^\mu f_\mu = u^\mu \frac{dP_\mu}{d\tau} = u^\mu \frac{d}{d\tau}(m u_\mu) \overset{\downarrow}{=} u^\mu\left(\frac{dm}{d\tau}u_\mu + m\frac{du_\mu}{d\tau}\right) = \underbrace{u^\mu u_\mu}_{-c^2}\frac{dm}{d\tau} + m\underbrace{u^\mu a_\mu}_{=0} = -c^2\frac{dm}{d\tau}$$

⇒ A force may change rest mass $\boxed{\frac{dm}{d\tau} = -\frac{u^\mu f_\mu}{c^2}}$ if $u^\mu f_\mu \neq 0$!

- If $u^\mu f_\mu = 0 \Rightarrow \frac{dm}{d\tau} = 0$ f^μ called pure force $\gamma\left(\frac{\bar{f}\bar{v}}{c}, \bar{f}\right)$

↓

acts on 3-acceleration but not on rest energy

Variational approach

- Free particle : no external forces

- Action : $S \in \mathbb{R}$ must be Lorentz invariant (not dependent on motion of observer)

  $\Rightarrow$ S must depend on Lorentz scalar: τ is the only one characterizing the particle

$$S = \int L\,dt = \alpha\int d\tau = \alpha\int \gamma^{-1}dt$$

$\beta << 1$ no external forces

$$L = \alpha\gamma^{-1} = \alpha\left(1-\bar\beta^2\right)^{1/2} = \alpha\left(1-\frac{1}{2}\bar\beta^2+\dots\right) \simeq \alpha - \frac{1}{2}\alpha\frac{\bar v^2}{c^2} \overset{!}{=} T + \cancel{V} = \frac{1}{2}m\bar v^2 \Rightarrow \alpha = -m_0c^2$$

$$\boxed{L = -m_0c^2\gamma^{-1} = -m_0c^2\left(1-\bar\beta^2\right)^{1/2}} \leftarrow$$ Lagrangian of a free particle

- Note: S has a clear physical meaning in S.R. : interval of proper time along world line

  $L \to L' = aL + b$ gives same eq. of motion $(\delta S = 0)$ a = time rescaling, b = time shift

- The metric is hidden in there!

$$-c^2d\tau^2 = ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \Rightarrow d\tau = \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} \qquad S = -m_0c\int\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$$

Very relevant in G.R.! $\eta \to g$

- Conjugate momentum :

$$\bar p = \frac{\delta L}{\delta \bar v} = \cancel{-}m_0\cancel{c^2}\frac{1}{\cancel{2}}\left(1-\bar\beta^2\right)^{-1/2}\cancel{(-)}\frac{1}{\cancel{c^2}}\cancel{2}\bar v = \gamma m_0\bar v \qquad \boxed{\bar p = \gamma m_0\bar v}$$ as seen before

- Hamiltonian :

$$H = \bar v\bar p - L = m_0\gamma\bar v^2 + m_0c^2\gamma^{-1} = \gamma m_0c^2\left(\bar\beta^2 + \overset{(1-\bar\beta^2)}{\gamma^{-2}}\right) = \gamma m_0c^2 \qquad \boxed{E = \gamma m_0c^2}$$

This confirms our interpretation of the 4-momentum components $\left(p^\mu\right) = \left(\frac{E}{c}, \bar p\right)$

- Equation of motion : (Euler-Lagrange eq.)

$$\frac{d}{dt}\frac{\delta L}{\delta \bar v} - \frac{\delta L}{\delta \bar x} = \frac{d}{dt}\left(\overset{\gamma(v)}{\gamma} m_0\bar v\right) = m_0\left[\dot\gamma\bar v + \gamma\dot{\bar v}\right] \qquad \dot\gamma = \cancel{+}\frac{1}{\cancel{2}}\left(1-\bar\beta^2\right)^{-3/2}(-\cancel{2})\bar\beta\dot{\bar\beta} = \gamma^3\frac{\bar v\dot{\bar v}}{c^2}$$

$$= m_0\left(\gamma^3\frac{\bar v\dot{\bar v}}{c^2}\cdot\bar v + \gamma\dot{\bar v}\right)$$

$$= m_0\dot{\bar v}\underbrace{\left(\gamma^3\frac{\bar v^2}{c^2} + \gamma\right)}_{>0} \overset{!}{=} 0 \Rightarrow \boxed{\dot{\bar v} = 0}$$ no acceleration as expected

$\bar x = \bar a t + \bar b$ move along straight line

(instantaneous rest frame $\bar v = 0$)

– Eq. of motion but directly from least action principle

$$\delta S = -m_0 c^2 \delta\int_A^B d\tau = -m_0 c^2 \delta \int_A^B \frac{1}{c}\left(-\eta_{\mu\nu}dx^\mu dx^\nu\right)^{\frac{1}{2}} = -m_0 c^2 \int_A^B \frac{1}{\cancel{2}c}\overbrace{\left(-\eta_{\mu\nu}dx^\mu dx^\nu\right)^{-1/2}}^{c^{-1}}(-\eta_{\mu\nu})(\cancel{2}\, dx^\mu \overset{\leftrightarrow}{\delta d} x^\nu)$$

$$= +m_0\int_A^B \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}d\tau \overset{\text{by part}}{\underset{\downarrow}{=}} m_0\left[\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\Big|_A^B - \int_A^B \eta_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\delta x^\nu d\tau\right]$$

$$\underset{=0}{}$$

$$= -m_0\int_A^B \eta_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\delta x^\nu d\tau \overset{!}{=} 0 \quad \forall\ \delta x^\nu \text{ and } \eta_{\mu\nu} = \text{const}$$

$$\Rightarrow \boxed{\frac{d^2x^\mu}{d\tau^2} = 0 \quad \rightarrow \quad x^\mu = a^\mu\tau + b^\mu} \Rightarrow \begin{cases} \text{no acceleration, motion along a straight line} \\ \text{no change in particle' energy} \quad m\frac{du^0}{d\tau} = \frac{dp^0}{d\tau} = 0 \end{cases}$$

note: here $\eta_{\mu\nu}$ = const !
not the case in G.R ! You will see how important this is!

Same... but slightly different

$$S = -mc^2\int d\tau = -mc^{\cancel{2}}\int \frac{1}{\cancel{c}}\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2}d\tau = -mc\int\left(-\eta_{\mu\nu}u^\mu u^\nu\right)^{1/2}d\tau$$

$$\delta S = +m\cancel{c}\int_A^B \frac{1}{\cancel{2}}\cancel{\overbrace{(-u_\nu u^\nu)^{-1/2}}^{c^{-1}}}(+)\cancel{2}u_\nu\delta u^\nu d\tau = m\int_A^B u_\nu \frac{\overset{\leftrightarrow}{\delta d}x^\nu}{d\tau}d\tau = m\int_A^B u_\nu\frac{d\delta x^\nu}{d\tau}d\tau \leftarrow \text{by part}$$

$$= m\left[\underset{=0}{u_\nu\delta x^\nu\Big|_A^B} - \int_A^B \frac{du_\nu}{d\tau}\delta x^\nu d\tau\right] \overset{!}{=} 0 \quad \forall\ \delta x^\nu \Rightarrow \text{4-velocity} \quad \boxed{a^\nu = 0}$$

– Dispersion relation for massive particles

group and phase velocities can not be the same for massive particles

$$v_{ph} = \frac{H}{p} \neq \frac{dH}{dp} = v_{gr} \quad \text{but} \quad \boxed{v_{ph}v_{gr}} = \frac{H}{p}\frac{dH}{dp} = \frac{cp}{\sqrt{1+p^2}}\ \frac{c\sqrt{1+p^2}}{p} = \boxed{c^2}$$

i.e. geometric mean of v_{ph} and v_{gr} is c ⇒ $v_{ph} > c$ if $v_{gr} < c$

$$\frac{H}{p}\frac{dH}{dp} = \frac{d(H^2)}{d(p^2)} = c^2 \quad \text{integrate} \quad d(H^2) = c^2 d(p^2) \Rightarrow H^2 = c^2p^2 + \underline{\underline{\text{const}}}$$

↑
rest mass as
integration constant.

Decay of particles

Example: body of mass M splits in two parts of mass m_1, m_2

$\Rightarrow M = m_1 + m_2$? NO!

- In rest frame of body : $E = Mc^2$
- Energy conservation : $Mc^2 = E_1 + E_2$ (1) split $\Rightarrow m_1, m_2$ move apart $\Rightarrow |\bar{p}_i| > 0$

  $\Downarrow$ <— $\Rightarrow E_i^2 = m_i^2 c^4 + c^2 \bar{p}_i^2 > m_i^2 c^4$

  $M > m_1 + m_2$!

  Spontaneous decay is possible

- if $M < m_1 + m_2$ $\Rightarrow$ decay is not possible (system is stable)

  $\Rightarrow$ no spontaneous decay (or energy conservation would be violated)

  $\Rightarrow$ to have decay, you need to give energy to the body of an amount $\Delta E = m_1 + m_2 - M$

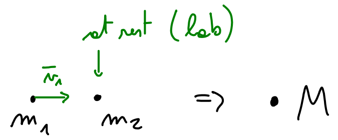
- Momentum conservation : $0 = \bar{p}_1 + \bar{p}_2$ $\Rightarrow$ $\bar{p}_1^2 = \bar{p}_2^2$ $\quad c^2 p_i^2 = E_i^2 - m_i^2 c^4$

  energies of the components E_1, E_2

  $E_1^2 - m_1^2 c^4 = E_2^2 - m_2^2 c^4$ ← plug (1)

  $\Rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 \qquad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2$

Inverse process



- $E = E_1 + E_2 = E_1 + m_2 c^2$
- $\bar{P} = \bar{p}_1 + \bar{p}_2 = \bar{p}_1$

$\Rightarrow$ Final velocity: $\bar{P} = \frac{E \bar{v}}{c^2} \Rightarrow \bar{v} = \frac{c^2 \bar{P}}{E} = \frac{c^2 \bar{p}_1}{E_1 + m_2 c^2}$ velocity of the merged body in the lab frame

$\Rightarrow$ Final mass M :

$$E^2 = M^2 c^4 + c^2 \bar{P}^2$$

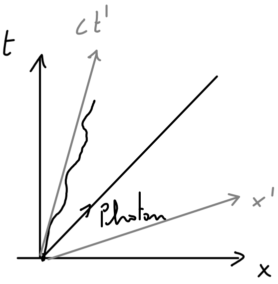
$$c^4 M^2 = E^2 - c^2 \bar{P}^2$$

$$= (E_1 + m_2 c^2)^2 - (E_1^2 - m_1^2 c^4)$$

$$= \cancel{E_1^2} + m_2^2 c^4 + 2 E_1 m_2 c^2 - \cancel{E_1^2} + m_1^2 c^4$$

$$= c^4 \left(m_1^2 + m_2^2 + \frac{2 E_1 m_1}{c^2} \right) \Rightarrow \boxed{M^2 = m_1^2 + m_2^2 + \frac{2 E_1 m_1}{c^2}}$$

Photons



- Photons travel with the speed of light

$dx = c\,dt \qquad 0 = -c^2 dt^2 + dx^2 \Rightarrow ds^2 = 0$

null 4-interval, null geodesic $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0$

- **For massive particles** the proper time is not identically zero: $ds^2 < 0$ (i.e. $d\tau^2 \neq 0$!)

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} d\tau^2 = \eta_{\mu\nu} u^\mu u^\nu d\tau^2 < 0}$$ (interval along particle world-line $\bar{x}(\tau)$)

direction of motion given by particles is given by u^μ can be parameterized by τ } $dx^\mu = u^\mu d\tau$ (shift)

Because of that we used $p^\mu = m u^\mu$

- **For photons** $ds^2 = 0 \Rightarrow d\tau^2 = -\frac{ds^2}{c^2} = 0$ proper time on the light-cone is identically zero

$\Rightarrow u^\mu = \frac{dx^\mu}{d\tau}$ 4-velocity is not well defined!

we must use another affine parameter (not τ)

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} d\lambda^2 = \eta_{\mu\nu} K^\mu K^\nu d\lambda^2}$$

$K^\mu \equiv \frac{dx^\mu}{d\lambda}$ wave vector (sort of 4-velocity for photons)

$K_\mu K^\mu = 0$ null-type because it has to be along the light-cone

$$= -(K^0)^2 + \underbrace{(K^1)^2 + (K^2)^2 + (K^3)^2} = -(K^0)^2 + |\bar{K}|^2 = 0 \Rightarrow K^0 = \|\bar{K}\| \qquad \boxed{(K^\mu) = K^0(1, \bar{e})}$$

- we can use K^μ to define the momentum of photons

$(p^\mu) = \left(\frac{E}{c}, \bar{P}\right)$ we can use this because we know photons have an E and a $\bar{P}$ use same construction of massive particles

$$= \left(\frac{E}{c}, P\bar{e}\right) = \frac{E}{c}(1, \bar{e}) = \frac{\hbar\omega}{c}(1, \bar{e}) = \hbar(K^\mu) \Rightarrow \boxed{(K^\mu) = \frac{\omega}{c}(1, \bar{e}) = \frac{2\pi}{\lambda}(1, \bar{e})} = \left(\frac{2\pi}{\lambda}, \bar{K}\right)$$

$(E = cP)$ $(E = h\nu = \hbar\omega)$ energy is quantized "4-frequency"

$$\boxed{P = \frac{h\nu}{c} = \frac{\hbar\omega}{c} = \frac{h}{\lambda}} \qquad \omega = \frac{\nu}{2\pi} \quad \lambda = \frac{c}{\nu}$$

$$\boxed{m = 0!} \qquad E^2 = \underset{=0}{\underset{\uparrow}{m^2c^4}} + c^2p^2$$

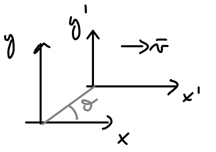
They must be massless!

- Momentum of a photon

moves along null lines => $p_\mu p^\mu = 0$ $\quad \bar{p} = (p^0, p^1, 0, 0)$ => $\frac{p^1}{p_0} = 1$ alternative way to state $v=c$

– Lorentz transform of 4-frequency :

- $(k^\mu) = \left(\frac{2\pi}{\lambda}, \vec{k}\right) = \frac{2\pi}{\lambda}(1, \cos\vartheta, \sin\vartheta, 0)$
- $(k^{\mu'}) = \left(\frac{2\pi}{\lambda'}, \vec{k}'\right) = \frac{2\pi}{\lambda'}(1, \cos\vartheta', \sin\vartheta', 0)$



$k^{\mu'} = \Lambda^{\mu'}{}_{\alpha} k^{\alpha}$ $\quad (\mu=0)$ time: $\frac{\lambda}{\lambda'} = \gamma(1-\beta\cos\vartheta)$ ← Doppler effect

$(\mu=i)$ space: $\tan\vartheta' = \frac{\tan\vartheta}{\gamma(1-\beta\sec\vartheta)}$ ← Aberration of light

$$\mu=0 \qquad k^0 = \Lambda^0{}_\alpha k^\alpha = \Lambda^0{}_0 k^0 + \Lambda^0{}_1 k^1 + \Lambda^0{}_2 k^2 + \Lambda^0{}_3 k^3$$

$$= \gamma k^0 - \beta\gamma k^1 + 0 + 0 = \frac{2\pi}{\lambda}(\gamma - \beta\gamma\cos\vartheta) \qquad \frac{2\pi}{\lambda'} = \frac{2\pi}{\lambda}\gamma(1-\beta\cos\vartheta)$$

Toward G.R. : linking gravity to the metric of space-time

- All objects fall with the same acceleration (if same initial conditions) even if ≠ mass and ≠ substance

inertial mass

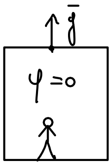
$\cancel{m_i}\,\bar{a} = -\cancel{m_G}\,\bar{\nabla}\varphi$ i.e. $m_i = m_G$ (eg. $m\bar{a} = -q\bar{E}$ $q \neq m$!)

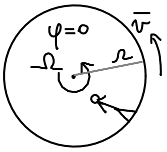
"gravitational charge"

$\Rightarrow$ $m_G = m_i$ is an inertial business!

- Idea: equivalence principle

a non inertial frame is equivalent to a gravitational field





locally equivalent to $\bar{g} = -\bar{\nabla}\varphi$
but physically very different

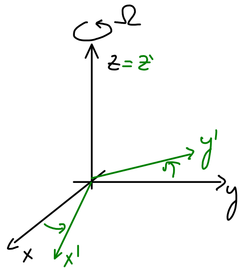
- Can we use this idea to construct a covariant theory of gravity?! Yes!

- eg. rotating frame

- weak field limit $\varphi \Longleftrightarrow g_{\mu\nu}$

Example: rotating system

- Equivalence principle : accelerated frame ↔ gravity
- Free particle : no field, i.e. no external force
- Inertial frame : $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$
- In a rotating system : $\begin{cases} x = x'\cos(\Omega t) - y'\sin(\Omega t) \\ y = x'\sin(\Omega t) + y'\cos(\Omega t) \\ z = z' \end{cases}$



$$dx = dx'\cos(\Omega t) + x'\sin(\Omega t)\Omega dt - dy'\sin(\Omega t) + y'\cos(\Omega t)\Omega dt$$
$$dy = dy'\sin(\Omega t) - x'\cos(\Omega t)\Omega dt + dy'\cos(\Omega t) + y'\sin(\Omega t)\Omega dt$$

$$ds^2 = -c^2dt^2 + \overset{\searrow}{d}x^2 + \overset{\searrow}{d}y^2 + \overset{\searrow}{d}z^2$$
$$= \big[c^2 - \Omega^2(x'^2 + y'^2)\big]\underset{\substack{\uparrow \\ \text{time} \\ g_{00}}}{dt^2} \underbrace{- dx' - dy' - dz'^2}_{\substack{\text{space} \\ g_{ii}}} \underbrace{+ 2\Omega y'dx'dt - 2\Omega x'dy'dt}_{\substack{\text{off-diagonal (mixed)} \\ g_{10}\ g_{01}\ g_{20}\ g_{02}}}$$
$$= g_{\mu\nu}\,dx^\mu dx^\nu$$

→ centrifugal force

→ locally as a gravitational field

} ↓ equivalence principle

⇒ Associate gravity to the metric of space-time ! $g_{\mu\nu} \neq \eta_{\mu\nu}$

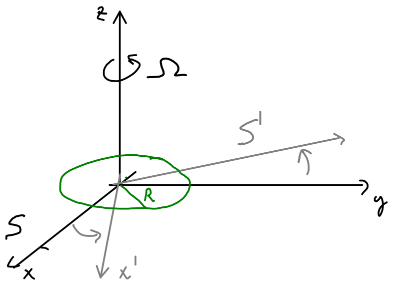
- Careful! Here the space is still Minkowski! (flat)

· $g_{\mu\nu} \neq \eta_{\mu\nu}$ only because of the "wierd" coord. system we used not because of an intrinsic property of space-time

· In fact non-locally, very different behavejor then with a grav. field

↳ in a grav. field : for $r \to \infty$ frome source of field $\bar{g} = -\bar{\nabla}\varphi \to 0$

↳ here $r \to \infty$ $\bar{g} \to \infty$!



Note:

measure the circonference of a circle in the x-y plane

S = inertial ($\Omega = 0$) $C = 2\pi R$

S = non-inertial ($\Omega \neq 0$) $C \neq 2\pi R$! ≠ geometry!

↑ because of Lorentz contraction

"curved space"

Connecting the metric to a gravitational field

- Let's try to find the metric associated to a gravitational field
- Take a non relativistic particle in a given fixed gravitational field

$L = -mc^2 + \frac{1}{2} m \bar{v}^2 - m\varphi$ note $m_i = m_G$! freedom to add a const $\varphi \to \varphi + A$, e.g. $A = -c^2$

$$S = \int L \, dt = -mc^2 \int \left(1 - \frac{\bar{v}^2}{2c^2} + \frac{\varphi}{c^2}\right) dt \overset{!}{=} -mc^2 \int d\tau$$

↑ Equivalence principle : free particle $S = -mc^2 \int d\tau$ (inertial mass)

but with some specific metric

$\eta_{\mu\nu} \to g_{\mu\nu}$!

$$d\tau = \frac{1}{c}\left(-g_{\mu\nu} dx^\mu dx^\nu\right)^{1/2} \overset{!}{=} \left(1 - \frac{\bar{v}^2}{2c^2} + \frac{\varphi}{c^2}\right) dt \simeq \left(1 - \frac{\bar{v}^2}{c^2} + \frac{2\varphi}{c^2}\right)^{1/2} dt$$

$\frac{\bar{v}^2}{2c^2} - \frac{\varphi}{c^2} \ll -1$ i.e. non relativistic particle in weak field

$$g_{\mu\nu} dx^\mu dx^\nu = -c^2\left(1 - \frac{\bar{v}^2}{c^2} + \frac{2\varphi}{c^2}\right) dt^2 = \left(1 + \frac{2\varphi}{c^2}\right) c^2 dt^2 + \frac{d\bar{x}^2}{\cancel{dt^2}} \cancel{dt^2}$$

← identify $g_{\mu\nu}$

$$\Rightarrow \boxed{g_{00} = 1 + \frac{2\varphi}{c^2}, \quad g_{0i} = 0 = g_{i0}, \quad g_{ij} = \delta_{ij}}$$

← no coordinat transformation that can make it η everywher simultaneously, you can only locally

small correction of $\eta_{00} = 1$

In reality you have a correction of the same order of g_{00}

$g_{ii} = 1 - \frac{2\varphi}{c^2}$ (coming from higher orders)

Without it you get the wrong gravitational lensing

- Limit of the approach:
  - we assumed a given and fixed φ (φ was not 'predicted / associated to a source)
  - not the correct value for g_{ii} terms

- But....
  - By assuming a free particle we can associate gravity to a curved space-time !!

    welcome to General Relativity !

units : $\left[\frac{G}{c}\right] = \frac{m}{Kg}$ ← geometry ← mass

G enters through the weak field limit

c " " relativity

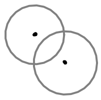
Part III

Charged particles - E.M. fields - Emission/absorption processes

Electrodynamics: main concepts

- It considers electro-magnetic fields and charged particles
- How we want it? 1) must satisfy Coulomb's law

  2) must satisfy superposition principle i.e Total field = $\sum_i$ fields of particles



  => Fields eq.s must be linear differential eq.s "linear theory"

  (sum of their solution are solutions as well)

  => Lagrangian of the field must be quadratic

  (you get field eq.s by $\partial S = 0$, "$\delta y^2 = 2y$" ← linear)

  3) Vased on 6 degrees of freedom: $\bar{E}, \bar{B} \in \mathbb{R}^3$ => 6 parameters

- $\bar{E}, \bar{B}$ depend on (t,x,y,z) => we need a covariant theory, i.e $(x^\mu) \in \mathbb{M}$

  => covariant theory => $\bar{E}, \bar{B}$ -> tensor

- Fields can be 4-vectors or tensors

  not enough for 6 numbers

  Rank-2 : ($\mathbb{M}$)

  16 components $\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}$ $F_{\mu\nu}$

  10 if symmetric $\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}$ $F_{\mu\nu} = F_{\nu\mu}$

  yes! => 6 if anti-symmetric $\begin{pmatrix}0&\cdot&\cdot&\cdot\\ &0&\cdot&\cdot\\ (-)& &0&\cdot\\ & & &0\end{pmatrix}$ $F_{\mu\nu} = -F_{\nu\mu}$

- Anti-symmetry is ensured by if F is expressed as derivatives of a 4-vector

  $(A^\mu) = (\phi, \bar{A})$ 4-potential : ϕ = scalar potential , $\bar{A}$ = vector potential

  $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ Field tensor! $(F^{\mu\nu}) = \begin{pmatrix} 0 & \bar{E} \\ -\bar{E} & \mathcal{B} \end{pmatrix}$ $(F_{\mu\nu}) = \begin{pmatrix} 0 & -\bar{E} \\ \bar{E} & \mathcal{B} \end{pmatrix}$

  $\bar{E} \equiv -\frac{1}{c}\dot{\bar{A}} - \bar{\nabla}\phi$

  $\bar{B} \equiv \bar{\nabla} \times \bar{A}$ $\mathcal{B}_{ij} \equiv \epsilon_{ija} B^a$ i=1,2,3 (spatial)

- Source of the field: $(j^\mu) \equiv (qc, \bar{j})$ q = charge $\bar{j} \equiv q\bar{v}$ = current

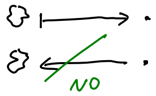
  ↑ 4-current

- Final goal: get field equations $\partial S_f = 0$ Maxwell's equations

  (eq. of motion of the field)

Particle in an electromagnetic field

- Use a test particle to investigate the field
- The field is a physical entity
- Consider effect of field on particle (field acting on particle)
  Neglect effect of particle on field
- Consider a fixed field, i.e. $\delta_0 A^\mu = 0$



- From mostly empirical consideration we define the action

$S = S_p + S_{pf}$ total action of particle in field

$S_p = -mc^2 \int d\tau$ isolated particle

S_{pf} : interaction particle-field

$(j^\mu) = (qc, q\bar{v}) = q\left(\frac{dx^\mu}{dt}\right)$

$$\boxed{S_{pf} = \overset{\text{charge}}{\frac{q}{c}} \int A_\mu dx^\mu} = \frac{1}{c}\int A_\mu \underbrace{q\frac{dx^\mu}{dt}}_{j^\mu} dt = \boxed{\frac{1}{c}\int A_\mu j^\mu dt} = \boxed{\frac{q}{c}\int A_\mu u^\mu d\tau}$$

convenience (units)

4-potential

$A(x^\mu)$ given at the time-space where the particle is, i.e. x^μ

- q is the only quantity (const => invariant) characterizing a charged particle
- A^μ does not belong to the particle
  it is the potential experienced by a particle which is assumed to exist there (fixed)
  => the fact that A^μ is not invariant is irrelevant

- <u>Meaning of ϕ and $\bar{A}$ look at the Lagrangian</u>

$$S = \int L\,dt = \int\left(-mc^2 d\tau + \frac{1}{c}A_\mu j^\mu dt\right) = \int\left(-mc^2\gamma^{-1} + \frac{1}{c}A_0 j^0 + \frac{1}{c}A_i j^i\right) dt$$

$j^0 = qc$
$j^i = qv^i$

$$= \int \underbrace{\left(-mc^2\gamma^{-1} - \phi q + \frac{1}{c}\bar{A}\,\bar{j}\right)}_{L} dt$$

Identifying the Electric and magnetic fields: eq. of motion (3D)

$$L = -mc^2\gamma^{-1} - \phi q + \frac{1}{c}\bar{A}\,\bar{j} \qquad \gamma^{-1} = \left(1-\frac{\bar{v}^2}{c^2}\right)^{1/2} \qquad \bar{j} = q\bar{v} \qquad \text{(split } A^0 \text{ from } A^i\text{)}$$

- Generalized momentum

$$\bar{P} = \frac{dL}{d\bar{v}} = +m\cancel{c^2}\,\frac{1}{\cancel{2}}\left(1-\frac{\bar{v}^2}{c^2}\right)^{-1/2}\cancel{(-)}\,\cancel{2}\,\frac{\bar{v}}{\cancel{c^2}} + \frac{1}{c}\bar{A}q = \underbrace{\gamma m\bar{v}}_{\bar{P}} + \frac{q}{c}\bar{A} = \bar{P} + \frac{q}{c}\bar{A}$$

- Equation of motion, Euler-Lagrange eq.s:

$$\frac{d}{dt}\overset{(1)}{\frac{\delta L}{\delta\bar{v}}} - \overset{(2)}{\frac{\delta L}{\delta\bar{x}}} = 0 \qquad \frac{d\bar{P}}{dt} + \frac{q}{c}\frac{\delta\bar{A}}{\delta t} + \cancel{\frac{q}{c}(\bar{v}\bar{\nabla})}\bar{A} + q\bar{\nabla}\phi - \cancel{\frac{q}{c}(\bar{v}\bar{\nabla})}\bar{A} - \frac{q}{c}\bar{v}\times(\bar{\nabla}\times\bar{A}) = 0$$

$$(1) = \frac{d\bar{P}}{dt} + \frac{q}{c}\frac{d\bar{A}}{dt} = \frac{d\bar{P}}{dt} + \frac{q}{c}\left(\frac{\delta\bar{A}}{\delta t} + \frac{\delta\bar{A}}{\delta\bar{x}}\frac{d\bar{x}}{dt}\right) = \frac{d\bar{P}}{dt} + \frac{q}{c}\frac{\delta\bar{A}}{\delta t} + \frac{q}{c}(\bar{v}\bar{\nabla})\bar{A}$$

$$(2) = -q\bar{\nabla}\phi + \frac{q}{c}\bar{\nabla}(\bar{A}\bar{v}) = -q\bar{\nabla}\phi + \frac{q}{c}(\bar{v}\bar{\nabla})\bar{A} + \frac{q}{c}\bar{v}\times(\bar{\nabla}\times\bar{A})$$

$$\uparrow$$

$$= \underbrace{\cancel{(\bar{A}\bar{\nabla})\bar{v}}}_{=0} + (\bar{v}\bar{\nabla})\bar{A} + \bar{v}\times(\bar{\nabla}\times\bar{A}) + \underbrace{\cancel{\bar{A}\times(\bar{\nabla}\times\bar{v})}}_{=0} \quad \leftarrow \bar{v} = \text{const}$$

$$\Rightarrow \quad \bar{F} = \frac{d\bar{P}}{dt} = -q\left(\frac{1}{c}\frac{\delta\bar{A}}{\delta t} + \bar{\nabla}\phi\right) + \frac{q}{c}\bar{v}\times(\bar{\nabla}\times\bar{A}) \quad \Rightarrow \quad \boxed{\begin{aligned}\bar{E} &\equiv -\frac{1}{c}\frac{\delta\bar{A}}{\delta t} + \bar{\nabla}\phi \\ \bar{B} &\equiv \bar{\nabla}\times\bar{A}\end{aligned}}$$

$$= q\bar{E} + \frac{q}{c}\bar{v}\times\bar{B}$$

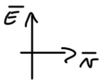
Lorentz force!

(eq. of motion)

work: $\frac{dE_{kin}}{dt} = \bar{v}\frac{d\bar{P}}{dt} = q\bar{E}\bar{v} + \frac{q}{c}\underbrace{\cancel{\bar{v}(\bar{v}\times\bar{B})}}_{=0}$

only $\bar{E}$ exerts work on the particle

if $\bar{E}\bar{v} = 0 \Rightarrow$ no work



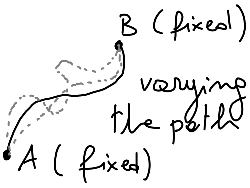
$$\uparrow$$

$$\left(\frac{d}{dt}\left(\frac{1}{2}m\bar{v}^2\right) = \frac{1}{\cancel{2}}\underbrace{m\cancel{2}\bar{v}}\dot{\bar{v}} = \bar{v}\frac{d\bar{P}}{dt}\right)$$

Equations of motion, covariant approach

$$S=\int_A^B \underbrace{-mc^2 d\tau}_{(1)} + \underbrace{\frac{q}{c} A_\mu u^\mu d\tau}_{(2)}$$

$\delta S = 0$ Least action principle



$$\delta(1) = -\int_A^B m\, a_\nu\, \delta x^\nu d\tau$$

$$\delta(2) = \frac{q}{c}\int_A^B \delta(A_\mu u^\mu)\, d\tau = \frac{q}{c}\int_A^B \left(\delta A_\mu u^\mu + A_\mu \overset{\curvearrowleft\curvearrowright}{\delta \frac{dx^\mu}{d\tau}}\right) d\tau$$

$$= \frac{q}{c}\int_A^B \overset{\swarrow}{\delta A_\mu} u^\mu d\tau + \frac{q}{c}\left[\underbrace{\cancel{A_\mu \delta x^\mu \Big|_A^B}}_{=0} - \int_A^B \overset{\swarrow}{\frac{dA_\mu}{d\tau}} \delta x^\mu d\tau\right]$$

$$= \frac{q}{c}\int \left(\underset{\smile}{\partial_\nu} A_{\underset{\smile}{\mu}} \delta x^{\hat\nu} u^{\hat\mu} - \partial_\nu A_\mu u^\nu \delta x^\mu\right) d\tau \qquad \nu \leftrightarrow \mu$$

$$= \frac{q}{c}\int \big(\underbrace{\partial_\mu A_\nu - \partial_\nu A_\mu}\big) u^\nu \delta x^\mu d\tau$$

$$\delta A_\mu = \partial_\nu A_\mu \delta x^\nu$$

$$\frac{dA_\mu}{d\tau} = \partial_\nu A_\mu \frac{dx^\nu}{d\tau} = \partial_\nu A_\mu u^\nu$$

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \;!$$

The electro-magnetic field is one object (not 2 independent fields)

$$\delta S = \int \left(-m a_\mu + \frac{q}{c} F_{\mu\nu} u^\nu\right) \delta x^\mu d\tau = 0 \qquad \forall\, \delta x^\mu$$

$$\Rightarrow \boxed{m\frac{du_\mu}{d\tau} = \frac{q}{c} F_{\mu\nu} u^\nu}$$

4 eq. of motions

Lorentz 4-force

$\mu = 0$: work $\qquad \frac{d}{dt}(\gamma m c^2) = q\bar{E}\bar{v}$

$\mu = i$: Lorentz force $\qquad m\frac{d}{dt}(\gamma \bar{v}) = q\bar{E} + \frac{q}{c}\bar{v}\times\bar{B}$

↳ $u^0 = \gamma c \quad u^i = \gamma v^i \quad d\tau = \gamma^{-1} dt$ ↗

- Alternative: use Euler-Lagrange eq.s

remember: $L = -\overset{\downarrow}{m c^2} + \frac{q}{c}\overset{\swarrow}{A_\mu} u^\mu \qquad -c^2 = u_\mu u^\mu$ and $\overset{\downarrow}{A_\mu(x^\nu)}$

$$\frac{d}{d\tau}\frac{\partial L}{\partial u^\mu} - \frac{\partial L}{\partial x^\mu} = \frac{d}{d\tau}\left(m u_\mu + \frac{q}{c}A_\mu\right) - \frac{q}{c}\partial_\mu A_\nu u^\nu = m\frac{du_\mu}{d\tau} + \frac{q}{c}\partial_\nu A_\mu \underbrace{\frac{dx^\nu}{d\tau}}_{u^\nu} - \frac{q}{c}\partial_\mu A_\nu u^\nu$$

$$= m\frac{du_\mu}{d\tau} + \frac{q}{c}F_{\nu\mu} u^\nu = 0$$

↑ $F_{\nu\mu} = -F_{\mu\nu}$

$$m\frac{du_\mu}{d\tau}=\frac{q}{c}F_{\mu\nu}u^\nu \qquad u_0=-\gamma c$$

$$m\partial_0=\frac{q}{c}\left(-F_{00}u^0+F_{01}u^1+\dots+F_{03}u^3\right)=\frac{q}{c}\left[0+(\delta_0A_1-\delta_1A_0)u^1+\dots+(\delta_0A_3-\delta_3A_0)u^3\right]$$

$$=\frac{q}{c}\left[\delta_0A_1u^1+\dots+\delta_0A_3u^3+\delta_1A_0u^1+\dots+\delta_3A_0u^3\right]$$

$$m\cancel{\gamma}\frac{du_0}{dt}=\frac{q}{c}\left[\frac{d\bar{A}}{c\,dt}\bar{u}-\bar{\nabla}\phi\cdot\bar{u}\right]=\frac{q}{c}\bar{u}\left(\frac{1}{c}\frac{d\bar{A}}{dt}-\bar{\nabla}\phi\right)=\frac{q}{c}\bar{u}\bar{E}=\frac{q}{c}\frac{dx^\mu}{d\tau}E_\mu=\frac{q}{c}\cancel{\gamma}\bar{v}\bar{E}$$

$$\frac{d}{dt}(m\gamma c)=\frac{q}{c}\bar{v}\bar{E} \qquad \frac{d}{dt}(mc^2\gamma)=\frac{dE}{dt}=q\bar{v}\bar{E} \qquad \checkmark$$

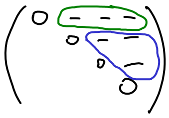
$$m\cancel{\gamma}\frac{du_i}{dt}=\frac{q}{c}\left(-F_{i0}u^0+F_{i1}u^1+\dots+F_{i3}u^3\right)=\frac{q}{c}\left[-\underset{\substack{\uparrow\\(\delta_iA_0-\delta_0A_i)}}{F_{i0}}u^0+(\delta_iA_1-\delta_1A_i)u^1+\dots+(\delta_iA_3-\delta_3A_i)u^3\right]$$

A closer look at the lectro-magnetic field tensor

- Components: $F_{\mu\nu} = \delta_\mu A_\nu - \delta_\nu A_\mu$

a) From anti-symmetry of $F_{\mu\nu}$

$F_{\alpha\alpha} = \delta_\alpha A_\alpha - \delta_\alpha A_\alpha = 0 \Rightarrow$ all diagonal elements $= 0$



$F_{\alpha\beta} = - F_{\beta\alpha}$

b) $A = (\phi, \bar{A})^T \Rightarrow F_{0i} = \delta_0 A_i - \delta_i A_0 = -\frac{1}{c}\frac{\delta A_i}{\delta t} - \delta_i \phi \equiv E_i$ $\quad \boxed{\bar{E} = -\left(\frac{1}{c}\frac{\delta \bar{A}}{\delta t} + \bar{\nabla}\phi\right)}$

c) $\left.\begin{array}{l} F_{12} = \delta_1 A_2 - \delta_2 A_1 \\ F_{13} = \delta_1 A_3 - \delta_3 A_1 \\ F_{23} = \delta_2 A_3 - \delta_3 A_2 \end{array}\right\}$ $\boxed{\bar{B} = \bar{\nabla} \times \bar{A}}$ $\quad B_{ij} = \epsilon_{ija} B^a$

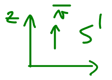
$$\Rightarrow \text{All together:} \quad (F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\bar{E} \\ \bar{E} & \mathcal{B} \end{pmatrix} \qquad (F^{\mu\nu}) = \begin{pmatrix} 0 & \bar{E} \\ -\bar{E} & \mathcal{B} \end{pmatrix}$$

$\mathcal{B}$

- Transformation of F

$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\alpha} \Lambda^{\nu'}{}_{\beta} F^{\alpha\beta}$ for $\bar{v} = \bar{v}\, \bar{e}_z \Rightarrow$

(as any tensor)



$$\boxed{\begin{array}{lll} E'_x = \gamma(E_x + \beta B_y) & E'_y = \gamma(E_y - \beta B_x) & E'_z = E_z \\ B'_x = \gamma(B_x - \beta E_y) & B'_y = \gamma(B_y + \beta E_x) & B'_z = B_z \end{array}}$$

Boost $\perp \bar{e}_z$ unchanged

- Associated invariants

$$\boxed{\begin{array}{l} F_{\mu\nu} F^{\mu\nu} = -2(\bar{E}^2 - \bar{B}^2) \\ (*F)_{\mu\nu} F^{\mu\nu} = -2 \bar{E}\bar{B} \end{array}}$$

$\Lambda F \Lambda F = FF$

$\leftarrow$ from orthonormality of $\Lambda \Rightarrow \bar{E}^2 - \bar{B}^2 = \bar{E}'^2 - \bar{B}'^2$

$\leftarrow$ $(*F)$ = Hodge dual tensor $\Rightarrow \bar{E}\bar{B} = \bar{E}'\bar{B}'$

$(*F)_{\mu\nu} F^{\mu\nu} = \varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$

Consequences:

frame in which

$$\text{a) if } \left\{ \begin{array}{lll} (\bar{E}^2 - \bar{B}^2) \lesseqgtr 0 & \Rightarrow & \bar{E}^2 \gtreqless \bar{B}^2 \\ -2\bar{E}\bar{B} = 0 & \Rightarrow & \bar{E} \perp \bar{B} \end{array} \right\} \forall \text{ frame}$$

eg. electro.-mag. waves

b) if you want to transform from a pure $\bar{E}$ (i.e. $\bar{B}=0$) to a pure $\bar{B}$ (i.e. $\bar{E}=0$)

$$\Rightarrow \bar{E}^2 - \underset{=0!}{\bar{B}^2} = \underset{=0!}{\bar{E}'^2} - \bar{B}'^2 \Rightarrow \underset{\geq 0}{\bar{E}^2} = -\underset{\geq 0}{\bar{B}'^2} \Rightarrow \bar{E} = 0 = \bar{B}$$ only this trivial solution!

c) if $\bar{E}\bar{B}=0$ and $\begin{cases} \bar{E}^2 > \bar{B}^2 & \Rightarrow \exists \Lambda \mid \bar{B}'=0 \\ \bar{E}^2 < \bar{B}^2 & \Rightarrow \exists \underset{\uparrow \text{Lorentz. trasf.}}{\Lambda} \mid \bar{E}'=0 \end{cases}$

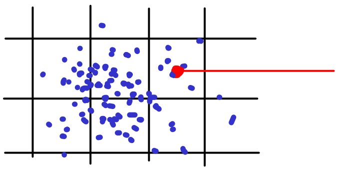
Many charged particles: 4-current density

- One charged particle

$d\tau = \gamma^{-1} dt$

$$S_{pf} = \frac{q}{c}\int A_\mu dx^\mu = \frac{1}{c}\int A_\mu j^\mu dt \qquad j^\mu \equiv q\frac{dx^\mu}{dt} = \gamma^{-1} q u^\mu \qquad \bar{j} = (cq, \bar{j})^T \quad \bar{j} \equiv q\bar{v}$$

- More particles



$dq = \rho dV$ charge within volume dV

$\rho =$ charge density

- $q_j =$ const is invariant but ρ is not invariant (volums transforms)
- $dq = \rho dV$ " " : $\rho \equiv \frac{dq}{dV}$ $\quad dq = \frac{dq}{\cancel{dV}}\cancel{dV}$ invariant

- Toward the continuous limit

Charge of volume V : $Q_V(\bar{r}) = \sum_j q_j = \int_V q_j \delta(\bar{r} - \bar{r}_j) dV \simeq \int_V \rho(\bar{r}) dV$

Action of volume V : $dS_{pf} = \frac{dq}{c}\int A_\mu dx^\mu = \frac{\rho dV}{c}\int A_\mu dx^\mu = \frac{dV}{c}\int A_\mu \rho \frac{dx^\mu}{dt} dt = \frac{dV}{c}\int A_\mu j^\mu dt$

4-density current $\quad (j^\mu) \equiv (\rho c, \bar{j}) \quad \bar{j} = \rho\bar{v} \qquad [j] = \left[\frac{C}{m^3}\frac{m}{s}\right] = \left[\frac{C}{m^2 s}\right]$ charge flux

$$\Rightarrow \boxed{S_{pf}} = \frac{1}{c}\int A_\mu j^\mu dt\, dV = \boxed{\frac{1}{c^2}\int A_\mu j^\mu d\Omega} \qquad dt\,dV = \frac{c\,dt}{c} dV = \frac{1}{c} dx^0 dx^1 dx^2 dx^3 \equiv \frac{d\Omega}{c}$$

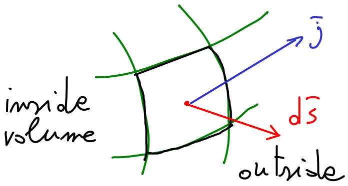
Action of collection of charged particles (fluid/gas) : plasma

Continuity equation

- Change of charge in a volume $\dot{Q} = \frac{\partial}{\partial t}\int_V \rho \, dV$ ρ = charge density

- It is given by the amount of charges that enter/exit that volume, i.e flux

flux : $\bar{j} = \rho \bar{v} \left[\frac{C}{m^2 s}\right]$



Convention: $\boxed{\bar{j}\, d\bar{s} > 0}$
if the flux is pointing outside the volume (loss of charges)
⇒ define $d\bar{s}$ pointing outside vol.

$dq = j_i \, ds^i dt$ charge passing across the surface $d\bar{s}$ during time dt

$\boxed{\dot{Q} = -\oint j_i \, ds^i}$ integral over the entire surface enclosing a volume
Since $\oint j_i ds^i > 0$ is the charge that exit the volume ⇒ need $\dot{Q} < 0$

Gauss theorem (3D) $\oint \bar{j} d\bar{s} = \int \bar{\nabla} \bar{j} \, dV$

$$\Rightarrow \frac{\partial}{\partial t}\int \rho \, dV = -\oint j_i \, ds^i = -\int \partial_i j^i \, dV \quad \Rightarrow \quad \int_V \left(\dot{\rho} + \partial_i j^i\right) dV = 0 \quad \text{true } \forall \text{ volume}$$

$$\boxed{\frac{c}{c}\frac{\partial \rho}{\partial t} + \bar{\nabla}\bar{j} = 0} \longrightarrow \boxed{\partial_\mu j^\mu = 0}$$

$$(\partial_\mu) = \left(\frac{\partial}{\partial x^\mu}\right) = \left(\frac{\partial}{c\,\partial t}, \bar{\nabla}\right)$$
$$(j^\mu) = (c\rho, \bar{j})$$

Continuity equation
it expresses charge conservation (!)
i.e. charges move from one volume to another, they are not created/destroyed

Continuity eqaution and Gauge invariance

$S_{pf} = \frac{1}{c^2}\int A_\mu j^\mu d\Omega$ $\quad A_\mu \rightarrow \tilde{A}_\mu = A_\mu + \frac{\delta f}{\delta x^\mu}$

i.e. add the gradient of an arbitrary function $f(x^\nu)$ "x" space + time"
⇒ no change in eq. of motion

$$\tilde{S}_{pf} = \frac{1}{c^2}\int \left(A_\mu j^\mu + \frac{\delta f}{\delta x^\mu} j^\mu\right) d\Omega$$

$$= S_{pf} + \frac{1}{c^2}\int \frac{\delta f}{\delta x^\mu} j^\mu d\Omega \qquad \frac{\partial}{\delta x^\mu}(f j^\mu) = j^\mu \partial_\mu f + f \overbrace{\partial_\mu j^\mu}^{=0}$$

Because of charge conservation
$\partial_\mu j^\mu = 0$

$$= S_{pf} + \frac{1}{c^2}\int \partial_\mu (f j^\mu) d\Omega$$

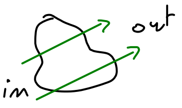
← 4-D Gauss theorem
S_μ = 4-surface

$$= S_{pf} + \underbrace{\frac{1}{c^2}\oint f j^\mu dS_\mu}_{\text{const.}}$$

⇒ $\delta \tilde{S}_{pf} = \delta S_{pf} + 0$ (least action principle)

No effect on the eq. of motion (obtain same $F_{\mu\nu}$)

flux across a closed surface
⇒ Q = const.



⇒ For charged particles : Gauge invariance ⟷ charge conservation

- you can always choose f such that $\tilde{\Phi} = 0$ ⇒ $(A_\mu) = (0, \vec{A})$

  but in general you can not impose $\tilde{\vec{A}} = \vec{0}$ (there are 3 conditions, not one)

- The field is characterized by its effect on trajectory of particles

  ⇒ different potentials can correspond to the same field!

Dynamic of the field : summary

- Variables of the field are $(A^\mu) = (\phi, \bar{A})$
- Lagrangian density $\mathcal{L} = -\frac{1}{16\pi c} F^{\mu\nu} F_{\mu\nu} + \frac{1}{c^2} A_\mu j^\mu$ $S = \int L\,dt = \int \mathcal{L}\,dt\,dV$
- Eq. of motion : $\partial S = \int_a^b \mathcal{L}\,d\Omega \overset{!}{=} 0 \Rightarrow \partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu$

  → continuity eq. : $\partial_\mu \partial_\nu F^{\mu\nu} = 0 \Rightarrow \partial_\mu j^\mu = 0$

- In Lorentz gauge, i.e. $\partial_\mu A^\mu \overset{!}{=} 0$: $\Box A^\mu = +\frac{4\pi}{c} j^\mu$

- Solution : a) homogeneous $\Box A^\mu = 0$ → wave function (D'Alembert eq.)

  $(\partial_\nu F^{\mu\nu} = 0 \leftrightarrow \partial_{[\alpha} F_{\beta\gamma]} = 0)$ → ∃ electromagnetic waves

  b) Inhomogeneous $\Box A^\mu = \frac{4\pi}{c} j^\mu$

$$A^\mu(t, \bar{x}) = \frac{1}{c} \int d^3x' \int dt'\, G(t-t', \bar{x}-\bar{x}')\, j^\mu(t', \bar{x}')$$

$$G(t-t', \bar{x}-\bar{x}') = \underbrace{\frac{1}{|\bar{x}-\bar{x}'|}}_{\text{Coulomb}} \delta\Big(\underbrace{t-t'-\frac{|\bar{x}-\bar{x}'|}{c}}_{\text{Back light cone}}\Big)$$

$$f = \frac{q}{Rc(1-\bar{e}\bar{\beta})^3}$$

⇒ Lienard-Wiechert potentials : $\phi(\bar{r}, t) = \frac{q}{R(1-\bar{e}\bar{\beta})}$ $\bar{A} = \phi \bar{\beta}$

- In far away observer: $\bar{E}_{far} = f\, \bar{e} \times [(\bar{e}-\bar{\beta}) \times \dot{\bar{\beta}}]$ $\bar{B}_{far} = f\, \bar{e} [\dot{\bar{\beta}} + \bar{e} \times (\beta \times \dot{\bar{\beta}})]$

Dynamic of the field

- A particle is described by (x^μ) $\Rightarrow$ x^μ are the variable
- A 4-potential is described by $(A^\mu) = (\phi, \bar{A})$ $\Rightarrow$ ϕ and $\bar{A}$ are the variables
- Equation of motion of a field $\rightarrow$ how the values of the field evolve
- Derive eq. of motion (not where the field is going...)

- Classical scalar field $A(q^\mu)$ q^μ = 4-vector (generalized coordinate)
- " vector " $A^\mu(q^\mu)$ e.g. $(A^\mu) = (\phi, \bar{A})^T$
- Lagrangian density $\mathcal{L}(A^\mu, \partial_\nu A^\mu, q^\mu) \Rightarrow S = \iint \mathcal{L} \, dV dt = \frac{1}{c} \int \mathcal{L} \, d\overset{dx^0dx^1dx^2dx^3}{\Omega}$
- Least action principle $\delta S = 0 \;\models\!\!\Rightarrow$ $\boxed{\frac{\delta}{\delta q^\nu}\left(\frac{\delta \mathcal{L}}{\delta A^\mu_{,\nu}}\right) - \frac{\delta \mathcal{L}}{\delta A^\mu} = 0}$ Euler Lagrange equations

A) Lagrangian density of the field must be:

1) reproduce Coulomb force
2) of quadratic form (to have a linear theory $\rightarrow$ superposition principle)
3) invariant under Lorentz transformations (scalar)

$\Rightarrow$ It can only be:

$$\mathcal{L}_f = -\frac{1}{16\pi c} F^{\mu\nu} F_{\mu\nu} \qquad F^{\mu\nu} F_{\mu\nu} = 2(\bar{B}^2 - \bar{E}^2)$$ invariant (field strength)

$\left(\bar{E} = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\phi \quad \bar{B} = \bar{\nabla} \times \bar{A}\right)$

$16\pi c$ $\rightarrow$ to set Coulomb's units

$-$ $\rightarrow$ to have a minimum in S (for least action principle)
(in fact: E^2 contains a $\dot{A}^2$ that in principle can be large (positive) enough)

$$\models\!\!\Rightarrow \boxed{\mathcal{L} = -\frac{1}{16\pi c} F^{\mu\nu} F_{\mu\nu} + \frac{1}{c^2} A_\mu j^\mu} = \overset{①}{\mathcal{L}_f} + \overset{②}{\mathcal{L}_{pf}}$$

① = self interaction

② = coupling field-matter $\leftarrow$ const because of charge conserved.

B) Looking for the eq. of motion

Least action principle : $S = \int \left(\frac{1}{c^2} A_\mu j^\mu - \frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} \right) d\Omega = \int_a^b \left[\iiint_{-\infty}^{\infty} \mathcal{L}\, dV \right] dt$

$$\partial S = \int \left(\frac{1}{c^2} \underset{①}{\partial (A_\mu j^\mu)} - \frac{1}{16\pi c} \partial (\underset{②}{F_{\mu\nu} F^{\mu\nu}}) \right) d\Omega$$

a, b fixed

① $\partial (A_\mu j^\mu) = j^\mu \partial A_\mu$ because the derivation is done given a fixed j^μ (i.e. $\partial j^\mu =$ we let vari only the potential A

② $\partial (F_{\mu\nu} F^{\mu\nu}) = \partial F_{\mu\nu} \cdot F^{\mu\nu} + F_{\mu\nu} \partial F^{\mu\nu} = 2 F^{\mu\nu} \partial F_{\mu\nu}$

$= 2 F^{\mu\nu} \partial (\delta_\mu A_\nu - \delta_\nu A_\mu)$

use anty-symmetry

$= 2 F^{\mu\nu} (\delta_\mu \partial A_\nu - \delta_\nu \partial A_\mu) = 2 F^{\mu\nu} \delta_\mu \partial A_\nu - 2 F^{\mu\nu} \delta_\nu \partial A_\mu$

$= -2 F^{\nu\mu} \delta_\mu \partial A_\nu - 2 F^{\mu\nu} \delta_\nu \partial A_\mu$ ← relabeling in 1° term ν -> μ

$= -4 F^{\mu\nu} \delta_\nu \partial A_\mu$

$$= \int \left(\frac{1}{c^2} j^\mu \partial A_\mu + \frac{1}{4\pi c} \underbrace{F^{\mu\nu} \delta_\nu \partial A_\mu}_{} \right) d\Omega$$

integrate by part (on dx^0 part) ⊛

$$⊛ = \int \left[\int F^{\mu\nu} \delta_\nu \partial A_\mu dx^0 \right] dV$$

$$= \int \left[F^{\mu\nu} \partial A_\mu \Big|_a^b - \int_a^b \delta_\nu F^{\mu\nu} \partial A_\mu dx^0 \right] dV$$

= 0 Because A_μ is fixed at "a, b" (least action princ.)

$$= - \int \delta_\nu F^{\mu\nu} \partial A_\mu d\Omega$$

$$= \int \left(\frac{1}{c^2} j^\mu - \frac{1}{4\pi c} \delta_\nu F^{\mu\nu} \right) \partial A_\mu d\Omega \overset{!}{=} 0$$

↖ arbitrary

$$\Rightarrow \boxed{\delta_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu}$$ Maxwell's equations (eq. of motion of the field)

- 4 inhomogeneous linear differential eq.
  ↳ as expected by construction (for superposition principle)

- They fully describe the $F^{\mu\nu}$

- They "contain" the continuity eq. $\underset{\text{symmetric}}{\delta_\mu} \overset{=0}{\underset{\text{anti-symmetric}}{\delta_\nu F^{\mu\nu}}} = \frac{4\pi}{c} \delta_\mu j^\mu \Rightarrow \boxed{\delta_\mu j^\mu = 0}$

Explicit form

$F^{\mu\nu} = \begin{pmatrix} 0 & \bar{E} \\ -\bar{E} & B \end{pmatrix}$ $F_{\mu\nu} = \begin{pmatrix} 0 & -\bar{E} \\ \bar{E} & B \end{pmatrix}$

- $\boxed{\delta_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu}$ 4 equations, 2° couple of Maxwell's eq.

· time ($\mu=0$):

$$\delta_\nu F^{0\nu} = \delta_0 \overset{=0}{F^{00}} + \delta_i F^{0i} = \bar{\nabla} E = \frac{4\pi}{\cancel{c}} \cancel{c} \rho \quad \boxed{\bar{\nabla}\bar{E} = 4\pi\rho} \; ①$$

· space ($\mu=1$):

$$\delta_\nu F^{1\nu} = \delta_0 F^{10} + \delta_j F^{1j} = \frac{1}{c}\frac{\delta F^{10}}{\delta t} + \underset{=0}{\cancel{\delta_1 F^{11}}} + \delta_2 F^{12} + \delta_3 F^{13} = \frac{4\pi}{c} j^1$$

$$\frac{1}{c}\frac{\delta E_x}{\delta t} - \frac{\delta B_z}{\delta y} + \frac{\delta B_y}{\delta z} = \frac{4\pi}{c} j^1 \qquad \left(\frac{\delta B_z}{\delta y} - \frac{\delta B_y}{\delta z} = -(\bar{\nabla}\times\bar{B})_x \right)$$

$$\boxed{c\bar{\nabla}\times\bar{B} = \frac{\delta\bar{E}}{\delta t} + 4\pi\bar{j}} \; ② \qquad \text{(by joining } \mu = 1,2,3\text{)}$$

Continuity equation from ②

$$\bar{\nabla}(\bar{\nabla}\times\bar{B}) = 0 = \frac{1}{c}\frac{\delta\bar{\nabla}\bar{E}}{\delta t} + \frac{4\pi}{c}\bar{\nabla}\bar{j} = \frac{1}{c}\frac{\delta(4\pi\rho)}{\delta t} + \frac{4\pi}{c}\bar{\nabla}\bar{j} = 0 \implies \frac{\delta\rho}{\delta t} + \bar{\nabla}\bar{j} = 0 \quad ✓$$

- $\boxed{\delta_\nu F^{\mu\nu} = 0}$ ← associated homogeneous equation (for $j^\mu = 0$: $\rho = 0$ $\bar{j} = 0$)

This equation has non trivial solutions, i.e. it can have $F^{\mu\nu} \neq 0$ because of the antisymmetry of $F^{\mu\nu}$

In fact: $\delta_\alpha F_{\beta\gamma} + \delta_\beta F_{\gamma\alpha} + \delta_\gamma F_{\alpha\beta} \equiv \boxed{\delta_{[\alpha} F_{\beta\gamma]} = 0}$ identically satisfied because $F^{\mu\nu}$ is antisymmetric

· These are $4\times4\times4 = 64$ equations
but the non-trivial ones are 4 (in fact non trivial only if $\alpha \neq \beta \neq \gamma$
corresponding to $\delta_\nu F^{\mu\nu} = 0$

$\Rightarrow$ $\mu=0$: $\boxed{\dot{\bar{B}} + c\bar{\nabla}\times\bar{E} = 0}$ $\mu=i$: $\boxed{\bar{\nabla}\bar{B} = 0}$ Gauss-Faraday laws

In Lorentz gauge

②

- A^μ are not univocal : different potentials can lead to the same E.M field
- In fact: $A^\mu \to \tilde{A}^\mu = A^\mu + \partial^\mu f$ lead to the same eq. of motion of particles
- Exaples: f such that $\phi = 0$ or $\overset{\text{(Coulomb gauge)}}{\bar\nabla \bar A = 0}$ ⇐ (not invariant)

  not $\bar A = 0$ because these are 3 conditions, not one!

- Lorentz gauge : $\boxed{\partial_\nu A^\nu = \frac{1}{c}\frac{\partial \phi}{\partial t} + \bar\nabla \bar A = 0}$ satisfied ∀ frame (i.e. invariant condition)

$$\Rightarrow \underbrace{\partial_\nu F^{\mu\nu}}_{①} = \frac{4\pi}{c} j^\mu \qquad ① = \partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \overset{\text{Lorentz gauge}}{\overset{\downarrow}{\overset{=0}{\cancel{\partial^\mu \partial_\nu A^\nu}}}} - \partial_\nu \partial^\nu A^\mu = -\Box A^\mu$$

$$\partial_\nu \partial^\nu = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \bar\nabla\right) \equiv \Box \quad \text{D'Alembert operator}$$

$$\boxed{\Box A^\mu = -\frac{4\pi}{c} j^\mu}$$

$\bar\nabla \bar A' = 0 = \bar\nabla \bar A + \bar\nabla \bar\nabla f \Rightarrow \bar\nabla^2 f \overset{!}{=} -\bar\nabla A$ Gauss gauge

$\partial_\nu A'^\nu = 0 = \partial_\nu A^\nu + \partial_\nu \partial^\nu f \Rightarrow \Box f \overset{!}{=} -\partial_\nu A^\nu$ Lorentz gauge

Dynamics of the field: summay

- Variables of the field: $\boxed{(A^\mu) = (\phi, \bar{A})}$
- Lagrangian density: $L = \int \mathcal{L}\, dV$

-> reproduce Coulomb force + Quadratic form + Invariant under Lorentz

$$\boxed{\mathcal{L} = -\frac{1}{16\pi c} F^{\mu\nu}F_{\mu\nu} + \frac{1}{c^2} A_\mu j^\mu} = \overset{①}{\mathcal{L}_f} + \overset{②}{\mathcal{L}_{pf}}$$

① self interaction

② coupling field matter

const. (charge cons.)

quadratic, invariant: $F_{\mu\nu}F^{\mu\nu} = 2(\bar{B}^2 - \bar{E}^2)$ field strength

to have a minimum

- Eq. of motion: $\delta S = \int_a^b \mathcal{L}\, d\Omega = \int \left(\frac{1}{c^2} j^\mu - \frac{1}{4\pi c}\partial_\nu F^{\mu\nu}\right) \underset{}{\overset{\text{arbitrary (i.e. } \forall \delta A_\mu)}{\delta A_\mu}}\, d\Omega = 0$

$$\Rightarrow \boxed{\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu}$$

eq. of motion of the field (2° couple Max. eq.s)
4 inhomogeneous linear diff. eq.s

↓

continuity eq. $\underbrace{\partial_\mu\partial_\nu F^{\mu\nu}}_{=0} = \frac{4\pi}{c}\partial_\mu j^\mu \Rightarrow \partial_\mu j^\mu = 0$

$=0$ $\partial_\mu\partial_\nu$ symmetric, $F^{\mu\nu}$ anti-symmetric

$(\mu=0)$: $\boxed{\bar\nabla \bar E = 4\pi\rho}$ $(\mu=i)$: $\boxed{c\bar\nabla\times\bar B = \frac{\partial E}{\partial t} + 4\pi\bar j}$

↳ $\bar\nabla(\bar\nabla\times\bar B) = 0 = \frac{\partial}{\partial t}(4\pi\rho) + 4\pi\bar\nabla\bar j \rightarrow \partial_\mu j^\mu = 0$ continuity

- Associated homogeneous equation $\boxed{\partial_\nu F^{\mu\nu} = 0}$ => solution in vacuum!

non trivial solution ($F^{\mu\nu} \neq 0$) because $\partial_{[\alpha}F_{\beta\gamma]} = 0$ because of anti-symmetry of F

$(\mu=0)$: $\boxed{\frac{\partial \bar B}{\partial t} + c\bar\nabla\times\bar E = 0}$ $(\mu=i)$: $\boxed{\bar\nabla \bar B = 0}$ Gauss-Faraday laws

- Apply Lorentz gauge: $\partial_\mu A^\mu = 0 \Rightarrow \Box A^\mu = -\frac{4\pi}{c} j^\mu$: $\Box A^\mu = 0$ Plane wave (vacuum)

③

Solving Maxwell's eq.

- Homogeneous eq. $\Box A^{\mu} = 0$ $\Box = -\frac{1}{c^2}\frac{\delta^2}{\delta t} + \bar{\nabla}^2$ D'Alambert operator

D'Alambert equation

Wave equation : $-\frac{1}{c^2}\frac{\delta^2 A^{\mu}}{\delta t} + \bar{\nabla}^2 A^{\mu} = 0$

↓

solution = plane wave propagating with velocity = c

- The field must be time evolving !
- $\Box A^{\mu} = 0$ has non trivial solution, i.e. $A^{\mu} \neq 0$ is allowed

i.e you can have a potential $F^{\mu\nu}$ in vacuum !
(see also ⊛ below) ↑ (no charges)

⇒ Existence of electromagnetic waves!

* * *

⊛ - $\boxed{\delta_{[\alpha} F_{\beta\gamma]} = 0}$ ($\delta_{[\alpha} F_{\beta\gamma]}$ is identically zero because of the antisymmetry of F)

↑

cyclic indexes $\delta_{\alpha} F_{\beta\gamma} + \delta_{\beta} F_{\gamma\alpha} + \delta_{\gamma} F_{\alpha\beta} \equiv \delta_{[\alpha} F_{\beta\gamma]}$

- Inhomogeneous eq.:

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \rightarrow \Box A^\mu = -\frac{4\pi}{c} j^\mu \quad \text{(in Lorentz gauge)} \circledast$$

time ($\mu=0$): $\Box A^0 = -\frac{4\pi}{\cancel{c}}\cancel{c}\rho$ $\quad -\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \bar{\nabla}^2\phi = -4\pi\rho$ (1)

space ($\mu=i$): $\Box A^i = -\frac{4\pi}{c}\bar{j}$ $\quad -\frac{1}{c^2}\frac{\partial^2 A^i}{\partial t^2} + \bar{\nabla}^2 A^i = -\frac{4\pi}{c} j^i$ (2)

- Constant field: (1) → $\bar{\nabla}^2\phi = -4\pi\rho$ charges are source of $\phi \Rightarrow \bar{E}$

(2) → $\bar{\nabla}^2\bar{A} = -\frac{4\pi}{c}\bar{j}$ currents " " " $\bar{A} \Rightarrow \bar{B}$

$\bar{E} = -\frac{1}{c}\dot{\bar{A}} - \bar{\nabla}\phi = \bar{\nabla}\phi$; $\bar{B} = \bar{\nabla}\times\bar{A}$

- Solve by means of Green's functions

Solution to solve $\circledast$ → superposition principle

solve by (retarded) Green's functions

in observer's frame

$$\boxed{A^\mu(t,\bar{x}) = \frac{1}{c}\underbrace{\int d^3x' \int dt'}_{\text{(linear superposition)}} \underbrace{G(t-t', \bar{x}-\bar{x}')\, j^\mu(t',\bar{x}')}_{\text{(point sources)}}}$$

← 4-potential in $\bar{x}$ at t (for observer)

"of all"

$$G(t-t', x-x') = \frac{1}{|\bar{x}-\bar{x}'|}\underbrace{\delta\left(t-t'-\frac{|\bar{x}-\bar{x}'|}{c}\right)}_{=1 \text{ if } \left(t-t'-\frac{|\bar{x}-\bar{x}'|}{c}=0\right)}$$

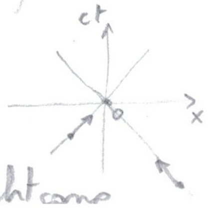
Coulomb's force

$\Downarrow$

$c\,dt = |d\bar{x}|$

↑

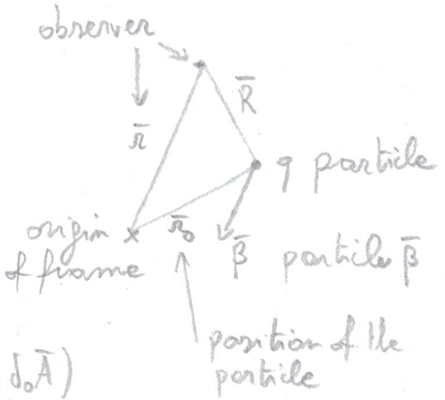
i.e. from back light cone



D) Lineard-Wiechert Potentials

- plug in solution $A^\mu(t,\bar{x})$ one charged particle $(j^\mu) = q\begin{pmatrix} c \\ \bar{v} \end{pmatrix}\delta(\underbrace{\bar{r}-\bar{r}_0(t)}_{\bar{R}})$

$$\Longrightarrow \boxed{\begin{aligned} \phi(\bar{r},t) &= \frac{q}{R(1-\bar{e}\bar{\beta})} \\ \bar{A}(\bar{r},t) &= \frac{q\bar{\beta}}{R(1-\bar{e}\bar{\beta})} = \phi\bar{\beta} \end{aligned}} \qquad t' = t - \frac{R}{c}$$



$\Downarrow$

derive $\bar{E}$ and $\bar{B}$ (they depend on $\bar{\nabla}\phi, \partial_0\bar{A}$) ⊛

- Potential (A^μ) is in terms of retarded coordinates!

but we need ∂_μ in terms of observer's coordinates

$\Rightarrow$ transform ∂_μ accordingly

- Approximation for observers far away (drop all terms which "decay" stronger than R^{-1}, i.e. R^{-2}, ...)

$$\Longrightarrow \boxed{\begin{aligned} \bar{E}_{far} &= \frac{q}{Rc(1-\bar{e}\bar{\beta})^3} \; \bar{e}\times\left[(\bar{e}-\bar{\beta})\times\dot{\bar{\beta}}\right] \\ \bar{B}_{far} &= \quad '' \qquad\quad \bar{e}\times\left[\dot{\bar{\beta}}+\bar{e}\times(\bar{\beta}\times\dot{\bar{\beta}})\right] \end{aligned}}$$

$\bar{e} \equiv \frac{\bar{R}}{R}$ direction from which photons come

Energy-Momentum Tensor : summary

- Define action of system $S = \int \mathcal{L}(q, q_{,\nu})\, dV\, dt$ $\quad q^{\mu}(x^{\nu})$ generalized coordinate
- List action principle $\delta S = 0$
- Euler Lagrange eq. $\frac{\delta \mathcal{L}}{\delta q} - \frac{\delta}{\delta x^{\nu}} \frac{\delta \mathcal{L}}{\delta q_{,\nu}} = 0 \qquad q \to q^{\mu}$
- Look for conserved quantity: $\frac{\delta \mathcal{L}}{\delta x^{\mu}} = 0$ (energy-momentum)

$\Rightarrow$ From its definition it follows a conserved quantity $\boxed{\frac{\delta T^{\nu}_{\mu}}{\delta x^{\nu}} = 0}$

$$\boxed{T^{\nu}_{\mu} = q^{\alpha}_{,\mu} \frac{\delta \mathcal{L}}{\delta q^{\alpha}_{,\nu}} - \delta^{\nu}_{\mu} \mathcal{L}}$$

↳ Not uniquely defined $\longrightarrow T^{\mu\nu} + \frac{\delta}{\delta x^{\sigma}} \underset{}{\overset{\text{antisymmetric}}{\psi^{\mu\nu\sigma}}}$ satisfies $\frac{\delta T^{\nu}_{\mu}}{\delta x^{\nu}} = 0$

↳ Not necessarily symmetric

- Make T^{μ}_{ν} unique by imposing 4-angular momentum in "standard" form

$$M^{\alpha\beta} = \int (x^{\alpha} dp^{\beta} - x^{\beta} dp^{\alpha}) \quad \Rightarrow \quad \text{conserved if } \boxed{T^{\mu\nu} = T^{\nu\mu}}$$

make it so with a proper $\psi^{\mu\nu\sigma}$

$$p^{\alpha} = \int T^{\alpha 0} dV = \text{const}$$ 4-momentum of the system

energy density

momentum density $\quad S^i$ = energy density flux

$$(T^{\mu\nu}) = \begin{pmatrix} \boxed{\varepsilon} & S_x/c & S_y/c & S_z/c \\ - & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ - & - & \sigma_{yy} & \sigma_{yz} \\ - & - & - & \sigma_{zz} \end{pmatrix} \quad \Rightarrow \quad \boxed{\partial_{\nu} T^{\nu}_{\mu} = 0}$$ is the energy-momentum conservation of a field!

Stress tensor 3-energy density flux $\boxed{\sigma^{ij}}$

Energy-Momentum tensor

- $\mathcal{L}$: no explicit dependency on x^μ ⇒ closed system (total energy, momentum, mass are conserved)

- Action

$$S = \int \underbrace{\mathcal{L}(q^\alpha, q^\alpha_{,\nu})\, dV}_{L}\, dt = \frac{1}{c}\int \mathcal{L}\, d\Omega$$

$d\Omega = c\,dt\,dV = dx^0 dx^1 dx^2 dx^3$

$q^\alpha(x^\nu)$: generalized coordinates

$\mathcal{L}$: lagrangian density

- Least action principle

$$\partial S = 0 \quad \Rightarrow \quad \frac{\delta \mathcal{L}}{\delta q^\alpha} - \frac{\delta}{\delta x^\nu}\left(\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\right) = 0$$

Euler-Lagrange eq. ⊛ $\left(\frac{\delta f}{\delta x^\nu} \equiv \partial_\nu f \equiv f_{,\nu}\right)$

(now use $q^\alpha \to q$ for beauty)

- Look for a conserved quantity regarding the system as a whole (we have seen $\partial_\mu j^\mu =$ charge conservation

$$\frac{\delta \mathcal{L}}{\delta x^\mu} = \frac{\delta \mathcal{L}}{\delta q}\frac{\delta q}{\delta x^\mu} + \frac{\delta \mathcal{L}}{\delta q_{,\nu}}\frac{\delta q_{,\nu}}{\delta x^\mu} = \left(\frac{\delta}{\delta x^\nu}\frac{\delta \mathcal{L}}{\delta q_{,\nu}}\right)\frac{\delta q}{\delta x^\mu} + \frac{\delta \mathcal{L}}{\delta q_{,\nu}}\frac{\delta q_{,\mu}}{\delta x^\nu} = \frac{\delta}{\delta x^\nu}\left(\frac{\delta \mathcal{L}}{\delta q_{,\nu}}\, q_{,\mu}\right)$$

(derivative of a product)

$$\frac{\delta q_{,\nu}}{\delta x^\mu} = \frac{\delta^2 q}{\delta x^\nu \delta x^\mu} = \frac{\delta^2 q}{\delta x^\mu \delta x^\nu} = \frac{\delta q_{,\mu}}{\delta x^\mu}$$

$$\delta^\nu_\mu \frac{\delta \mathcal{L}}{\delta x^\nu}$$

Plug in ⊛ : this is what "tells" the equation to follow the eq. of motion

$$\Rightarrow \quad \delta^\nu_\mu \frac{\delta \mathcal{L}}{\delta x^\nu} = \frac{\delta}{\delta x^\nu}\left(\frac{\delta \mathcal{L}}{\delta q_{,\nu}}\, q_{,\mu}\right)$$

$$\frac{\delta}{\delta x^\nu}\underbrace{\left(\frac{\delta \mathcal{L}}{\delta q_{,\nu}}\, q_{,\mu} - \delta^\nu_\mu \mathcal{L}\right)}$$ $= 0$

$$\boxed{\frac{\delta T^\nu{}_\mu}{\delta x^\nu} = 0}$$

Energy-Momentum conservation

$$\boxed{T^\nu{}_\mu \equiv \frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\, q^\alpha_{,\mu} - \delta^\nu_\mu \mathcal{L}}$$

These are the quantities of the overall system that are conserved!

↳ it is the Legendre transform of $\mathcal{L}$ ⇒ like Hamiltonian $\left(H = \frac{\delta L}{\delta \dot{q}^\alpha}\dot{q}^\alpha - L\right)$

$\Rightarrow$ $\boxed{\partial_\nu T^\nu{}_\mu = 0}$ represents energly-momentum conservation for a system!
$\downarrow$
(field)

Note:

- $T^{\mu\nu}$ is not unique

in fact: $\tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{\partial}{\partial x^\sigma}\psi^{\mu\nu\sigma}$ with $\psi^{\mu\nu\sigma} = -\psi^{\mu\overline{\sigma\nu}}$ ← antisymmetric with respect to σ, ν

still satisfies $\partial_\nu \tilde{T}^{\mu\nu} = 0$ because of the antysimmetry of ψ: $\frac{\partial^2 \psi^{\mu\nu}}{\partial x^\nu \partial x^\alpha} \overset{\text{identically}}{=} 0$

- In general it is not diagonal but you can make it so with a proper ψ

$\Rightarrow$ by imposing this specific additional condition, you make it unique:

- Set the 4-angular momentum tensor to be of the form

$$M^{\alpha\beta} = \int \left(x^\alpha \underset{\uparrow}{dp^\beta} - x^\beta \underset{\uparrow}{dp^\alpha}\right) = \frac{1}{c}\int \left(x^\alpha T^{\beta\sigma} - x^\beta T^{\alpha\sigma}\right) dS_\sigma$$

- Conservation of $M^{\alpha\beta}$: $\frac{\partial}{\partial x^\sigma}\left(x^\alpha T^{\beta\sigma} - x^\beta T^{\alpha\sigma}\right) \overset{!}{=} 0$ $\left(\frac{\partial x^\alpha}{\partial x^\sigma} = \delta^\alpha_\sigma\right)$

$$\Rightarrow \delta^\alpha_\sigma T^{\beta\sigma} - \delta^\beta_\sigma T^{\alpha\sigma} = T^{\beta\alpha} - T^{\alpha\beta} = 0$$

$\Rightarrow$ By wanting the 4-angular momentum with that form, you automaticully impos $T^{\mu\nu}$ to be symmetric

Energy density

$$(T^{\mu\nu}) = \begin{pmatrix} \boxed{\mathcal{E}} & \boxed{S_x/c \quad S_y/c \quad S_z/c} \\ - & \sigma_{xx} \quad \sigma_{xy} \quad \sigma_{xz} \\ - & - \quad \sigma_{yy} \quad \sigma_{yz} \\ - & - \quad - \quad \sigma_{zz} \end{pmatrix}$$

energy density flux $\vec{S}$ $\left[\frac{d\mathcal{E}}{dt\,dA}\right]$

momentum density

stress tensor σ^{ij}
momentum density flux
$\left[\frac{dP}{dt\,dA}\right] \Rightarrow$ pressure

$\boxed{\text{Meaning of } T^{\mu\nu}}$

$$\left(H = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \right)$$

- $\boxed{T^{00}} = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} =$ Hamiltonian density (Legendre transform of $\mathcal{L}$)

  $\overset{!}{=} \boxed{\mathcal{E}}$ energy density $\left(\dot{q} = \frac{\partial q}{\partial t} \right)$ $q \to q^\mu$ for brevity

- $\boxed{\partial_\nu T^{\mu\nu} = 0}$ 4 equations: (a) $\frac{\partial T^{00}}{c\partial t} + \partial_j T^{0j} = 0$ (b) $\frac{\partial T^{i0}}{c\partial t} + \partial_j T^{ij} = 0$

like $\frac{\partial \rho}{\partial t} + \partial_i j^i = 0$ <u>they are continuity equations</u>

$\Rightarrow$ cT^{0j} and cT^{ij} are the fluxes of T^{00} and T^{i0} respectively!

(a) $\frac{\partial \mathcal{E}}{\partial t} + c\partial_j T^{0j} = 0 \longrightarrow \boxed{cT^{0j} \equiv \bar{S}}$ <u>energy density flux</u> $\left(\begin{matrix}\text{Pointing}\\ \text{vector}\end{matrix}\right)$

A closer look: $\int (a)\, dV$

$$\int_V \frac{\partial \mathcal{E}}{\partial t} dV = \underbrace{\boxed{\frac{\partial}{\partial t}\int_V \mathcal{E}\, dV}}_{\left(\begin{matrix}\text{time variation of}\\ \text{energy within } V\end{matrix}\right)} = -c\int_V \partial_j T^{0j} dV \overset{\text{Gauss t.}}{=} \underbrace{\boxed{-c\oint T^{0j} df_j}}_{\left(\begin{matrix}\text{energy flux}\\ \text{across surface } \bar{f} \text{ defining } V\end{matrix}\right)}$$

$$\frac{E}{s} = [cT^{0j}]\, m^2 \to [cT^{0j}] = \frac{E}{m^2 s}$$

Analogy to one particle

$$\left(\frac{T^{0\mu}}{c}\right) = \left(\frac{\mathcal{E}}{c}, \frac{\bar{S}}{c^2}\right) \longleftrightarrow (P^\mu) = \left(\frac{E}{c}, \bar{P}\right) \quad \text{4-momentum of a particle}$$

$\Rightarrow$ $\frac{T^{0\mu}}{c}$ represents the 4-momentum <u>density</u> of the system

recall: particle $\bar{P} = \frac{E}{c^2}\bar{v}$ $c^2\bar{P} = E\bar{v}$ $\left[E\frac{m}{s}\right]$ energy flux

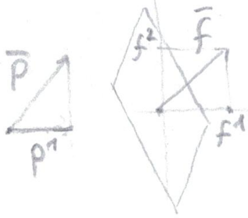
system $p^i = \frac{T^{0i}}{c}$ $c^2 p^i = cT^{0i} = S^i$ energy density flux

$$p^i = \frac{S^i}{c^2} = \frac{T^{0i}}{c}$$

(b) $\longrightarrow$

$\downarrow$

(b) $\frac{\partial p^i}{\partial t} + c\,\partial_j T^{ij} = 0$ $\boxed{cT^{ij} \equiv c\sigma^{ij}}$ momentum density flux



(momentum density (i-th component)
across surface (j-th component) per unit time)

σ^{ij} = stress tensor

A closer look : $\int (b)\, dV$

$$\underbrace{\frac{\partial}{\partial t}\int_V p^i\, dV}_{\text{(time variation of momentum)}} = \underbrace{-c\oint T^{ij}\, dF_j}_{\text{(flux of momentum across surface } \bar{f}\text{)}}$$

$\Downarrow$

force

$N = [cT^{ij}]\, m^2 \rightarrow [cT^{ij}] = \frac{N}{m^2}$ Pressure

"transfer of momentum"

All together

$cT^{00} = \varepsilon$ energy density

$cT^{0i} = \bar{S}$ " " flux = 3-momentum

$cT^{ij} = c\sigma^{ij}$ momentum density flux (pressure)

$$(T^{\mu\nu}) = \begin{pmatrix} \varepsilon & \boxed{\bar{S}/c} \\ \boxed{\frac{\bar{S}}{c}} & \boxed{\sigma^{ij}} \end{pmatrix}$$

$\boxed{\partial_\mu T^{\mu\nu} = 0}$ analogous to energy-momentum conservation

$\uparrow$

generalization of 4-momentum, it allows for more degrees of freedom

- Note: Despite $T^{\mu\nu}$ is not unique for a given system, The 4-momentum p^μ is unique!

Energy-Momentum tensor: summary

- Look for conserved quantities

– $S = \int L\, dt = \int \mathcal{L}\, dt\, dV = \frac{1}{c}\int \mathcal{L}\, d\Omega \;\rightarrow\; \delta S = 0 \;\rightarrow\; \frac{\delta}{\delta x^\nu}\left(\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\right) - \frac{\delta \mathcal{L}}{\delta q^\alpha} = 0$

– $\frac{\delta \mathcal{L}}{\delta x^\mu} = 0 \Rightarrow$ identify conserved quantities $\quad \mathcal{L}(q^\alpha, q^\alpha_{,\nu}) \quad q(x^\mu)$

E.-L. eq.

$$\frac{\delta \mathcal{L}}{\delta x^\mu} = \frac{\delta \mathcal{L}}{\delta q^\alpha}\,\delta_\mu q^\alpha + \frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\,\delta_\mu q^\alpha_{,\nu} \Rightarrow \delta_\nu\underbrace{\left(\frac{\delta \mathcal{L}}{\delta q_{,\nu}}\, q_{,\mu} - \delta^\nu_\mu \mathcal{L}\right)}_{\boxed{\delta_\nu T^\nu{}_\mu = 0}} = 0$$

Energy-Momentum conservation !

– Meaning:

$\boxed{T^{00}} = \dot q \frac{\delta \mathcal{L}}{\delta \dot q} - \mathcal{L} = \mathcal{E}$ Hamiltonian density (Legendre transform of $\mathcal{L}$)

$\left(\frac{T^{0\mu}}{c}\right) = \left(\frac{\mathcal{E}}{c}, \frac{\vec{S}}{c^2}\right) \leftrightarrow (p^\mu) = \left(\frac{E}{c}, \vec{p}\right) \Rightarrow \frac{T^{0\mu}}{c}$ = 4-momentum density of system

$= \left(\frac{T^{00}}{c}, \frac{T^{0i}}{c}\right)$

$p^i = \frac{E}{c^2} v^i \Rightarrow c^2 p^i = E v^i \; [E \frac{m}{s}]$ energy flux

$p^i = \frac{T^{0i}}{c} \Rightarrow c^2 p^i = c T^{0i} \equiv S^i$ " density flux

$\hookrightarrow \frac{S^i}{c^2} = p^i$

$$\delta_\nu T^{\mu\nu} = 0 \begin{cases} \mu = 0: & \frac{\delta T^{00}}{c\,\delta t} + \delta_i T^{0i} = 0 \quad \frac{\delta \mathcal{E}}{\delta t} + c\,\delta_i T^{0i} = 0 \Rightarrow c T^{0i} \equiv S^i \\ \mu = i: & \frac{\delta T^{i0}}{c\,\delta t} + \delta_j T^{ij} = 0 \quad \overset{p^i}{\frac{\delta S^i}{c^2 \delta t}} + \delta_j T^{ij} = 0 \Rightarrow T^{ij} = \sigma^{ij} \end{cases}$$

energy density flux

momentum " "

$$(T^{\mu\nu}) = \begin{pmatrix} \mathcal{E} & S_x/c & S_y/c & S_z/c \\ \cdot & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \cdot & \cdot & \sigma_{yy} & \sigma_{yz} \\ \cdot & \cdot & \cdot & \sigma_{zz} \end{pmatrix}$$

$\vec{S}$ = energy-density flux $\left[\frac{dE}{dt\,dA}\right]$

σ_{ij} = stress tensor $\left[\frac{dP}{dt\,dA}\right]$ "pressure"

– Energy-Momentum conservation: $\delta_\mu T^{\mu\nu} = 0$

Energy-momentum tensor of E.M. field: summary

- Specify $\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$ in $T^{\mu}_{\ \nu} = \frac{\delta \mathcal{L}}{\delta(\delta_\mu q)} \delta_\nu q - \mathcal{L} \delta^{\mu}_{\nu}$

$\hookrightarrow$ Energy density $\varepsilon \equiv T^{00} = \frac{\bar{E}^2 + \bar{B}^2}{8\pi}$

- Energy momentum conservation $\delta_\nu T^{\mu\nu} = 0$ (4 eq. $\begin{cases} \text{energy} \\ \text{momentum} \end{cases}$)

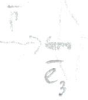
- Energy current flux : $\mu = 0$ $\frac{\delta \varepsilon}{\delta t} + \bar{\nabla} \bar{S} = 0$ $\bar{S} = \frac{c}{4\pi} \bar{E} \times \bar{B}$ Pointing vect.

- Explicit E, B from retarded potentials

$\hookrightarrow$ Larmor formula :
$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c(1 - \bar{e}\bar{\beta})^5} \left| \bar{e} \times \left[(\bar{e} - \bar{\beta}) \times \dot{\bar{\beta}} \right] \right|^2$$
$$P = \frac{2e^2}{3c} \gamma^6 \left[\dot{\beta}^2 - (\bar{\beta} \times \dot{\bar{\beta}})^2 \right] \underset{\beta \ll 1}{\approx} \frac{2e^2}{3c} \dot{\beta}^2$$

- Electromagnetic spectra : $\int \frac{dP}{dt} dt = \int \frac{d\hat{P}}{d\omega} d\omega$ (Parseval)

- Momentum of intensity ("things you can measure")
  - Intensity $I = \frac{cU}{4\pi} = \frac{|\bar{S}|}{4\pi} \leftarrow |\bar{S}| = I \cdot 4\pi$
  - Flux $F \equiv \int I \cos\theta \, d\Omega$
  - Radiation pressure $dP_i = T_{ij} \, dA^j$ $P_{Rad} = \frac{1}{2\pi} \left(\frac{\bar{E}^2}{2} - E_3^2 \right)$



$$= \int \frac{I}{c} \cos^2\theta \, d\Omega$$

Energy-Momentum Tensor of the E.M. field

- Field is described by its Lagrangian density $\mathcal{L}(A^\mu, A^\mu_{,\nu})$ (closed system)
- Apply Legendre transformation: Lagrangian → Hamiltonian
- Identify conserved quantity $\boxed{\partial_\nu T^{\mu\nu}=0}$

$\left.\right\}\ T^{\mu\nu}$

$$\boxed{T^\mu{}_\nu = A^\alpha_{,\nu}\frac{\partial\mathcal{L}}{\partial A^\alpha_{,\mu}} - \delta^\mu_\nu \mathcal{L}}$$

$A^\alpha(x^\beta)$ are the generalized coordinates q^α

plug in

- Consider only the field (i.e. $j^\mu=0$): $\boxed{\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}\ \text{"}+0\text{"}}$

$$\boxed{T^\mu{}_\nu = \frac{1}{4\pi}\left(F^{\mu\lambda}F_{\nu\lambda} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\delta^\mu_\nu\right)}$$

- Express $F^{\mu\nu}$ in terms of $\bar{B}$ and $\bar{E}$

$$\boxed{T^{00} = \frac{\bar{E}^2+\bar{B}^2}{8\pi} \equiv \mathcal{E}}$$

energy density of the field

- Look at energy-momentum conservation explicitely: $\partial_\nu T^{\mu\nu}=0$

$\mu=0$: $\frac{\partial\mathcal{E}}{\partial t} + \bar{\nabla}\bar{S} = 0$ "continuity eq. for energy density"

Change in energy density in unit time

Energy-current density: Flux, i

$$cT^{0i} = \boxed{\bar{S} = \frac{c}{4\pi}\bar{E}\times\bar{B}}$$

plug

Pointing vector

Flux: energy received by an observer per unit time and area

- $\frac{\bar{S}}{c}\left[\frac{dE}{\cancel{dt}\,dA}\cdot\frac{\cancel{dt}}{dl}\right]$ momentum current

⑤

- Express $\bar E, \bar B$ produced by a charge particle ⇒ retarded potentials (for observe

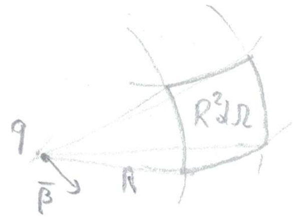
$$\Rightarrow \bar S = \frac{c}{4\pi} \overset{\downarrow}{\bar E} \times \overset{\downarrow}{\bar B} = \frac{q^2}{\underbrace{4\pi R^2} c(1-\bar e\bar\beta)^6} \left| \bar e \times \left[(\bar e - \bar\beta) \times \dot{\bar\beta} \right] \right|^2 \bar e$$

Flux in the direction $\bar e$

$$\Rightarrow dE = \bar S \bar e \underbrace{R^2 d\Omega}_{\text{area}} dt$$

Emitted energy

↖ use retarded times: $dt = (1-\bar e\bar\beta)dt'$



$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{q^2}{4\pi c(1-\bar e\bar\beta)^5} \left| \bar e \times \left[(\bar e - \bar\beta) \times \dot{\bar\beta} \right] \right|^2} \; ⊛$$

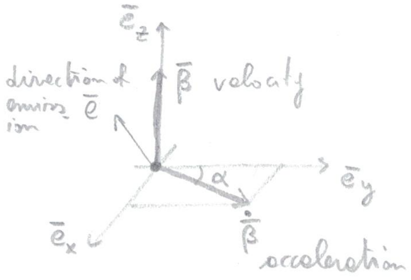
Power per solid angle

$$\left(P \equiv \frac{dE}{dt} \right)$$

Larmor formula !!!
(relativistic)

↑ acceleration!

- Integrate $\int ⊛ d\Omega$ ⇒ total power



- use: $\bar a \times (\bar b \times \bar c) = (\bar a \bar c)\bar b - (\bar a \bar b)\bar c$
- use: convenient coordinates:

$$\bar\beta = \beta(0,0,1)^T$$

$$\dot{\bar\beta} = \dot\beta(\sin\alpha\, \bar e_x + \cos\alpha\, \bar e_y)^T$$

$$\bar e = (\cos\phi \sin\theta,\ \sin\phi \sin\theta,\ \cos\theta)^T$$

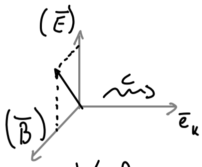
$$\Rightarrow \boxed{P = \frac{2e^2}{3c}\gamma^6 \left[\dot\beta^2 - (\bar\beta \times \dot{\bar\beta})^2 \right]}$$

look! γ^6 !

look! $\dot{\bar\beta}$!

$$\boxed{P \approx \frac{2e^2}{3c}\dot\beta^2} \quad (\beta \ll 1)$$

Momentum of intensity



Electromagnetic wave $\Rightarrow$ $\left.\begin{array}{l}\bar E\,\bar B = 0\\ E^2 - \bar B^2 = 0\end{array}\right\}$ $\Rightarrow$ $\bar E \times \bar B = |\bar E|^2\, \bar e_k$ $\forall$ frame !

(use invariants of the field)

$\uparrow$ direction of propagation of the e.m. wave

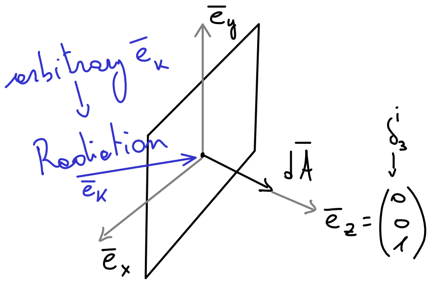
- $\boxed{T^{00}}$: $\varepsilon = U = \dfrac{\bar E^2 + \overset{\downarrow}{\bar B}{}^2}{8\pi} = \dfrac{\bar E^2}{4\pi}$ energy density

- $\boxed{T^{0i}}$: $\bar S = \dfrac{c}{4\pi}\,\bar E \times \bar B = \dfrac{c\,|\bar E|^2}{4\pi}\,\bar e_k = c\,U\bar e_k$ pointing vector (energy density flux

Intensity: $I \equiv \dfrac{|\bar S|}{4\pi} = \dfrac{cU}{4\pi}$ $\Rightarrow$ $\bar S = 4\pi I\, \bar e_k$

Luminosity: $$L \equiv \int \overset{\swarrow}{\bar S}\, d\overset{\swarrow}{\bar A} = 4\pi \int \underbrace{I\,\bar e_k}_{\bar S}\, \underbrace{\bar e_\Omega R^2 d\Omega}_{d\bar A} = 4\pi R^2 \underbrace{\int I \cos\theta\, d\Omega}_{F \equiv L/4\pi R^2}$$

- $\boxed{T^{ij} = \sigma^{ij}}$: Maxwell stress-energy tensor (momentum current density)



Force on the surface $d\bar A$:

$$dF_i = \overset{\downarrow}{T_{ij}}\, dA^j = T_{ij}\, dA\, \delta^j_3 = \overset{\downarrow}{T_{i3}}\, dA = \frac{1}{2\pi}\left(\frac{\bar E^2}{2}\delta_{i3} - E_i E_3\right) dA$$

$$\Rightarrow \boxed{P_i^{RAD} \equiv \frac{dF_i}{dA} = \frac{1}{2\pi}\left(\frac{\bar E^2}{2}\delta_{i3} - E_i E_3\right)} \quad \left\{\begin{array}{ll} P_1^{RAD} = 0 = P_2^{RAD} & \text{components along the surface} \\ P_3^{RAD} = \dfrac{1}{2\pi}\left(\dfrac{\bar E^2}{2} - E_3^2\right) & \text{component orthogonal to the surface}\end{array}\right.$$

- Example : Isotropic radiation

$\rightarrow$ Average over angles

$E_3 = E\cos\theta$

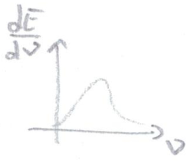
$$p^{RAD} = \langle P_3^{RAD}\rangle = \frac{1}{2\pi}\left(\frac{\langle \bar E^2\rangle}{2} - \langle E_3^2\rangle\right) = \frac{1}{2\pi}\overset{\downarrow}{\langle E_3^2\rangle} = \frac{1}{2\pi}\cdot \overset{U}{\boxed{\frac{1}{4\pi}\int E^2}}\cos^2\theta\, d\Omega = \frac{1}{2\pi}\int U\cos^2\theta\, d\Omega = \frac{U}{3}$$

$\uparrow$ $\langle \bar E^2\rangle = 3\langle E_3^2\rangle$ because of isotropy

$\nearrow$ same result obtained by considering light as a quantum gas !

Spectrum of the emission from a charged particle

- Spectrum: energy per unit frequency ω , (eg. Plank spectrum



)

- We have Larmor f.: $$\frac{dP}{d\Omega}=\frac{d^2E}{d\Omega\, dt}=\frac{q^2}{4\pi c(1-\bar{e}\bar{\beta})^6}\left|\bar{e}\times(\bar{e}-\bar{\beta})\times\dot{\bar{\beta}}\right|^2$$

Here dt, not dt'! (observer) you convert to dt' later

$$\Rightarrow \frac{dE}{d\Omega}=\int\left|\frac{dP}{d\Omega}\right|dt$$

$$=\frac{q^2}{4\pi c}\int_{-\infty}^{\infty}\Bigg|\overbrace{\frac{\bar{e}\times[(\bar{e}-\bar{\beta})\times\dot{\bar{\beta}}]}{(1-\bar{e}\bar{\beta})^3}}^{f(t)}\Bigg|^2 dt$$

$$=\frac{q^2}{4\pi c}\int_{-\infty}^{\infty}|f(t)|^2dt=\frac{q^2}{4\pi c}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2d\omega=\int_{-\infty}^{\infty}\frac{d^2E}{d\Omega\, d\omega}d\omega$$

$$\Rightarrow \boxed{\frac{d^2E}{d\Omega d\omega}=\frac{q^2}{8\pi^2 c}|\hat{f}(\omega)|^2}$$

Parseval theorem: $\int_{-\infty}^{\infty}|f(t)|^2dt=\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\frac{d\omega}{2\pi}$

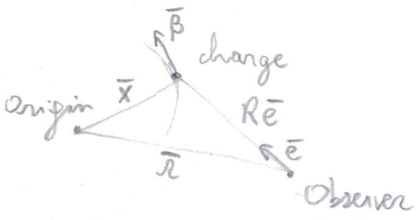
Fourier transform $\hat{f}(\omega)\equiv\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt$

(*) →

- Use retarded times in fourier transf.

$$t'=t-\frac{R}{c} \Rightarrow dt=(1-\bar{e}\bar{\beta})dt'$$

- Decompose distances



$$\bar{R}\equiv\bar{r}-\bar{x}$$

$$R=\bar{R}\bar{e}=\bar{r}\bar{e}-\bar{x}\bar{e}$$

→ gives a constant phase factor $e^{i\omega\bar{e}\bar{r}}$ to be ignored

$$\Rightarrow \boxed{\frac{d^2E}{d\Omega d\omega}=\frac{q^2\omega^2}{8\pi^2 c}\left|\int_{-\infty}^{\infty}[\bar{e}\times(\bar{e}\times\bar{\beta})]e^{-i\omega(t'-\frac{\bar{e}\bar{x}}{c})}dt'\right|^2}$$

spectrum per unit angle

- Non relativistic case integrated over all angles

$$E=\int P dt=\frac{2q^2}{3c^3}\int|\bar{a}(t)|^2dt=\frac{2q^2}{3c^3}\int|\hat{\bar{a}}(\omega)|^2\frac{d\omega}{2\pi} \Rightarrow \boxed{\frac{dE}{d\omega}=\frac{q^2}{3\pi c^3}|\hat{\bar{a}}|^2=\frac{q^2\omega^4}{3\pi c^3}|\hat{\bar{x}}|^2}$$

Spectrum

$$\left(\text{here: } \bar{a}=\ddot{\bar{x}}=-\int\frac{d\omega}{2\pi}\omega^2\hat{\bar{x}}(\omega)e^{i\omega t}\right)\longrightarrow$$

(*)

observer

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt \, \frac{\bar{e}\times[(\bar{e}-\bar{\beta})\times\dot{\bar{\beta}}]}{(1-\bar{e}\bar{\beta})^3} e^{-i\omega t}$$

Account for retardation: $t' = t - \frac{R}{c}$

(!)

$$= \int_{-\infty}^{\infty} dt' \, \underbrace{\frac{\bar{e}\times[(\bar{e}-\bar{\beta})\times\dot{\bar{\beta}}]}{(1-\bar{e}\bar{\beta})^2}} e^{-i\omega(t'+\frac{R}{c})}$$

$$= \int_{-\infty}^{\infty} dt' \frac{d}{dt'}\left[\frac{\bar{e}\times(\bar{e}\times\bar{\beta})}{(1-\bar{e}\bar{\beta})}\right] e^{-i\omega(t'+\frac{R}{c})}$$

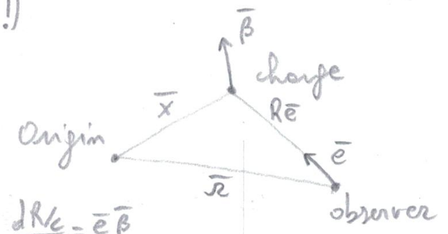
integrate by part

$$= \underbrace{\frac{\bar{e}\times(\bar{e}\times\bar{\beta})}{(1-\bar{e}\bar{\beta})} e^{-i\omega(t'+\frac{R}{c})}\Bigg|_{-\infty}^{\infty}}_{(1)} - \int_{-\infty}^{\infty} \frac{\bar{e}\times(\bar{e}\times\bar{\beta})}{(1-\bar{e}\bar{\beta})} \frac{d}{dt'} e^{-i\omega(t'+\frac{R}{c})} dt'$$

(1) = 0 ; contribution from distant (infinite) past and future can be neglecte

(!)

$$(2) = -i\omega\left(1+\frac{dR/c}{dt'}\right) e^{-i\omega(t'+\frac{R}{c})}$$



decompose distances : (!)

$\bar{R} = \bar{r} - \bar{x}$ $\quad R = \bar{e}\bar{R} = \underbrace{\bar{e}\bar{r}} - \bar{e}\bar{x}$ $\quad \frac{dR/c}{dt'} = \bar{e}\bar{\beta}$

gives a const phase factor $e^{i\omega\bar{e}\bar{r}}$ that can be ignored

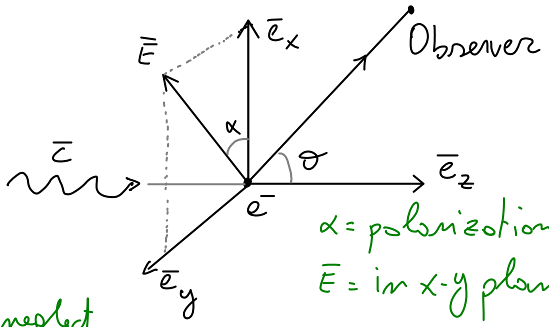
$$= -i\omega(1-\bar{e}\bar{\beta})\, e^{-i\omega(t'-\frac{\bar{e}\bar{x}}{c})}$$

$$= i\omega \int_{-\infty}^{\infty} [\bar{e}\times(\bar{e}\times\bar{\beta})] e^{-i\omega(t'-\frac{\bar{e}\bar{x}}{c})} dt'$$

Thomson cross section

- e^- : non relativistic, $q=-e$
- monocromatic polarized wave

=> scattering of incoming radiation



α = polarization angle

$\bar{E}$ = in x-y plane

1) Lorentz force: $m\overset{\checkmark}{\ddot{x}} = mc\dot{\bar{\beta}} = -e\bar{E} - \overbrace{\frac{e}{c}\bar{v}\times\bar{B}}^{\text{neglect}}$ $\quad \beta \ll 1 \Rightarrow |\bar{E}| \gg \frac{|\bar{v}\times\bar{B}|}{c}$

2) Larmor formula: $\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}|\bar{e}\times\dot{\bar{\beta}}|^2 = \frac{e^4}{4\pi m^2c^3}|\bar{e}\times\bar{E}|^2$

3) Set coordinates: $\bar{e} = \begin{pmatrix}\sin\vartheta\\0\\\cos\vartheta\end{pmatrix}$ $\quad \bar{E} = E\begin{pmatrix}\cos\alpha\\\sin\alpha\\0\end{pmatrix}$ $\Rightarrow$ $\bar{e}\times\bar{E} = E\begin{pmatrix}-\sin\alpha\cos\vartheta\\\cos\alpha\cos\vartheta\\\sin\alpha\sin\vartheta\end{pmatrix}$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{e^4E^2}{4\pi m^2c^3}\left(1-\sin^2\vartheta\cos^2\alpha\right)}$$

4) Differential cross section: $\quad P = |\bar{S}|\sigma \quad$ Power = flux · area

$\frac{dP}{d\Omega} = |\overset{\checkmark}{\bar{S}}|\frac{d\sigma}{d\Omega}$ $\quad \bar{S} = \frac{c}{4\pi}|\bar{E}|^2\bar{e}_z$ $\quad \Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \underbrace{\frac{e^4}{m_e^2c^4}}_{r_e^2\sim 10^{-28}m^2}\left(1-\sin^2\vartheta\cos^2\alpha\right)}$

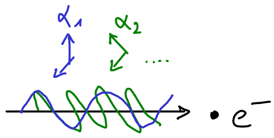
are associated to the e^-

other meaning of r_e: $\quad \underset{\substack{\uparrow\\\text{electric energy at } r_e}}{\frac{e^2}{r_e}} \overset{!}{=} \underset{\substack{\uparrow\\\text{rest energy}}}{m_ec^2} \Rightarrow r_e = \frac{e^2}{m_ec^2}$ classical radious of e^-

5) Assume unpolarized light

=> incoming waves have random angles

=> do average over all polarization angles α



$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_\vartheta = \frac{r_e^2}{2\pi}\int_0^{2\pi}\left(1-\sin^2\vartheta\cos^2\alpha\right)d\alpha = \frac{r_e^2}{2}\left(1+\cos^2\vartheta\right)$$

6) Total cross section

$$\sigma = \int\left\langle\frac{d\sigma}{d\Omega}\right\rangle_\alpha d\Omega = \cancel{2}\pi\frac{r_e^2}{\cancel{2}}\int_{-1}^{1}\left(1+\cos^2\vartheta\right)d\cos\vartheta = \frac{8\pi}{3}r_e^2 \equiv \sigma_T \approx 10^{-29}m^2$$

- A particle perspective, photon: (events rate) = (particles flux) $\cdot$ (cross section)

- E.g. Ions: $\sigma_i \propto r_i^2 = \left(\frac{q_i^2}{m_i c^2}\right)^2 = \left(\frac{Z^2 e^2}{m_i c^2}\right)^2 = \frac{Z^4}{m_i^2} r_e$

Z = charge number
m_i = mass of ion

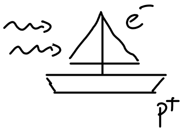
Hidrogen: ion = p^+

$$\left.\begin{array}{l} - m_i = 2000\, m_e \quad Z = 1 \\ - m_e \end{array}\right\} \Rightarrow \quad F_{RAD} = \frac{S}{c}\sigma \qquad F_{e^-} = (2000)^2 F_i \quad (!)$$

↳ the reason: p^+ "oscillate much less" => less re-emission = scattered

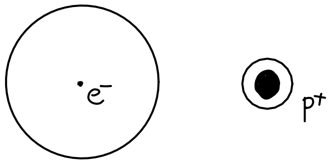
- Are then the e^- decouple from the p^+? No!

$e^- \longleftrightarrow p^+$ => coupled
coulomb



$\Rightarrow \quad m = m_p + m_e$ not m_e

this is the "inertial mass" of the plasma
σ_T (electron) is the cross section to be used



$$\frac{|\vec{E}|}{E} = \sin^2\alpha\cos^2\theta + \cos^2\alpha\cos^2\theta + \sin^2\alpha\sin^2\theta = \sin^2\alpha\overbrace{(\cos^2\theta + \sin^2\theta)}^{1} + \cos^2\alpha\cos^2\theta$$
$$= 1 - \cos^2\alpha + \cos^2\alpha\cos^2\theta = 1 - \cos^2\alpha(1 - \cos^2\theta) = 1 - \cos^2\alpha\sin^2\theta$$

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_\theta = \frac{r_e}{2\pi}\int_0^{2\pi}(1 - \sin^2\theta\cos^2\alpha)\,d\alpha = \frac{r_e}{2\pi}\left[\alpha - \frac{1}{2}\sin^2\alpha(\alpha + \sin\theta\cos\theta)\right]\Bigg|_0^{2\pi}$$
$$= \frac{r_e}{\cancel{2\pi}}\left(\cancel{2\pi} - \frac{1}{2}\sin^2\theta\cdot\cancel{2\pi}\right) = \frac{r_e}{2}(2 - 1 + \cos^2\theta) = \frac{r_e}{2}(1 + \cos^2\theta)$$

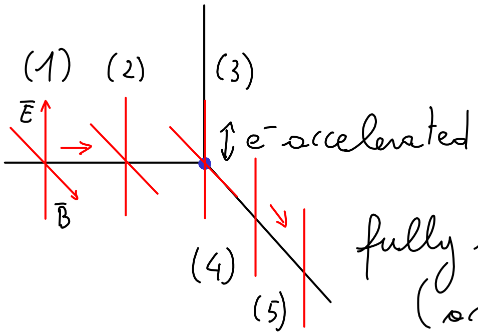
- Thomson scattering and polarization

- One scattering event does not give raise to polarized light

- But for an incoming radiation field with a quadrupole momentum you get polarized scattered light !

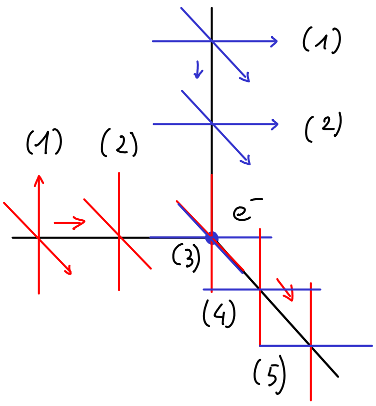
↑ not an isotropic field

A) 1 monocromatic wave



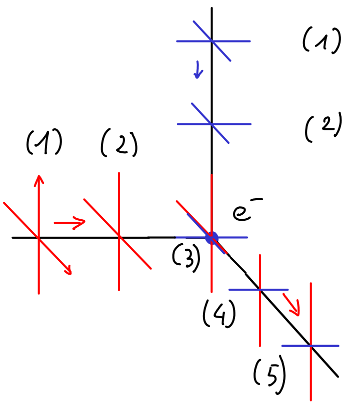
fully linearly polarized radiation (according to scattering angle)

B) Isotropic radiation field



NO polarization

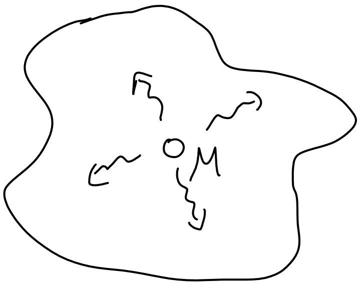
C) Radiation field with quadruple momentum in intensity distribution



linear polarization (not full)

E.g. Cosmic Microwave Background radiation (CMB)

Eddington luminosity, matter accreation



- Ionized gas "cloud"
- Luminouse spherically symmetric source : "star"

- Momentum current density

$$\frac{\bar S}{c} = \frac{|\bar E|^2}{4\pi}\bar e = \frac{L}{4\pi R^2 c}$$

spherical symmetry

luminosity: $L = \int \bar S \, d\bar A = 4\pi R^2 \frac{c}{4\pi}|\bar E|^2$

- Radiation pressure on electron in plasma :

$$\bar F_{RAD} = \bar P_{RAD}\,\sigma_T = \frac{L}{4\pi R^2 c}\sigma_T \bar e$$

- Gravity:

$$F_G = -\frac{GMm}{R^2}\bar e$$

=> Eddington luminosity :

$$\bar F_{RAD}(L_e) + \bar F_G \overset{!}{=} 0 \Rightarrow \frac{L_e}{4\pi \cancel{R^2} c}\sigma_T \bar e = \frac{GMm}{\cancel{R^2}}\bar e \Rightarrow \boxed{L_e = \frac{4\pi G M m c}{\sigma_T}}$$

(1) ions

(2) electrons

(1) neglect mass of e^-

(2) " cross section of ions

Remember: in astrophysics
~ 70% Hydrogen
~ 30% Helium
~ Rest Metals
} depends on environment

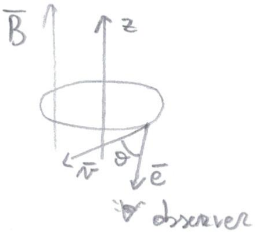
Synchrotron radiation

- **e^- in $\bar{B}$ only**

- $\underline{\bar{E}=0}$ $\Rightarrow$ $\gamma = \text{const}$ ← $\bar{B}$ does not make work

  $\Rightarrow$ $\frac{d}{dt}(\gamma\bar{v}) = -\frac{e}{mc}\,\bar{v}\times\bar{B}$ ← Lorentz force

- $\underline{\bar{B}\equiv B\bar{e}_z}$ $\Rightarrow$



$$\gamma\dot{v}_z = 0 \qquad \ddot{v}_z = 0$$

$$\gamma\dot{v}_y = -\frac{e}{mc}v_x B \qquad \ddot{v}_y = -\frac{eB}{\gamma mc}\overset{\downarrow}{\dot{v}_x} = \left(\frac{eB}{\gamma mc}\right)^2 v_y$$

$$\gamma\dot{v}_x = -\frac{e}{mc}v_y B \qquad \ddot{v}_x = -\frac{eB}{\gamma mc}\dot{v}_y = \left(\frac{eB}{\gamma mc}\right)^2 v_x$$

$\Rightarrow$ $\ddot{v}_i = \left(\frac{eB}{\gamma mc}\right)^2 v_i \quad (i=1,2)$ $\quad \boxed{\omega_L \equiv \frac{eB}{\gamma mc}} = \frac{18\,\text{Hz}}{\gamma}\left(\frac{B}{\mu G}\right)\left(\frac{m_e}{m}\right)$ Larmor frequency

- **Emission (Larmor f.)**

  Set orbit on a fixed plane $v_z = 0$, $\bar{\beta}\perp\dot{\bar{\beta}}$ (circular orbit)

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c(1-\bar{\beta}\bar{e})^5}\left|\bar{e}\times\left((\bar{e}\times\bar{\beta})\times\dot{\bar{\beta}}\right)\right|^2$$

- **Account for beaming** (relat. $e^- \Rightarrow \beta \dot{\approx} 1$)

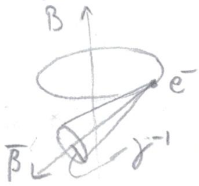
  - Maximum intensity at $\theta=0$: $\quad (1-\bar{\beta}\bar{e})^{-1} \overset{\theta=0}{=} (1-\beta)^{-1} \overset{(1-\beta^2)=(1-\beta)(1+\beta)}{=} \frac{\overset{\approx 2}{1+\beta}}{1-\beta^2} \approx 2\gamma$

  - Within which θ is the intensity $> \frac{1}{2}$ max?

$$(1-\bar{\beta}\bar{e})^{-1} = (1-\beta\cos\theta)^{-1} \overset{!}{\gtrsim} \frac{1}{2}\underset{\uparrow\theta=0}{(1-\beta)^{-1}} \Rightarrow \left.\begin{array}{l}\cos\theta \geqslant 2-\frac{1}{\beta} \quad (\approx 1 \Rightarrow \theta \text{ is small}) \\ \cos\theta \approx 1-\frac{\theta^2}{2}\end{array}\right\} \theta \leq \sqrt{2\left(\frac{1-\beta}{\underset{\approx 1}{\beta}}\right)} \simeq$$

$$\dot{\simeq}\sqrt{2\,\frac{1-\beta^2}{\underset{\approx 2}{1+\beta}}} \simeq \gamma^{-1} \quad (!)$$

$\Rightarrow$ $\boxed{\text{"all" emission on the orbital plane}}$



← Relat. beaming

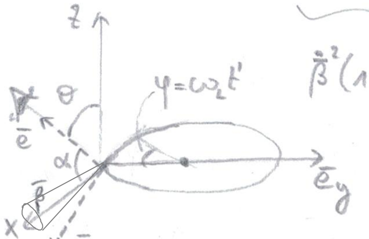
Total emitted power

$$U_B = \frac{B^2}{8\pi} \qquad \sigma_T = \frac{8\pi}{3}\frac{e^4}{m_e^2 c^4}$$

$$\boxed{P} = \frac{2e^2}{3c}\gamma^6\Big[\dot{\bar\beta}^2 - \underbrace{(\overset{\text{orthogonal}}{\bar\beta \times \dot{\bar\beta}})^2}_{=(\bar\beta\dot{\bar\beta})^2}\Big] = \frac{2e^2}{3c}\gamma^4\dot\beta^2 = \frac{2e^2}{3c}\gamma^4\left(\frac{eB}{\gamma mc}\right)^2 = \boxed{c\gamma^2\sigma_T U_B}$$

$$\dot{\bar\beta}^2(1-\bar\beta^2) = \dot\beta^2\gamma^{-2}$$

$$\dot\beta^2 = \frac{\dot{\bar v}^2}{c^2} = \frac{1}{c^2}(\dot v_x^2 + \dot v_y^2) = \frac{1}{c^2}\omega_L^2(v_x^2+v_y^2) = \omega_L^2\beta^2 \overset{\beta\simeq 1}{\approx} \omega_L^2$$



$\bar e_\perp \equiv \bar e \times \bar e_y$: orthogonal to $\bar e$ and $\bar e_y$

line of sight observer orbit

$$\bar e = (\sin\theta, 0, \cos\theta)^T \qquad \bar x = (\sin\varphi, \cos\varphi, 0)^T$$

$$\bar\beta = \beta(\cos\varphi, -\sin\varphi, 0)^T \qquad \dot{\bar\beta} = \beta\dot\varphi(-\sin\varphi, -\cos\varphi, 0)^T$$

Spectrum

- Spectral emission of a charged particle

$$\boxed{\frac{d^2E}{d\Omega\, d\omega}} = \frac{e^2\omega^2}{4\pi c}\left|\int_{-\infty}^{\infty}[\bar e \times(\bar e \times \bar\beta)]\, e^{-i\omega\left(t' - \frac{\bar e \bar x}{c}\right)} dt'\right|^2$$

- Here:

$$\boxed{\underset{a}{\bar e} \times(\underset{b}{\bar e} \times \underset{c}{\bar\beta})} = (\underset{a}{\bar e}\underset{c}{\bar\beta})\underset{b}{\bar e} - \underbrace{(\underset{a}{\bar e}\underset{b}{\bar e})}_{=1}\underset{c}{\bar\beta} = \beta\begin{pmatrix} -\cos\varphi\cos^2\theta \\ \sin\varphi \\ \cos\varphi\sin\theta\cos\theta \end{pmatrix} \qquad \bar e_y = \begin{pmatrix} 0 \\ \sin\varphi \\ 0 \end{pmatrix}$$

① Decompose in 2 basis : $\bar e_y \qquad \bar e_\perp \equiv \bar e \times \bar e_y = \begin{pmatrix} -\cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$ [to evaluate polarization]

$$= \boxed{\beta\cos\varphi\cos\theta\, \bar e_\perp + \sin\varphi\, \bar e_y}$$

② Approximate trigonometry and "θ → α" + only small α are relevant (because of beaming)

$$\sin\theta = \sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha \cong 1 - \frac{\alpha^2}{2} \qquad \cos\theta = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha \approx \alpha$$

$$\sin\varphi = \sin(\omega_L t') \approx \omega_L t' - \frac{\omega_L^3 t'^3}{6} \qquad \cos\varphi \approx 1 - \frac{\omega_L^2 t'}{2}$$

$$\bar e \times(\bar e \times \bar\beta) = \boxed{\beta\alpha\, \bar e_\perp + \omega_L t'\, \bar e_y}$$ (1° order only)

3) Now fourier phase

$$\boxed{\psi} \equiv \omega\left(t' - \frac{\bar{e}\bar{x}}{c}\right) = \omega\left(t' - \frac{x \sin\theta \sin\varphi}{c}\right) \qquad \left(\beta = \frac{x\omega_L}{c} \quad \beta \simeq 1 \quad \Rightarrow x = \frac{c\beta}{\omega_L}\right)$$

$$= \omega\left(t' - \frac{\beta \sin\theta \sin\varphi}{\omega_L}\right)$$

$$\left(\beta \simeq 1 \qquad 1-\beta^2 = (1+\beta)(1-\beta) \simeq 2(1-\beta), \quad 1-\beta^2 = \gamma^{-2} \quad \Rightarrow \left(1-\beta \simeq \frac{1}{2\gamma^2}\right)\right)$$

$$\simeq \omega\left[t' - \frac{\beta}{\cancel{\omega_L}}\left(1-\frac{\alpha^2}{2}\right)\cancel{\omega_L} t'\left(1-\frac{\omega_L^2 t'^2}{6}\right)\right]$$

$$= \omega t'\left[1-\beta\left(1-\frac{\alpha^2}{2}\right)\left(1-\frac{\omega_L^2 t'^2}{6}\right)\right]$$

$$= \omega t'\left(1-\beta+\beta\frac{\omega_L^2 t'^2}{6}+\beta\frac{\alpha^2}{2}-\beta\frac{\alpha}{2}\frac{\omega_L^2 t'}{6}\right) \qquad \left(\alpha\,\omega_L^2 t'^2 = \alpha \overset{\text{small}}{\varphi^2} \simeq 0\right)$$

$$= \frac{\omega t'}{2}\left(\frac{1}{\gamma^2}+\frac{\omega_L^2 t'^2}{3}+\alpha^2\right) \quad \leftarrow \text{extract } \left(\frac{1}{\gamma^2}+\alpha^2\right)$$

$$= \frac{\omega t'}{2}\left(\frac{1}{\gamma^2}+\alpha^2\right)\left(1+\frac{\tau^2}{3}\right) \qquad \tau \equiv \frac{\omega_L t'}{(\gamma^{-2}+\alpha^2)^{1/2}} \rightarrow (*) \; dt'$$

$$= \frac{\omega}{2}\frac{\tau(\gamma^{-2}+\alpha^2)^{1/2}}{\omega_L}(\gamma^{-2}+\alpha^2)\left(1+\frac{\tau^2}{3}\right) \qquad \xi \equiv \frac{\omega}{3\omega_L}(\gamma^{-2}+\alpha^2)^{3/2}$$

$$= \boxed{\frac{3\xi\tau}{2}\left(1+\frac{\tau^2}{3}\right)}$$

τ : rescaled retarded time

ξ : adimensional frequency

~~$\Rightarrow$ Plug ψ and τ and $\bar{e}\times(\bar{e}\times\bar{\beta}) = \beta\alpha\,\bar{e}_\perp + \omega_L t'\,\bar{e}_y$ in spectrum~~

$$\boxed{\frac{d^2E}{d\Omega\, d\omega}} = \frac{e^2\omega^2}{4\pi c}\left|\hat{g}_\perp(\omega)\,\bar{e}_\perp + \hat{g}_\parallel(\omega)\,\bar{e}_y\right|^2$$

$$\hat{g}_\perp(\omega) = \int_{-\infty}^{\infty}\beta\alpha\, e^{-i\psi}\left(\frac{(\gamma^{-2}+\alpha^2)^{1/2}}{\omega_L}d\tau\right) = \frac{\overset{\simeq 1}{\beta}\alpha(\gamma^{-2}+\alpha^2)^{1/2}}{\omega_L}\underbrace{\int_{-\infty}^{\infty} e^{-i\psi}d\tau}_{\equiv \frac{2}{\sqrt{3}}K_{1/3}(\xi)}$$

$$\hat{g}_\parallel(\omega) = \int_{-\infty}^{\infty}\cancel{\omega_L}\frac{(\gamma^{-2}+\alpha^2)^{1/2}}{\cancel{\omega_L}}\tau e^{-i\psi}\frac{(\gamma^{-2}+\alpha^2)^{1/2}}{\cancel{\omega_L}}d\tau = \frac{(\gamma^{-2}+\alpha^2)}{\omega_L}\underbrace{\int_{-\infty}^{\infty}\tau e^{-i\psi}d\tau}_{\equiv \frac{2i}{\sqrt{3}}K_{2/3}(\xi)}$$

~~Bessel functions~~

$$= \frac{e^2\omega^2}{4\pi c}\left|\frac{(\gamma^{-2}+\alpha^2)}{\omega_L}\frac{2}{\sqrt{3}}\left[\frac{\alpha}{(\gamma^{-2}+\alpha^2)^{1/2}}K_{1/3}(\xi)\,\bar{e}_\perp + i\,K_{2/3}(\xi)\right]\right|^2$$

$$= \boxed{\frac{e^2\omega^2}{3\pi c\,\omega_L^2}(\gamma^{-2}+\alpha^2)^2\left(\underbrace{\frac{\alpha^2}{(\gamma^{-2}+\alpha^2)}K_{1/3}^2(\xi)}_{(\perp)} + \underbrace{K_{2/3}^2(\xi)}_{(\parallel)\;(!)}\right)}$$

- ~~Because of beaming, only component on orbital plan is relevant ($|\theta| < \gamma^{-1}$)~~
  ~~$\Rightarrow$ Strong polarization!!~~

- Plug $\bar e\times(\bar e\times\bar\beta)$, $\psi(\xi,\tau)$, τ :

$$\frac{dE^2}{d\Omega\, d\omega} = \frac{e^2\omega^2}{4\pi c}\left|\int_{-\infty}^{\infty}\overbrace{\left[\bar e\times(\bar e\times\bar\beta)\right]}^{\downarrow}\, e^{-i\omega\overbrace{\left(t'-\frac{\bar e\bar x}{c}\right)}^{\downarrow -\psi}}\;\overset{\tau}{\overset{\downarrow}{dt'}}\right|^2$$

$$= \frac{e^2\omega^2}{4\pi c}\left|\int_{-\infty}^{\infty}\left(\beta\alpha\,\bar e_\perp + \omega_L\,\overset{t'(\tau)}{\overset{\downarrow}{t'}}\,\bar e_y\right) e^{-i\frac{3\xi\tau}{2}\left(1+\frac{\tau^2}{3}\right)}\;\overbrace{\frac{(\gamma^{-2}+\alpha^2)^{1/2}}{\omega_L}\,d\tau}^{dt'}\right|^2$$

$$= \frac{e^2\omega^2}{4\pi c}\left|\int_{-\infty}^{\infty}\beta\alpha\,\bar e_\perp\, e^{-i\psi}\frac{(\gamma^{-2}+\alpha^2)^{1/2}}{\omega_L}d\tau + \int_{-\infty}^{\infty}\cancel{\omega_L}\,\frac{(\gamma^{-2}+\alpha^2)^{1/2}}{\cancel{\omega_L}}\,\tau\,\bar e_y\, e^{-i\psi}\,\frac{(\gamma^{-2}+\alpha^2)^{1/2}}{\omega_L}d\tau\right|^2$$

$$= \frac{e^2\omega^2}{4\pi c}\left|\underbrace{\beta\alpha\frac{(\gamma^{-2}+\alpha^2)^{1/2}}{\omega_L}\underbrace{\int_{-\infty}^{\infty}e^{-i\psi(\tau)}d\tau}_{\equiv\frac{2}{\sqrt{3}}K_{1/3}(\xi)}}_{\hat g_\perp}\cdot\bar e_\perp + \underbrace{\frac{(\gamma^{-2}+\alpha^2)}{\omega_L}\underbrace{\int_{-\infty}^{\infty}\tau\, e^{-i\psi(\tau)}d\tau}_{\equiv\frac{2i}{\sqrt{3}}K_{2/3}(\xi)}}_{\hat g_y}\cdot\bar e_y\right|^2$$

Bessel functions

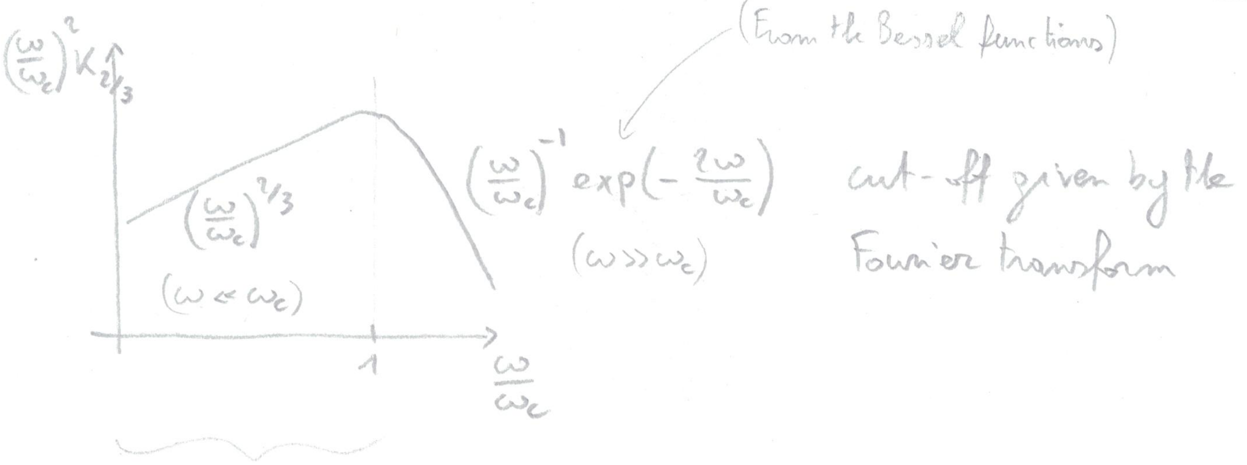
$$= \frac{e^2\omega^2}{4\pi c}\left|\hat g_\perp(\omega)\,\bar e_\perp + \hat g_y(\omega)\,\bar e_y\right|^2$$

$$= \frac{e^2\omega^2}{3\pi c\,\omega_L^2}(\gamma^{-2}+\alpha^2)^{1/2}\left(\underbrace{\frac{\alpha^2}{(\gamma^{-2}+\alpha^2)}K^2_{1/3}(\xi)}_{(\perp)} + \underbrace{K^2_{2/3}(\xi)}_{(/\!/)}\right)\quad \textcircled{!}$$

- Because of beaming, only component on orbital plane (//) is relevant $(|\theta|\lesssim\gamma^{-1})$

$\Rightarrow$ Strong linear polarization (!!)

- For $\alpha = 0$ (Plane of the orbit) => ⊥ component $= 0$



- very broad spectrum because it comes from the F. transform of a short puls (because of beaming)

  - Pulse in t space ⟹ broad in ω space

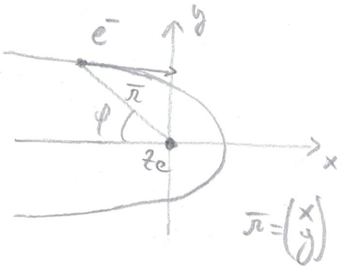
$$(\alpha=0) \quad \frac{d^2E}{d\Omega d\omega} = \frac{e^2\omega^2}{3\pi c \omega_L^2}\gamma^{-4} K^2_{2/3}(\xi) \qquad \xi = \frac{\omega}{3\omega_L}\gamma^{-3}$$

$$= \frac{e^2\omega^2 3}{\pi c \omega_c^2}\gamma^2 K^2_{2/3}\left(\frac{\omega}{\omega_c}\right)$$

$$\omega_c \equiv 3\omega_L\gamma^3 \overset{!}{=} \frac{3eB\gamma^2}{mc}$$

cut off frequency from Bessel function

Bremsstrahlung



- Scatter of e^- with ion Ze, $\beta \ll 1$

- Keplerian motion, we want spectrum
  $\left(\frac{dE}{d\omega} = \frac{2e^2\omega^4}{3c^3}|\hat{r}|^2\right)$

① Trajectory of e^-

- $$E_{e^-} = \frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{Ze^2}{r} \quad (1)$$
  Kinetic + ang. momentum + electric pot.
  (total energy of e^-)

- Potential in $V \propto r^{-1}$ ⇒ Keplerian motion : Hyperbolic orbit

  $\Rightarrow r(\varphi) = p(1+\varepsilon\cos\varphi)^{-1}$ $\quad p = \ell^2/(Ze^2 m)$ orbital parameter

  $$E = \frac{Ze^2}{2a} \Longleftarrow \left[\begin{array}{ll} \varepsilon^2 = 1 + 2Ep/(Ze^2) & \text{eccentricity} \\ p = a(\varepsilon^2 - 1) & a = \text{"semi major axis"} \end{array} \right.$$

- Combine with ① $\Rightarrow$ $$\dot{r}^2 = \frac{2Ze^2}{mr^2}\left[\frac{r^2}{2a} + r - \frac{a(\varepsilon^2-1)}{2}\right] \quad (2)$$

- Convenient to express it as a function of eccentric anomaly ψ

  $$r(\psi) = a(\varepsilon\cosh\psi - 1) \Rightarrow \dot{r}(\psi) = a\varepsilon\sinh(\psi)\dot{\psi} \quad (3)$$

- Combine (2) and (3) $\Rightarrow$ $$\dot{\psi}^2 = \frac{Ze^2}{ma^3(\varepsilon\cosh\psi - 1)}$$

- Relate $\psi \to$ time : $\dot{\psi} = \frac{d\psi}{dt} \Rightarrow dt = \frac{d\psi}{\dot{\psi}}$

  $$\Rightarrow t = \int_0^t dt' = \int_0^\psi \frac{d\psi'}{\dot{\psi}} = \tau(\varepsilon\sinh\psi - \psi) \quad \text{with } \tau = \sqrt{\frac{ma^3}{Ze^2}}$$

- Relate to energy at ∞ (Boundary conditions) $\quad E_\infty = \frac{mv_\infty^2}{2} = \frac{Ze^2}{2a} \Rightarrow a = \frac{Ze^2}{mv_\infty^2}$

② Going after the spectrum ⇒ Fourier space

(!)

$$\frac{dE}{d\omega} = \frac{2e^2\omega^4}{3c^3}|\hat{x}|^2$$

$$\left[\begin{array}{l} t = \tau(\varepsilon \sinh\psi - \psi) \\ r(\psi) = a(\varepsilon\cosh\psi - 1) \\ \cos\varphi = \dfrac{\varepsilon - \cosh\psi}{\varepsilon\cosh\psi - 1} \end{array}\right] \longrightarrow \begin{cases} x(\psi) = r\cos\varphi = a(\varepsilon - \cosh\psi) \\ y(\psi) = \sqrt{r^2 - x^2} = a\sqrt{\varepsilon^2 - 1}\,\sinh\psi \end{cases} \qquad \bar{r} \equiv \begin{pmatrix} x \\ y \end{pmatrix}$$

$$dt = \tau(\varepsilon\cosh\psi - 1)\,d\psi$$

$$\left(r(\varphi) = r(\psi) \Rightarrow P(1 + \varepsilon\underset{=}{\cos\varphi}) = a(\varepsilon\cosh\psi - 1) \right)$$

$$e^{i\omega t} = e^{i\omega\tau(\varepsilon\sinh\psi - \psi)}$$

③a Easier to compute F. transf. of $\bar{v}$ instead of $\bar{x}$

plug $x(\psi)$

$$\hat{v} = -i\omega\hat{r} \Rightarrow \hat{x}(\omega) = \frac{i}{\omega}\hat{\dot{x}} = \frac{i}{\omega}\int_{-\infty}^{+\infty} dt\,\underset{\uparrow\ \frac{dx}{d\psi}\frac{d\psi}{dt}}{\dot{x}}\,e^{i\omega t} = \frac{i}{\omega}\int d\psi\,\frac{dx}{d\psi}\,e^{i\omega t(\psi)}$$

$$\Rightarrow \hat{x}(\omega) = -\frac{ia}{\omega}\int d\psi\,\sinh(\psi)\,e^{i\omega\tau(\varepsilon\sinh\psi - \psi)} = \frac{\pi a}{\omega}\,\overset{\frac{1}{i\nu}\frac{d H_{i\nu}^{(1)}}{d\varepsilon}}{H_{i\nu}^{(1)'}}(i\nu\varepsilon)$$

$$\hat{y}(\omega) = \frac{ia\sqrt{\varepsilon^2 - 1}}{\omega}\int d\psi\,\cosh\psi\,e^{i\omega\tau(\varepsilon\sinh\psi - \psi)} = -\frac{\pi a\sqrt{\varepsilon^2 - 1}}{\omega\varepsilon}\,H_{i\nu}^{(1)}(i\nu\varepsilon)$$

③b Spectrum for Larmor formula

(Hankel functions H of order ν and derivative H', ν = ωτ)

$$\frac{dE}{d\omega} = \frac{2e^2\omega^4}{3c^3}|\hat{r}|^2 \qquad (\beta \ll 1)$$

$$= \frac{2\pi^2 a e^2\omega^2}{3c}\left\{\left[H_{i\nu}^{(1)'}(i\nu\varepsilon)\right]^2 - \left(1 - \frac{1}{\varepsilon^2}\right)\left[H_{i\nu}^{(1)}(i\nu\varepsilon)\right]^2\right\}$$

just because of properties of the H func.

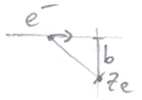
$$= \boxed{-i\,\frac{2\pi^2 a^2 e^2\omega}{3\tau\varepsilon c^3}\,\frac{d}{d\varepsilon}\left[\varepsilon H_{i\nu}^{(1)}(i\nu\varepsilon)\,H_{i\nu}^{(1)'}(i\nu\varepsilon)\right]}$$

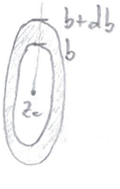
H : Hankel func.

immaginary ⇒ $iH_{i\nu}^{(1)} \in \mathbb{R}$, $H_{i\nu}^{(1)'} \in \mathbb{R}$

↳ For 1 e⁻ and 1 Z⁺ : go for a population now…

④ Combine contribution (integrate) of all e^- (fixed v_∞) and "add" to all ions eZ





of e^- within b and b+db

$$dN_e = \underbrace{n_e}_{\text{(number density)}} \underbrace{(2\pi b\, db)}_{(dA)} \underbrace{v_\infty dt}_{(dx)} = 2\pi n_e v_\infty a^2 \varepsilon\, d\varepsilon\, dt$$

$b \to \ell \to \varepsilon$

$\uparrow$ ang. momentum

from $p = \frac{\ell^2}{Ze^2 m}$ $\quad E = \frac{1}{2} m v_\infty^2$

$\varepsilon^2 = 1 + \frac{2Ep}{Ze^2}$ $\quad \ell = b m v_\infty$

$$\Rightarrow \boxed{\varepsilon^2 = 1 + \frac{b^2}{a^2}}$$

$$\boxed{\frac{d^3E}{d\omega\, dt\, dV}} = \int_0^1 \left(\frac{dE}{d\omega}\right) \cdot (2\pi n_e v_\infty a^2 \varepsilon\, d\varepsilon) \cdot (n_i)$$

← for each ion → Spectrum of a portion of ◻ dV ionized gas

$\equiv j(\omega)$

Emissivity

$$\left(\frac{dE}{d\omega}\right) = -i \frac{2\pi^2 a^2 e^2 \omega}{3\tau \varepsilon c^3} \frac{d}{d\varepsilon}\left[\varepsilon H_{i\nu}^{(1)}(i\nu\varepsilon)\, H_{i\nu}^{(1)\prime}(i\nu\varepsilon)\right]$$

$$a = \frac{Ze^2}{m v_\infty^2} \quad \tau = \frac{a}{v_\infty}$$

$$= \boxed{i \frac{4\pi^3 Z^2 e^6 n_e n_i}{3 m^2 c^3 v_\infty} \left(\frac{Ze^2 \omega}{m v_\infty^3}\right) H_{i\nu}^{(1)}(i\nu)\, H_{i\nu}^{(1)}(i\nu)}$$

$$\approx \frac{16\pi Z^2 e^6 n_i n_e}{3 m^2 c^3 v_\infty} \cdot \underbrace{\begin{cases} \ln\left(\frac{2 m v_\infty^3}{Z e^2 \omega}\right) & \omega \ll \tau^{-1} \quad \text{(almost constant)} \\ \pi/\sqrt{3} & \omega \gg \tau^{-1} \quad \text{(constant)} \end{cases}}_{\equiv \sqrt{3}\, g_{ff}(v_\infty, \omega) \text{ : Gaunt factor } (\approx 1)}$$

⑤ Average over all velocities v_∞ ⟶

⑤ Average over all velocities v_∞

- $$\left\langle \frac{g_{ff}(v_\infty,\omega)}{v_\infty} \right\rangle_v \equiv \bar{g}_{ff}(\omega)\langle v^{-1}\rangle_v = \bar{g}_{ff}(\omega)\int_{v_{min}(\omega)}^{\infty} v^{-1}\, \underset{\uparrow \text{ velocity distribution}}{P(v_\infty)}\, dv_\infty$$

To emit an photon, you need enogh energy from the incoming e⁻

$$\frac{m_e v_\infty^2}{2} \geq \hbar\omega \Rightarrow v_\infty \geq v_{min} = \sqrt{\frac{2\hbar\omega}{m_e}}$$

- Assume a thermalized population ⇒ Maxwell-Boltzmann distrib.

$$P(v_\infty)\,dv_\infty = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v_\infty^2 \exp\left(-\frac{m v_\infty^2}{2k_B T}\right) dv_\infty$$

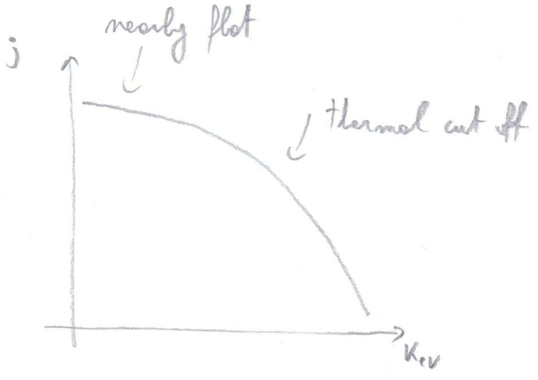
$$\Rightarrow \boxed{j(\omega) = \frac{16\pi^2}{3\sqrt{3}}\,\frac{Z^2 e^6 n_i n_e}{m^2 c^3}\,\bar{g}_{ff}(\omega)\sqrt{\frac{2m}{\pi k_B T}}\,\exp\left(-\frac{\hbar\omega}{k_B T}\right)}$$

(• For 1e⁻ → spectrum flat

• The shape of the spectrum is given by the population)

↓

(Brehmsstrahlung is good to measure T, n_e)



Radiation damping: summary

- E.M.ism is linear theory => incomplete
  - no back reaction

- Trick: derive an effective force: $E = \int \underset{\substack{\uparrow \\ \text{Larmor}}}{P}\, dt = -\int \bar{F}_{\text{rad}}\, d\bar{s} = \text{Work}$

- Eq. of motion of e^- in $\bar{B}$ => damped oscillator (Lorentz + damping) forces
  - damping factor $\gamma = \frac{2e^2\omega_0^2}{3mc^3}$
  - eq. of motion => $\dot{\bar{\beta}}$

  => emission (Larmor) P

  => cross section $\sigma = \frac{P}{|\bar{S}|}$

- Energy transfer $e^- \longleftrightarrow$ field
  - e^- rest frame => $F_L' = q\bar{E}' + 0$
  - $\langle \bar{B} \rangle = 0 = \langle \bar{E} \rangle$ => $\langle \ddot{\bar{x}} \rangle \propto \langle |\bar{E}'| \rangle = 0$

    But. $\langle \ddot{\bar{x}}^2 \rangle \propto \langle |\bar{E}'|^2 \rangle \neq 0$ $\Rightarrow$ emission (Larmor) $\Downarrow$ P_{emitted}
  - Λ back to observer

- $P_{\text{obs}} = |\bar{S}|\,\sigma_T$

$\Rightarrow P = P_{\text{em}} - P_{\text{obs}} > 0 \Rightarrow e^- \xrightarrow{\text{energy}} \text{field}$ Power transfer

$\Rightarrow \frac{dE}{dt} = mc^2 \frac{d\gamma}{dt}$ $\longleftarrow$ Back reaction on e^-

Radiation damping

- So far we ignored the effect of charges on the E.M. field
- Electromagnetism is a linear theory ⇒ can not deal with back-reaction
- it is incomplete!

∘ E.g. $\boxed{e^- \text{ in } \bar{B}}$ ⇒ circular orbit ⇒ emission

but in E.D. no loss of energy for e^- ⇒ emit forever: wrong!

↑ Back reaction of emitted photons on motion

- We can "fix" it with a "trick": effective force

$$E = \int_{-\tau/2}^{\tau/2} \overset{⊛}{P}\, dt = \overbrace{-\int_{\bar{s}_0}^{\bar{s}_1} \bar{F}_{rad}\, d\bar{s}}^{\text{work}}$$

- emitted energy in $\Delta t = \tau$
- effective force

$$⊛ \quad P = \frac{2q^2}{3c}\gamma^6 \underbrace{\left[\dot{\beta}^2 - (\bar{\beta}\times\dot{\bar{\beta}})^2\right]}_{\cdot\, \dot{\beta}^2 \left[\dot{\bar{e}}_\beta^2 - (\bar{\beta}\times\dot{\bar{e}}_\beta)^2\right]}$$

Larmor F. integrated over all angles

Plug P to solve integral

- Note: P is a homogeneous function: $f(a x) = a^k f(x) \quad a \in \mathbb{R}$ const.

⇒ Use Euler's theorem for homog. func.: $\bar{x}\dfrac{df(x)}{d\bar{x}} = k f(\bar{x})$

Here: $k = 2 \quad \underset{\text{variable}}{\underline{\bar{x} = \dot{\bar{\beta}}}} \quad a = \dot{\beta} \quad f \to P \Rightarrow \dot{\bar{\beta}}\dfrac{dP(\dot{\bar{\beta}})}{d\dot{\bar{\beta}}} = 2P(\dot{\bar{\beta}})$

$$-\int \bar{F}_{rad}\, d\bar{s} = \int \frac{1}{2}\dot{\bar{\beta}}\frac{dP(\dot{\bar{\beta}})}{d\dot{\bar{\beta}}}\, dt = \underbrace{\frac{1}{2}\bar{\beta}\frac{dP(\dot{\bar{\beta}})}{d\dot{\bar{\beta}}}\Bigg|}_{\text{Boundary term (omit)}} - \frac{1}{2}\int \bar{\beta}\frac{d}{dt}\frac{dP(\dot{\bar{\beta}})}{d\dot{\bar{\beta}}}\, dt \qquad \bar{\beta}\, dt = \frac{1}{c}\frac{d\bar{s}}{dt}dt = \frac{d\bar{s}}{c}$$

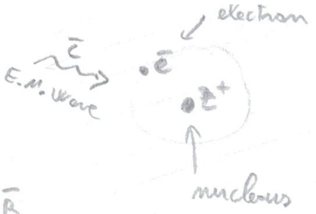
$$= -\int \underbrace{\frac{1}{2c}\frac{d^2P(\dot{\bar{\beta}})}{dt\, d\dot{\bar{\beta}}}}_{\equiv \bar{F}_{rad}} d\bar{s}\,.$$

$\ddot{\bar{\beta}} = \dfrac{\dddot{\bar{x}}}{c}$ very rare!

$$\Rightarrow \boxed{\bar{F}_{rad} = \frac{1}{2c}\frac{d}{dt}\frac{dP(\dot{\bar{\beta}})}{\delta\dot{\bar{\beta}}}}$$

- Non relat. $P = \dfrac{2q^2}{3c}\dot{\beta}^2 \Rightarrow \boxed{\bar{F}_{rad} = \dfrac{2q^2}{3c^2}\ddot{\bar{\beta}}}$

Example: bound electron + incoming e.m. wave



→ oscillator eq. $\ddot{\bar{x}} = -\omega_0^2 \bar{x} \quad \overset{\frac{1}{c}\frac{\partial}{\partial t}}{\Longrightarrow} \quad \underbrace{\ddot{\bar{\beta}}}_{\dddot{\beta}} = -\omega_0^2 \bar{\beta}$

→ Includ source term :

$$\begin{cases} \bar{F}_{rad} = -\dfrac{2e^2}{3c^2}\,\omega_0^2 \overbrace{\dot{\bar{\beta}}}^{\ddot{\beta}}{}^{\swarrow \frac{\dot{\bar{x}}}{c}} & \text{damping force} \\ \bar{F}_L = -\dfrac{e}{c}\,\bar{E}_0\, e^{i\omega t} & \text{Lorentz force (incoming EM wave)} \end{cases}$$

$$\Rightarrow \ddot{\bar{x}} + \omega_0^2 \bar{x} = \frac{\bar{F}_R + \bar{F}_L}{m} = -\underbrace{\frac{2e^2\omega_0^2}{3c^2 m}\frac{\dot{\bar{x}}}{c}}_{\gamma} - \frac{e}{cm}\bar{E}_0 e^{i\omega t}$$

$\left(\bar{F} = q\bar{E} + \frac{q}{c}\bar{v}\times\bar{B}\right)$! negligible

$$\ddot{\bar{x}} + \gamma\dot{\bar{x}} + \omega_0^2\bar{x} = -\frac{e}{mc}\bar{E}_0 e^{i\omega t} \qquad \gamma = \frac{2e^2\omega_0^2}{3mc^3}$$

damping factor (not the γ factor!)

- Solution of damped oscillator: $\bar{x} = (-) \Rightarrow \underset{\approx}{\dot{\bar{\beta}}} = -\omega^2\frac{\bar{x}}{c} = \frac{\omega^2 e \bar{E}_0 e^{i\omega t}}{mc(\omega_0^2 - \omega - i\gamma\omega^3)}$

$$\Rightarrow P = \frac{2e^2}{3c}|\dot{\bar{\beta}}|^2 = \frac{2e^4}{3m^2c^3}\bar{E}_0^2\frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \omega^6\gamma^2}$$

(non relativistic Larmor formula)

↑ Plug

incoming energy current $|\bar{S}| = c\dfrac{\bar{E}_0^2}{4\pi}$

▭ Scattering cross section : $\boxed{\sigma \equiv \frac{P}{|\bar{S}|} = \sigma_T \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \omega^6\gamma^2}}$ $\quad \sigma_T = \dfrac{8\pi e^4}{3m^2c^4}$

$P = \frac{dE}{dt} = \underbrace{\frac{dE}{dt\,dA}}_{|\bar{S}|} dA \Rightarrow dA = \frac{P}{|\bar{S}|}$

limits

- $\omega \gg \omega_0,\ \omega \gg \gamma$: $\frac{\omega^4}{\omega^4} \approx 1 \Rightarrow \sigma \approx \sigma_T$ (Thompson) "binding forces are irrelevant"
- $\omega \ll \omega_0,\ \omega \gg \gamma$: $\frac{\omega^4}{\omega_0^4} \Rightarrow \sigma \approx \sigma_T\left(\frac{\omega}{\omega_0}\right)^4$ (Rayleigh scattering)
- $\omega \approx \omega_0$: $\omega^2 - \omega_0^2 = (\omega-\omega_0)(\omega+\omega_0) \approx (\omega-\omega_0)\,2\omega_0$

$$\Rightarrow \sigma \approx \sigma_T\frac{\omega_0^2}{4(\omega-\omega_0^2)+\gamma^2} = \frac{2\pi^2e^2}{mc}\underbrace{\left[\frac{\gamma/(2\pi)}{(\omega-\omega_0)^2+\gamma^2/4}\right]}_{}$$

(Lorentz profile)

Free e^- ⟷ radiation field, net energy transfer

- Field on particle:
  - P absorbed by the particle: $P_{abs} = |\bar{S}|\, d\sigma_T = c\, U \sigma_T$
  - Lorentz force on particle : $\bar{F}_L' = e\bar{E}' + \frac{e}{c}\underbrace{\bar{v}\times\bar{B}'}_{=0}$ $\Rightarrow \ddot{x}' = c\dot{\beta}' = \frac{e\bar{E}'}{m}$ e^- rest frame
- Particle on field:
  - Emitted power $P_{em}' = \frac{2e^2}{3c}|\dot{\beta}'|^2$
- Average effect $\langle\dots\rangle$ over time $T \gg \omega^{-1}$

on average + and – cancel out

$\langle\bar{B}'\rangle = 0 \quad \langle\bar{E}'\rangle = 0 \quad \langle E_i' B_j'\rangle = 0$ wave ⟹ oscillation around zero

$\langle|\ddot{x}'|^2\rangle = \frac{e\langle E'^2\rangle}{m} > 0$ (!) ⟹ net emission

- Link E' (e^- rest frame) to the energy density U "seen" by the observer

→ Lorentz transform $\bar{E}' \to E$ $\quad E_x' = \gamma(E_x + \beta E_y) \quad E_y' = \gamma(E_y - \beta E_x) \quad E_z' = E_z$

$$|E'| = E_x'^2 + E_y'^2 + E_z'^2 = \gamma^2(1+\beta^2)(E_x^2+E_y^2) + E_z^2 \longleftarrow \langle E_i^2\rangle = \frac{4\pi U}{3}$$

$\left(T^{00}:\ U = \frac{E^2}{4\pi}\right)$ ↖ isotropy

$$= \frac{4\pi U}{3}\left[\gamma^2(1+\beta^2)\cdot 2 + 1\right]$$

$$= \frac{4\pi U}{3}\gamma^2\left(2 + 2\beta^2 + \gamma^{-2}\right) \qquad \gamma^{-2} = 1 - \beta^2$$

$dE' = \gamma dE$
$dt' = \gamma dt$

$$\Rightarrow \quad P_{em}' = \frac{8\pi e^2}{3m^2c^3}U\gamma^2\left(1+\frac{\beta^2}{3}\right) = c\,U\sigma_T\gamma^2\left(1+\frac{\beta^2}{3}\right) = P_{em}$$

Because $P = \frac{dE}{dt}$ is Lorentz invariant

- Net energy transfer e^- ⟷ field

$$\boxed{P_{net} = P_{em} - P_{abs} = \frac{4}{3}\beta^2\gamma^2 c\, U\sigma_T > 0}$$

e^- gives energy to the field!

- Compute the time scale of the back reaction

$E = \gamma m c^2$ electron

$\gamma^2 = (1-\beta^2)^{-1} \Rightarrow \beta^2 = 1-\gamma^{-2} = \gamma^{-2}(\gamma^2-1)$

$$\frac{dE}{dt} = \boxed{mc^2 \frac{d\gamma}{dt}} \overset{!}{=} -\frac{4}{3}\beta^2\gamma^2 c U \sigma_T = \boxed{-\frac{4}{3}(\gamma^2-1) c U \sigma_T}$$

- Solve for γ by separating the variables

$$d\gamma = -\frac{4U\sigma_T}{3mc}(\gamma^2-1)\,dt \qquad \boxed{\tau \equiv \frac{3mc}{4U\sigma_T}} \leftarrow \text{time scale of decay}$$

$$\Rightarrow \gamma(t) \rightarrow \boxed{\beta(t) = \frac{2\exp(-t/\tau)}{1+\exp(-\frac{2t}{\tau})}} \quad \text{exponential decay of velocity}$$

if $m \uparrow$ ⇒ "more inertia" ⇒ $\tau \uparrow$ less damped

if $U \uparrow$ ⇒ "more emission" ⇒ $\tau \downarrow$ more damped

- Typical path length: $\lambda \approx c\tau = \frac{3mc^2}{4U\sigma_T}$ (for initially relativistic e^-)

after λ the e^- has lost all its kinetical energy

Compton scattering

- In electrodynamics : light as E.M. wave

  when light hits on electron ⇒ Lorentz force

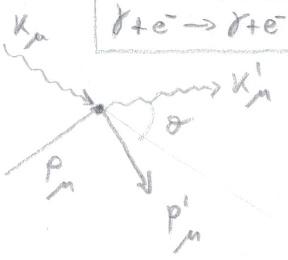
  No momentum transfer to e^- ⇐ $\begin{cases} \bar{B}: \text{does not make any work} \\ \bar{E}: \overset{e^-}{\text{oscillates}} \Rightarrow \text{on average no work done on the } e^- \end{cases}$

  ⇒ Fix with radiation damping

- Consider light as stream of photons γ

  scattering is a collision between particles

$$\boxed{\gamma + e^- \rightarrow \gamma + e^-}$$



$$\begin{cases} \text{photon :} & p_\gamma^\mu = \hbar K^\mu = \hbar \frac{\omega}{c}(1, \bar{e})^T \\ \text{electron :} & p_{e^-}^\mu = \left(\frac{E}{c}, \bar{p}\right) = \gamma m (c, \bar{v}) \end{cases}$$

- Energy-momentum conservation

Before : after

$$\boxed{p_{e^-}^\mu + p_\gamma^\mu = p_{e^-}^{\prime\mu} + p_\gamma^{\prime\mu}} \Rightarrow \mu = 0 : \quad E + \hbar\omega = E' + \hbar\omega' \qquad (1)$$

$$\mu = i : \quad \bar{p} + \frac{\hbar\omega}{c}\bar{e} = \bar{p}' + \frac{\hbar\omega'}{c}\bar{e}' \qquad (2)$$

$$(2): \quad c\bar{p}' = c\bar{p} + \hbar(\omega\bar{e} - \omega'\bar{e}')$$

$$(c\bar{p}')^2 = c^2\bar{p}^2 + \hbar^2(\omega\bar{e} - \omega'\bar{e}')^2 + 2c\bar{p}\hbar(\omega\bar{e} - \omega'\bar{e}')$$

$$E'^2 = c^2\bar{p}'^2 + m_0^2c^4 = \underbrace{c^2\bar{p}^2}_{E^2 - \cancel{m^2c^4}} + \hbar^2(\omega\bar{e} - \omega'\bar{e}')^2 + 2c\bar{p}\hbar(\omega\bar{e} - \omega'\bar{e}') + \cancel{m^2c^4}$$

Plug ↑

$$(1): \quad E' = E + \hbar(\omega - \omega')$$

$\bar{e}\bar{e}'$ ↓

$$\Rightarrow \boxed{E(\omega - \omega') = \hbar\omega\omega'(1 - \cos\theta) + c\bar{p}(\omega\bar{e} - \omega'\bar{e}')} \qquad (3)$$

- With e^- at rest (before scattering) ⇒ $\bar{p} = 0 \quad E = mc^2 \equiv E_0$

Ratio between energy of γ and e^- energies ↓

$$\Rightarrow E(\omega - \omega') = \hbar\omega\omega'(1 - \cos\theta) \longrightarrow \boxed{\frac{\omega'}{\omega} = \frac{1}{1 + \varepsilon(1 - \cos\theta)} \quad \varepsilon \equiv \frac{\hbar\omega}{E_0}}$$

- Average frequency change over all angles (isotropic unpolarized light)

$$\frac{\langle\Delta\omega\rangle_\theta}{\omega} = \frac{\langle\omega'-\omega\rangle_\theta}{\omega} = \frac{\langle\omega'\rangle-\omega}{\omega} = \frac{1}{\cancel{\omega}_1}\left\langle\frac{\overset{1}{\cancel{\omega}}}{1+\varepsilon(1-\cos\theta)}\right\rangle_\theta - 1$$

cross section for unpolarized (α) light

$$\left\langle\frac{1}{1+\varepsilon(1-\cos\theta)}\right\rangle_\theta = \frac{1}{\sigma_T}\int\frac{1}{1+\varepsilon(1-\cos\theta)}\left\langle\frac{d\sigma}{d\Omega}\right\rangle_\alpha d\Omega$$

$$= \frac{1}{\sigma_T}\frac{r_e^2}{2}\int\frac{(1+\cos^2\theta)\cdot(\sin\theta\, d\theta\, d\varphi)}{1+\varepsilon(1-\cos\theta)}$$

$\cos\theta = \mu$

$d\mu = \sin\theta\, d\theta$

$$= \frac{1}{\sigma_T}\frac{r_e^2}{\cancel{2}}\cancel{2}\pi\int_{-1}^{1}\frac{1+\mu^2}{1+\varepsilon(1-\mu)}d\mu$$

$$= \boxed{\frac{\pi r_e^2}{\sigma_T}\frac{\underbrace{\ln(1+2\varepsilon)}_{\text{expand } 2x-\frac{4x^2}{2}+\frac{x^3}{3}}(2\varepsilon^2+2\varepsilon+1)-2\varepsilon(1+\varepsilon)}{\varepsilon^3} - 1}$$

exact result

- For $\varepsilon = \frac{\hbar\omega}{E_0} \ll 1$

(one scattering event)

$$\boxed{\frac{\langle\Delta\omega\rangle}{\omega}} \approx \overbrace{\frac{8\pi r_e^2}{3\sigma_T}}^{=1}(1-\varepsilon) - 1 = -\varepsilon = \boxed{-\frac{\hbar\omega}{m_e c^2}}$$

- Relative energy loss by photon
  => energy gain by e^-! (momentum transfer to e^-)

Note: cross section → probability of the scattering to occour !!

Total energy change of one γ with more e^- (ionized gas => more than 1 scattering)

$$\langle\Delta\omega\rangle = -\frac{\hbar\omega^2}{m_e c^2} \xrightarrow{E=\hbar\omega} \langle\Delta E_\gamma\rangle = -\frac{\hbar^2\omega^2}{m_e c^2}$$

energy loss of 1 photon in 1 scattering

$$\boxed{\frac{dE_\gamma^{(-)}}{dt} = -\underbrace{(c N_e \sigma_T)}_{\text{collision rate}}\frac{(\hbar\omega)^2}{m_e c^2}}$$

← Total power loss by 1 photon with all e^- (N_e number density)

collision rate : $\left[\frac{m}{s}\frac{1}{m^3}m^2\right] = \left[\frac{1}{s}\right]$

energy loss per second ($\gamma \longrightarrow e^-$)

Now gain of the γ from e^-

$\gamma \leftarrow e^-$

- Power gain by γ = Power loss by e^-
- From radiation damping (1 e^- with radiation field)

$$P^{e^-}_{net} = P_{em} - P_{abs} = \frac{4}{3}\beta^2\gamma^2 c \, U_\omega \sigma_T$$

$$U = \int U_\omega \, d\omega \quad \text{spectrum}$$

$$= \frac{4}{3}\beta^2\gamma^2 c \, n_\gamma(\omega)\hbar\omega \, \sigma_T > 0 \qquad U_\omega = n_\gamma(\omega)\cdot\hbar\omega$$

(power that goes to γ)

number density of γ_ω → spectral energy distribution of γ
e.g. Planck spectrum

⇒ Energy gain by 1 γ_ω from all e^- with β

$$P^{(+)}_\gamma = \frac{P^{e^-}_{net}}{n_\gamma(\omega)}\cdot n_e(\beta) = \frac{4}{3} n_e(\beta)\,\beta^2\gamma^2(\beta)\,c\,\hbar\omega\,\sigma_T$$

→ Integrate over all β (i.e. full e^- population)

$$\boxed{P^{(+)}_{\gamma\, tot}} = \frac{4}{3} c\,\hbar\omega\,\sigma_T \int_0^\infty n_e(\beta)\,\beta^2\gamma^2(\beta)\,d\beta \qquad N_e = \int_0^\infty n_e(\beta)\,d\beta$$

$$= \boxed{\frac{4}{3} c\,\hbar\omega\,\sigma_T N_e \langle\beta^2\gamma(\beta)\rangle_\beta} \qquad \frac{\int_0^\infty n_e(\beta)\,\beta^2\gamma^2(\beta)\,d\beta}{N_e} = \langle\beta^2\gamma^2(\beta)\rangle_\beta$$

Total power gain of 1 γ by all e^-

Net energy transfer per photon per unit time

$$\boxed{\frac{dE_\gamma}{dt}} = P^{(+)}_{\gamma\, tot} + P^{(-)}_\gamma = \boxed{c N_e \sigma_T \hbar\omega\left(\frac{4}{3}\langle\beta^2\gamma(\beta)\rangle_\beta - \frac{\hbar\omega}{m_e c^2}\right)}$$

④

Quantum gasses

e.g. photons

- <u>One particle</u> is fully characterized by : $(\cancel{m}, \bar{x}, \bar{p})$ $(E=cp)$!

$E = \frac{1}{2} m v^2 = \frac{p^2}{2m}$ (derived)

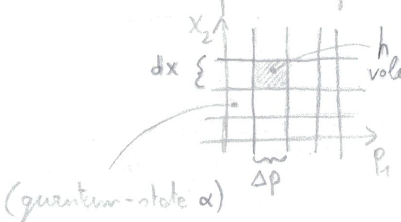
- <u>The phase-space</u>

$\bar{\xi} = (x, y, z, p_1, p_2, p_3)$

- <u>Classical gas</u> : continuous, any state (energy) is possible
- <u>Quantum gas</u> :

$\Rightarrow$ Heisemberg indetermination principle $\boxed{\Delta x \Delta p > \frac{\hbar}{2}}$

$\Rightarrow$ The phase-space is quantized in "cells" of size h

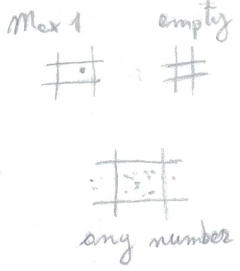


h volume $\Rightarrow$ in 3D the volume is h^3

. <u>Pauli exclusion principle</u>

YES for <u>Fermions</u> $N_\alpha = 0, 1$
ν: spin½
e^+, e^-, n, p

NO for <u>Bosons</u> $N_\alpha = 0, 1, \dots, \infty$
γ: spin 1



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· <u>In cosmology</u> :

- Adiabatic expansion (No heat transfer because of Isotropy)
- Thermal equilibrium (expansion $\Rightarrow$ instantaneous T.E.) $\left[\frac{\dot{R}}{H} \ll 1\right]$
- Ideal fluid (only direct collisions, no long term interactions)

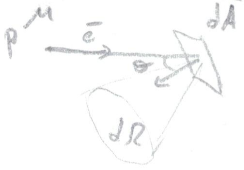
□ Entropy dominated by the photons (CMB)

$\Rightarrow$ NO S generation $\Rightarrow$ reversible $\Rightarrow$ adiabatic

Propagation and transport of radiation < photons >

- Radiation ~ fluid

- Stream of particles: carry energy-momentum



- <u>Specific intensity</u> $\frac{dE}{dt\,d\omega\,dA\,d\Omega} = I_\omega \cos\theta$

- <u>Relate this observable (I_ω) to photons</u>

- $P^\mu = \hbar K^\mu = \frac{\hbar\omega}{c}(1,\bar e)^T = (\frac{E}{c},\bar P)^T \Rightarrow \bar P \overset{(1)}{=} \frac{\hbar\omega}{c}\bar e$ $E = \hbar\omega = cP$

  $\langle P,P\rangle = 0 = \langle K,K\rangle$ because massless

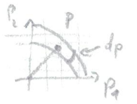
- <u>Phase-space volume element</u>: $d\Gamma = dx^3 dp^3 = (2\pi\hbar)^3$ $dx\,dp \geq \frac{\hbar}{2}$

  !isotropy! $\Rightarrow$ polar coord   quantized because of Heisemberg indetermind.

- <u># of cells</u>: $\frac{dx^3 dp^3}{(2\pi\hbar)^3} = \frac{dV\cdot \overset{(1)}{P^2}\,dp\,d\Omega}{(2\pi\hbar)^3} = dV\,\frac{\omega^2 d\omega\, d\Omega}{(2\pi c)^3}$ $dV = c\,dt\,\underbrace{d\bar A\,\bar e}_{dA\cos\theta}$

you are going for a $\frac{dE}{dt}$

- Energy carried by all γ with a certain $|\bar p|$



$dE = (\#\text{cell})\cdot(\text{energy of } 1\gamma \text{ in cell})\cdot(\text{number density of } \gamma \text{ in cell})$

$= \left(c\,dt\,dA\cos\theta\,\frac{\omega^2 d\omega\, d\Omega}{(2\pi c)^3}\right)\cdot(\hbar\omega)\cdot\left(\sum_{\alpha=1}^{2} n_{\alpha\bar p}\right)$ $\alpha = 1,2$ 2 degrees of polarization for photons

$$\Rightarrow \boxed{\frac{dE}{dt\,d\omega\,dA\,d\Omega} = \sum_{\alpha=1}^{2} n_{\alpha\bar p}\,\frac{\hbar\omega^3}{(2\pi)^3 c^2}\cos\theta} \equiv I_\omega\cos\theta$$

- For unpolarized light: $\underbrace{n_{1\bar p} = n_{2\bar p}}_{\text{factor of 2}} \Rightarrow \boxed{I_\omega = \frac{\hbar\omega^3}{4\pi^3c^2}\,n_{\alpha\bar p}}$

  you need to specify the distributi- e.g. Planck spectrum

Planck spectrum, summary

- Ensamble of quantum states, thermal equilibrium T ⇒ Boltzmannfactor
- Labelled by α (any quantum state)
- ε_α = energy of the state
- n_α = occupation number of state α

$$N = \sum_\alpha n_\alpha \,,\quad E(N) = \sum_\alpha \varepsilon_\alpha n_\alpha$$ total #, energy

$$\boxed{Z_{gc} = \prod_\alpha Z_\alpha \qquad Z_\alpha = \sum_{n_\alpha} \exp\left[-(\varepsilon_\alpha - \mu) n_\alpha / K_B T\right]}$$

Grand Canonical partition sum

- Distribution: $f_\alpha = \dfrac{e^{-(E(N_\alpha) - \mu N_\alpha)/KT}}{Z_{gc}}$
- $Z_{gc} = \sum_{N_\alpha} e^{-(E(N_\alpha) - \mu N_\alpha)/KT}$
- $\dfrac{1}{Z_{gc}} \equiv e^{\phi/KT} \rightarrow \phi(T,V,\mu) \equiv -KT \ln(Z_{gc})$ Free energy
- $\langle N_\alpha \rangle = -\dfrac{\delta \phi}{\delta \mu} = +\dfrac{KT}{Z_{gc}} \dfrac{\delta Z_{gc}}{\delta \mu}$ from [Euler eq.]

μ = chemical pot.

energy required to change the occup. number by unity

① **Fermi-Dirak** $n_\alpha \in \{0,1\}$ (⇒ Pauli exclusion) $\quad Z_\alpha^{FD} = 1 + \exp\left[-\dfrac{\varepsilon_\alpha - \mu}{K_B T}\right]$ Fermi-Dirak

spin 1/2: e^+, e^-, n, p, ν

② **Bose-Einstein** $n_\alpha \in [0, \infty)$ $\quad$ (Geometric series $\sum_{k=0}^{\infty} e^{-nx} = (1-e^{-x})^{-1}$) $\quad Z_\alpha^{BE} = \left[1 - \exp\left(-\dfrac{\varepsilon_\alpha - \mu}{K_B T}\right)\right]^{-1}$ Bose-Einstein

Spin 1: γ

$$\boxed{\bar{n}_\alpha = \frac{1}{Z_\alpha} \sum_\alpha n_\alpha \exp\left[\frac{-(\varepsilon_\alpha - \mu) n_\alpha}{K_B T}\right] = \frac{1}{K_B T} \frac{\delta}{\delta \mu} \ln Z_\alpha} \Rightarrow \begin{array}{l} ①\ \bar{n}_\alpha^{FD} = \\ ②\ \bar{n}_\alpha^{BE} = \end{array}\Bigg\} \frac{1}{\exp\left[\frac{(\varepsilon_\alpha - \mu)}{KT}\right] \pm 1} \begin{array}{l} \oplus \\ \ominus \end{array}$$

Mean occupation number

- $\varepsilon_\alpha = cp$ energy of photon ; at equilibrium, $\mu = 0$ "alternative approach"

$$\Rightarrow dU_{phot} = \underset{\substack{\uparrow \\ \text{2 polarizat.} \\ \text{states}}}{2} \cdot \underset{\substack{\uparrow \\ \text{number of} \\ \text{Phase space} \\ \text{cells}}}{\left(\frac{4\pi p^2 dp}{(2\pi\hbar)^3}\right)} \cdot \underset{\substack{\uparrow \\ \text{energy of a single photon}}}{cp} \cdot \underset{\substack{\uparrow \\ \text{distribution}}}{\bar{n}_{Phot}^{BE}}$$

Energy density in the all $\quad E = cp = \hbar\omega$

$$\frac{dU_{phot}}{d\omega} = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\frac{\hbar\omega}{KT}} - 1} \quad \left(\cdot \frac{c}{4\pi}\right) \Longrightarrow \boxed{I_\omega = \frac{\hbar}{4\pi^3 c^2} \frac{\omega^3}{e^{\frac{\hbar\omega}{K_B T}} - 1} \equiv B_\omega(T)}$$

by definition of I_ω

- $\boxed{U = \int_0^\infty \frac{dU}{d\omega} d\omega = aT^4}$ Stefan-Boltzmann law $\quad \boxed{n_\gamma(T) = \int_0^\infty \frac{dU_\omega}{d\omega} \frac{1}{\hbar\omega} d\omega}$ n° density

Quantum transition: summary

• Probability transition)

$$H = H^0 + H^1(t) \qquad H^0|i\rangle = E_i|i\rangle \quad H^0|f\rangle = E_f|f\rangle$$

$$\boxed{P_{fi} = \frac{1}{\hbar^2 T}\left|H^1_{fi}(\omega_{fi})\right|^2} \qquad H_{fi}(\omega_{fi}) = \int_0^T dt \langle f|H^1_{fi}(t)|i\rangle e^{i\omega_{fi}t} \qquad \omega_{fi} = \frac{E_f - E_i}{\hbar}$$

$$L = \left(-mc^2 + \frac{q}{c}A_\mu u^\mu\right)\sqrt{1-\beta^2}$$

$$H = \bar{P}\bar{v} - L \qquad - \qquad \bar{P} \to \hat{\bar{P}} = -i\hbar\bar{\nabla} \qquad x \to \hat{x}$$

Position operator

- keep A^μ as classical field, but $A^\mu(x) \Rightarrow \bar{P}\bar{A} \neq \bar{A}\hat{P}$
- Gauss gauge $\bar{\nabla}\bar{A} = 0$ and $\phi = 0$ gauge anyway

$\Downarrow$

• Hamiltonian

$$\boxed{H = \underbrace{\frac{P^2}{2m} + mc^2}_{\text{free particle}} + \underbrace{\frac{e}{mc}\bar{A}\bar{P}}_{H_1(t)\text{ perturb. Hamilt.}} + \underbrace{\frac{e}{2mc^2}\bar{A}^2}_{\sim 0 \text{ if } \omega\hbar \gg m_e c^2}}$$

In H there is a $\left(\bar{P} - \frac{q}{c}\bar{A}\right)^2 \Rightarrow (\bar{P}\bar{A})$ and $\bar{A}\bar{P} \neq 0$

• A → intensity)

$$\boxed{\bar{A}(\bar{x},t) = A(t)e^{i\bar{k}\bar{x}}\,\bar{e}} \quad + \quad \bar{S} \Rightarrow \bar{E} \Rightarrow \bar{A} \quad : \quad I_\omega = \frac{c}{4\pi T}\frac{\omega^2}{c^2}|\hat{\bar{A}}|^2$$

$$\Rightarrow \boxed{\Pi_{fi} = \frac{4\pi e^2}{m^2 c}\frac{I_\omega}{\omega_{fi}^2}\left|\langle f|e^{i\bar{k}\bar{x}}\,\bar{e}\bar{\nabla}|i\rangle\right|^2} = \frac{4\pi I_\omega}{c\hbar^2}|\langle f|\bar{e}\bar{d}|i\rangle|^2 \qquad \bar{d} = e\bar{x}$$

$\bar{p} = -i\hbar\bar{\nabla}$

dipole approx $e^{i\bar{k}\bar{x}} = 1 + i\bar{k}\bar{x} + \dots$

- **Bound-Bound:** $\hbar\omega_{fi} = |E_f - E_i| \qquad I_\omega \to I_\omega\delta(\omega - \omega_{fi})d\omega$ (line)

$$n_\omega = \frac{\Delta E_{tot}}{E_{phot}} = \frac{I_\omega d\omega}{\hbar\omega} \qquad \text{number of incoming photons}$$

$$\sigma_{fi} = \frac{P_{fi}}{n_\omega}$$

- **Bound free:**
  - Phase space cells: # of free accessible levels for free e^-
  - cell $d^3p \to d^3k$ to relate to incoming photon
  - wave function to plug in $\langle f|e^{i\bar{k}\bar{x}}\,\bar{e}\bar{\nabla}|i\rangle$

$\boxed{\text{Quantum Transitions}}$

Semi-classical approach!

E.M wave

- Quantum system

⇓

Transition

e.g. Atom

⇓

internal structure

- Transition probability

- Transition between internal configurations
- Quantum transitions : between states, bound - free, free - bound

- $\boxed{H = H^0 + H^1(t)}$ ← Hamiltonian of quantum mecc. system

time indep. ↑ time dependent

$|i\rangle$, $|f\rangle$ initial and final states eigenstates of H^0

$H^0|i\rangle = E_i|i\rangle \quad H^0|f\rangle = E_f|f\rangle$

- $\boxed{\Gamma_{fi} = \frac{1}{\hbar^2 T}\left|\hat{H}^1_{fi}(\omega_{fi})\right|^2}$ ← Probability of transition per unit time (rate)

$|i\rangle \longrightarrow |f\rangle$

System perturbed by $H^1(t)$

transition frequency $\boxed{\omega_{fi} = \frac{E_f - E_i}{\hbar}}$

$$= \int_0^T dt \, \underbrace{\langle f|\hat{H}^1_{fi}(t)|i\rangle}_{= \int d^3x \, \Psi_f^* \, \hat{H}^1(t) \, \Psi_i} e^{i\omega_{fi}t}$$

← time evolved transition matrix

$= \int d^3x \, \Psi_f^* \, \hat{H}^1(t) \, \Psi_i \;\}$ ← transition matrix element

T^{-1}: From average over time T

wave function

$\Psi_i = \langle x|i\rangle \quad \Psi_f = \langle x|f\rangle$

Position operator

Projection of the state $|i\rangle$ on the ~~[illegible]~~ eigenbasis of the desired operator

- plug $H^1(t)$ ⟶

Quantum transitions

- Small perturbations => Perturbation theory
- Not necessarily you have small changes in eigen-values and -states
- Transition $|\psi^0_m\rangle \to |\psi^0_n\rangle$ $(|m\rangle \to |n\rangle)$, you want: transition probability $P_{nm} = |\langle m|n\rangle|^2$

$$\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}(t)$$

· Perturbed Hamiltonian (time dependent perturbation)

$$|m(t)\rangle = |m\rangle e^{-iE_m t/\hbar}$$

· Time dependent eigenstate of $H^{(0)}$ $|m\rangle = |\psi_m\rangle$

$$|\psi_m(t)\rangle = \sum_K c_{mK}|K(t)\rangle$$
$$= \sum_K c_{mK}|K\rangle e^{\frac{-iE_K t}{\hbar}}$$

· Expand eigenstate $|\psi_m\rangle$ as a linear combination of eigenstates of H^0

c_{mK} tells you the "probability" of the particle to be in the K state

· Must obey Schrödinger eq. No sudden transition -> system evolves

$$\underset{(1)}{i\hbar\partial_t|\psi_m(t)\rangle} = \underset{(2)}{\hat{H}|\psi_m(t)\rangle}$$

$$(1) = i\hbar\partial_t \sum_K c_{mK}|K\rangle e^{-iE_K t/\hbar} = i\hbar\sum_K\left(\dot{c}_{mK} - c_{mK}\frac{iE_K}{\hbar}\right)|K\rangle e^{-iE_K t/\hbar} = \sum_K\left(i\hbar\dot{c}_{mK} + c_{mK}E_K\right)|K\rangle e^{-iE_K t/\hbar}$$

$$(2) = \left(\hat{H}^0 + \hat{H}^1(t)\right)\sum_K c_{NK}|K(t)\rangle = \sum_K c_{mK}\Big(\underbrace{\hat{H}^0|K(t)\rangle}_{E_K|K\rangle e^{\frac{-iE_K t}{\hbar}}} + \hat{H}^1(t)|K(t)\rangle\Big) = \sum_K c_{mK}\left(E_K + \hat{H}^1(t)\right)|K\rangle e^{\frac{-iE_K t}{\hbar}}$$

$$\sum_K\left(i\hbar\dot{c}_{mK} + \cancel{c_{mK}E_m}\right)|K\rangle e^{\frac{-iE_K t}{\hbar}} = \sum_K c_{mK}\left(\cancel{E_m} + \hat{H}^1(t)\right)|K\rangle e^{-iE_K t/\hbar}$$

· $\langle m|$ · initial unperturbed state (multiply from left)

$$\sum_K i\hbar\dot{c}_{mK}\underbrace{\langle m|K\rangle}_{\delta_{mK}} e^{\frac{-iE_K t}{\hbar}} = i\hbar\dot{c}_{mm} e^{\frac{-iE_m t}{\hbar}} = \sum_K c_{mK}\langle m|\hat{H}^1(t)|K\rangle e^{-iE_K t/\hbar}$$

δ_{mK} orthonormality of unperturbed eigenstates

$$\Rightarrow \boxed{\dot{c}_{mm} = -\frac{i}{\hbar}\sum_K c_{mK}\underbrace{\langle m|\hat{H}^1(t)|K\rangle}_{\text{transition matrix element}} e^{i\omega_{mK}t}} \qquad \frac{E_m - E_K}{\hbar} = \omega_{mK}$$

exact result

coupled linear differential eq. solve for c_{mK} → hard to solve exactly

- <u>Solving for c_{nm}</u>

- no need to follow the entire evolution of c_{nm} system in inital state
- enough to look at initial times => Solve it at 1° order

1) at $t=0$: System in initial state $|n\rangle$ => $c_{nn}(t=0)=1$ and $c_{nk}=0 \ \forall k\neq n$: $c_{nk}=\delta_{nk}$

2) For $t>0$ (but small with respect to the time scale for transition $|n\rangle \to |m\rangle$ to occure)
If weak perturbation => $|c_{nk}| \ll |c_{nn}| \ \forall k$

$$\Rightarrow \quad \dot{c}^{(1)\leftarrow 1^\circ \text{order}}_{nm} = -\frac{i}{\hbar}\langle m|\hat{H}^1(t)|n\rangle e^{i\omega_{mn}t} \quad \text{at } 1^\circ \text{ order}$$

we are neglecting the other coefficients c_{nk}
small but $\neq 0$

- integrate $\dot{c}(t) \to c(t)$:

$$\boxed{c^{(1)}_{nm}(T) = -\frac{i}{\hbar}\int_0^T \langle m|\hat{H}^1(t)|n\rangle e^{i\omega_{mn}t}\,dt}$$

valid only when $|c_{nk}| \ll |c_{nn}|$ $n\neq k$!

$P_{nm} = |c^{(1)}_{nm}|^2$ transition probability within T

$\Gamma_{nm} = \frac{P_{nm}}{T}$ transition rate

- Must specify $H^{(1)}(t)$ .....

- Hamiltonian : identify $H^{(1)}(t)$

- $L = -mc^2\gamma^{-1}$ Lagrangian of free particle

- $L = (-mc^2 + \underset{}{\overset{\text{interaction}}{\frac{q}{c}A_\mu u^\mu}})\gamma^{-1} = -mc^2\gamma^{-1} - \frac{q}{c}\phi c + \frac{q}{c}\bar{A}\bar{v}$ Lagrangian of charged particle in E.M field

$(A^\mu) = (\phi, \bar{A})$ $(u^\mu) = \gamma(c, \bar{v})$

- $\bar{P} = \frac{\delta L}{\delta \bar{v}} = \gamma m \bar{v} + \frac{q}{c}\bar{A}$ Conjugate momentum

- $H = \bar{P}\bar{v} - L$ Hamiltonian

$$= \gamma m \bar{v}^2 + \cancel{\frac{q}{c}\bar{A}\bar{v}} + mc^2\gamma^{-1} + \frac{q}{\cancel{c}}\phi\cancel{c} - \cancel{\frac{q}{c}\bar{A}\bar{v}}$$

$\bar{v} = \frac{\bar{p}}{m\gamma}$

$$= \frac{\bar{p}^2}{m\gamma} + \frac{mc^2}{\gamma} + q\phi$$

$$= \frac{1}{m\gamma}(\bar{p}^2 + m^2c^2) + q\phi$$

$\gamma^{-2} = 1 - \frac{\bar{v}^2}{c^2} = 1 - \frac{\bar{p}^2}{m^2\gamma^2c^2}$ $1 = \gamma^2 - \frac{\bar{p}}{m^2c^2}$

$\gamma = \left(1 + \frac{\bar{p}^2}{m^2c^2}\right)^{1/2} = \frac{1}{mc}(m^2c^2 + \bar{p}^2)^{1/2}$

$$= \underbrace{(c^2\bar{p}^2 + m^2c^4)^{1/2}}_{E} + q\phi$$

$\beta \ll 1$ $E = (c^2\bar{p}^2 + m^2c^4)^{1/2} \simeq \frac{\bar{p}^2}{2m} + mc^2$

$$\simeq \frac{\bar{p}^2}{2m} + mc^2 + q\phi$$

$\bar{P} = \bar{p} + \frac{q}{c}\bar{A}$

$$= \frac{1}{2m}\left(\bar{P} - \frac{q}{c}\bar{A}\right)^2 + mc^2 + q\phi$$

- Quantize Hamiltonian $H \to \hat{H}$ operator

$\bar{P} \to -i\hbar\bar{\nabla}$, $x \to \hat{x}$, $A^\mu(\underset{\underset{\hat{x}}{\uparrow}}{x})$ leave it as classic potential but $\hat{x}$ operator $\Rightarrow \hat{P}\hat{A} \neq \hat{A}\hat{P}$!

$$\left(\bar{P} - \frac{q}{c}\bar{A}\right)^2 \to \left(\hat{P} - \frac{q}{c}\bar{A}\right)^2 = \hat{P}^2 + \frac{q^2}{c^2}\bar{A}^2 - \frac{q}{c}(\underbrace{\hat{P}\bar{A}}_{(1)} + \bar{A}\hat{P})$$

"not $2\bar{A}\hat{P}$ or $2\hat{P}\bar{A}$"

$$(1): \hat{P}(\bar{A}\psi) = -i\hbar\bar{\nabla}(\bar{A}\psi) = -i\hbar\underbrace{\cancel{\psi\bar{\nabla}\bar{A}}}_{=0} - i\hbar\bar{A}\bar{\nabla}\psi = \underbrace{\cancel{(\hat{P}\bar{A})}}_{=0}\psi + \bar{A}(\hat{P}\psi)$$

$\hat{H}\psi$

Apply Coulomb gauge $\bar{\nabla}\bar{A} \overset{!}{=} 0$

$$\Rightarrow \boxed{\hat{H} = \frac{1}{2m}\left(\hat{P}^2 + \frac{e^2}{c^2}\bar{A}^2 - 2\frac{e}{c}\bar{A}\hat{P}\right) + mc^2} + \overset{\phi=0}{\cancel{q\phi}}$$

gauge ϕ away: no loss of generality

$q = -e$ electron charge

• Identify perturbed Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{P}^2 + \frac{e^2}{c^2}\bar{A}^2 - 2\frac{e}{c}\bar{A}\hat{P}\right) + mc^2$$

$$= \underbrace{\frac{\hat{P}^2}{2m} + mc^2}_{\hat{H}^{(0)} \quad (a)} \underbrace{- \frac{e}{mc}\bar{A}\hat{P}}_{(b)} \underbrace{+ \overset{\frown}{\frac{e^2}{2mc^2}\bar{A}^2}}_{\hat{H}^{(1)}(t) \quad (c)}$$

neglect (c) if energy E.M. wave ≪ e^- rest energy
$\hbar\omega \ll mc^2$

unperturbed
(free e^-)

perturbation
time dependent

Compare terms (b) and (c)

$\phi = 0$ gauge

$\nu = 2\pi\omega \quad \nu = c/\lambda$

$$\eta \equiv \frac{c}{b} \approx \frac{eA}{2cP}$$

- ▫ $\bar{E} = -\frac{1}{c}\dot{\bar{A}} - \cancel{\bar{\nabla}\phi} \;\longrightarrow\; \bar{E} = -\frac{1}{c} i\omega\bar{A} = -i\frac{2\pi}{\lambda}\bar{A} \qquad A \approx \frac{\lambda E}{2\pi}$
- ▫ $P \approx \alpha mc$ (for e^- in H atom) $\qquad \alpha \equiv \frac{e^2}{\hbar c}$ fine struct. const.

$$= \frac{e}{2c}\frac{\lambda E}{2\pi}\frac{1}{\alpha mc} = \frac{1}{4\pi\alpha}\frac{eE\lambda}{mc^2} \Rightarrow \text{if } eE\lambda \ll mc^2 \text{ ignore term (c)}$$

(work done by E.M. wave on e^-) (e^- rest energy)

$\Rightarrow$ Ignore term (c).

• Plug $\bar{A}$ of incoming E.M. wave

$$\Rightarrow \boxed{\hat{H}^{(1)}} = -\frac{e}{mc}\bar{A}\hat{P} = \frac{i\hbar e}{mc}\bar{A}\bar{\nabla}$$

$$= \boxed{\frac{i\hbar e}{mc} A(t) e^{i\bar{k}\bar{x}} \hat{e}\bar{\nabla}}$$

$$\bar{A}(\bar{x},t) = |A|\, e^{-i(\bar{k}\bar{x} - \omega t)}\, \bar{e}$$

$$= A(t)\, e^{i\bar{k}\bar{x}}\, \bar{e}$$

decompose in plane waves

Polarization direction

scalar time dependent amplitude

in Coulomb gauge $\quad \bar{e}\bar{k} = 0 \qquad \Box A = 0$

- Back to the Transition probability : Transition rate $\Gamma = P/T$

$$\Gamma_{fi} = \frac{1}{\hbar^2 T}\left|\int_0^T dt \langle f|\hat{H}^1_{fi}(t)|i\rangle e^{i\omega_{fi}t}\right|^2$$

$$= \frac{1}{\hbar^2 T}\left|\int_0^T dt \frac{i\hbar e}{mc} A(t)\langle f|e^{i\bar{k}\bar{x}}\hat{e}\bar{\nabla}\,|i\rangle e^{i\omega_{fi}t}\right|^2 \qquad \tilde{\bar{A}}(\omega_{fi}) = \int_0 \bar{A}(t)\, e^{i\omega_{fi}t}\, dt$$

$$= \frac{1}{\cancel{\hbar^2} T}\left|\int_0^T dt\, \bar{A}(t)\, e^{i\omega_{fi}t}\right|^2 \frac{\cancel{\hbar^2} e^2}{m^2c^2}\left|\langle f|e^{i\bar{k}\bar{x}}\hat{e}\bar{\nabla}|i\rangle\right|^2 = \frac{e^2}{m^2c^2T}\left|\tilde{\bar{A}}(\omega_{fi})\right|^2\cdot\left|\langle f|e^{i\bar{k}\bar{x}}\hat{e}\bar{\nabla}|i\rangle\right|^2$$

- Express A in terms of intensity (observable quantity)

$$I \equiv \langle|\bar{S}|\rangle = \frac{c\langle|\bar{E}^2|\rangle}{4\pi} = \frac{c}{4\pi T}\int_{-T/2}^{T/2}|\bar{E}|^2 dt = \frac{c}{4\pi T}\int_{-\infty}^{\infty}|\tilde{\bar{E}}|^2\frac{d\omega}{2\pi} \qquad I_\omega = \frac{\cancel{c}}{8\pi^2 T}\frac{\omega^2}{c^{\cancel{2}}}|\tilde{\bar{A}}|^2$$

$$\bar{S} = \frac{c}{4\pi}\bar{E}\times\bar{B} = \frac{c}{4\pi}\bar{E}^2\bar{e}_k$$

Parseval $T \to \infty$

$$\phi = 0 \qquad \bar{E} = -\frac{1}{c}\frac{\delta\bar{A}}{\delta t} - \cancel{\bar{\nabla}\phi} \qquad \tilde{\bar{E}} = -\frac{1}{c}i\omega\tilde{\bar{A}}$$

$$= \frac{8\pi^2 e^2}{m^2c\,\omega_{fi}^2} I_{\omega_{fi}}\left|\langle f|\underline{\underline{e^{i\bar{k}\bar{x}}}}\hat{e}\bar{\nabla}|i\rangle\right|^2$$

(fi!)

- Dipole approximation $e^{i\bar{k}\bar{x}} \simeq 1 + i\bar{k}\bar{x} - \frac{1}{2}(\bar{k}\bar{x})^2 + \dots$ neglect

(dipole) (quadrupole) (octopole)

$$\simeq \frac{8\pi^2 e^2}{m^2c\,\omega_{fi}^2} I_{\omega_{fi}}\left|\langle f|\hat{e}\bar{\nabla}|i\rangle\right|^2$$

- Use commutation roles + $\overset{f}{H}|i\rangle = \overset{f}{E_i}|i\rangle$ eigenstates

$$\bar{\nabla} = \frac{m}{\hbar^2}[\hat{x},\hat{H}_0] \Rightarrow \langle f|\bar{e}\bar{\nabla}|i\rangle = -\frac{m}{\hbar}\omega_{fi}\langle f|\bar{e}\hat{x}|i\rangle$$

(see *3)

$$= \frac{8\pi^2 e^2}{\cancel{m^2}c\,\cancel{\omega_{fi}^2}} I_{\omega_{fi}}\frac{\cancel{m^2}}{\hbar^2}\cancel{\omega_{fi}^2}\left|\langle f|\bar{e}\hat{x}|i\rangle\right|^2$$

- dipole operator $\hat{d} \equiv e\hat{x}$ charge of electron

for unpolarized light $\langle\bar{e}\hat{d}\rangle = \sqrt{\frac{1}{3}}$ average

$$= \boxed{\frac{8\pi^2 I_{\omega_{fi}}}{c\hbar^2}\left|\underbrace{\langle f|\bar{e}\hat{d}|i\rangle}_{\equiv\bar{d}_{fi}}\right|^2}$$

dipole matrix element

- if policromatic light $\Rightarrow$ $\boxed{I_{\omega_{fi}} = I_\omega\,\delta_D(\omega - \omega_{fi})}$ only those frequency plays a role for the transition

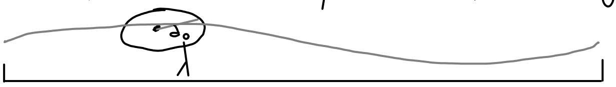
- **Validity of dipole approx: How small is the quadrupole $\bar{k}\bar{x}$?**

$\omega = kc$ dispersion relation E.M. wave $\Rightarrow$ $k = \frac{\omega_{fi}}{c} = \frac{E_f - E_i}{\hbar c}$

$r \approx a_0$ Bohr radious $\Rightarrow$ $r = \frac{\hbar^2}{me^2} = 5{,}3\, e^{-9}$ cm

- **Meaning 1:** $\bar{k}\bar{x} \approx kr = \frac{\overset{\searrow}{\omega_{fi}}}{c} a_0 = \frac{2\pi}{\cancel{c}} \frac{\cancel{c}}{\lambda} a_0 = 2\pi \frac{a_0}{\lambda}$ $\quad kr \ll 1$ if $\boxed{a_0 \ll \lambda}$

i.e. equivalent to assume that the field has no spatial dependency across the atom ($\lambda \gg a_0$)



– only time dependency is relevant

- **Meaning 2:** $\bar{k}\bar{x} \approx kr = \frac{\omega_{fi}}{c} \overset{\searrow}{a_0} = \frac{\omega_{fi}}{c} \frac{\hbar^2}{me^2} = \frac{\hbar\omega_{fi}}{\alpha mc^2}$ $\quad kr \ll 1$ if $\boxed{\hbar\omega \ll \alpha mc^2}$

$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ fine struct. constant.

- (i.e. Transition energy) ≪ (rest energy of e^-)
- e.g. Ionization of H atom:

$\Delta E = 13{,}6\ \text{eV}$, $\alpha mc^2 \simeq 3{,}7 \cdot 10^3\ \text{eV}$ $\Rightarrow$ yes: $\boxed{e^{i\bar{k}\bar{x}} \simeq 1}$

—— o ——

- **Higher orders** → Forbidden lines $\quad e^{i\bar{k}\bar{x}} \simeq 1 + i\bar{k}\bar{x} - \frac{1}{2}(\bar{k}\bar{x})^2 + \dots$

→ very, very low probabilities but visible in astro objects (e.g. nebulae) because of the super large number of atoms

- **Final Remark:**

This is for any quantum transition

→ not only between e^- energy levels in atom

→ also molecular transition (tipically in mm regime)

Probability P_{fi} → Transition cross section

② Bound-Bound transition

- $\hbar\omega_{fi} = |E_f - E_i|$ $E_f > E_i \Rightarrow$ absorbtion of a photon
  $E_f < E_i \Rightarrow$ emission " " "

- Only photons with ω_{fi} can be absorbed / emitted

$$I_\omega \rightarrow I_\omega \underbrace{\delta_D(\omega-\omega_{fi})}_{} d\omega$$ absorbed intensity of total incoming I_ω

Dirac's delta $[Hz^{-1}] \Rightarrow \delta_D(\omega-\omega_{fi})\,d\omega$ [dimensionless]

- Number of incoming photons with $[\omega, \omega+d\omega]$ per unit time and area

$$n_\gamma = \frac{I_\omega \overset{\cancel{\delta_D(\omega-\omega_{fi})}}{d\omega}}{\hbar\omega_{fi}} = \left(\frac{\text{Total energy}}{\text{energy of 1 photon}}\right)\left[\frac{1}{s\,m^2}\right] \qquad dI_\omega = \frac{dE}{dt\,dA\,d\omega\,d\Omega}$$

⇒ Absorbtion cross section for bound-bound transition

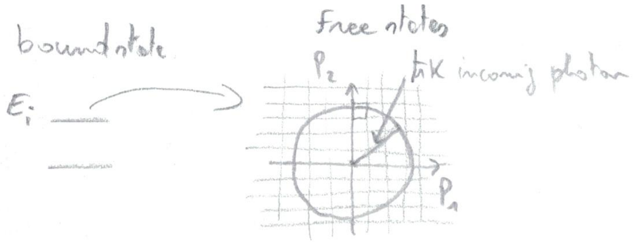
$$\boxed{\sigma_{fi} \equiv \frac{\overset{\text{number of possible final states}}{1}\cdot P_{fi}}{n_\gamma}} = \left(\frac{8\pi^2}{3c\hbar^2}\cancel{I_{\omega_{fi}}}\,\delta_D(\omega-\omega_{fi})\,\cancel{d\omega}\,|\hat{d}_{fi}|^2\right)\cdot\left(\frac{\hbar\omega_{fi}}{\cancel{I_\omega d\omega}}\right)$$

(transition probab. per unit time)

$$= \frac{8\pi^2\omega_{fi}}{3c\hbar}|\hat{d}_{fi}|^2\,\delta(\omega-\omega_{fi})$$

$$= \boxed{4\pi^2 r_e c\, f_{fi}\, \underbrace{\delta_D(\omega-\omega_{fi})}_{\phi(\omega-\omega_{fi})}} \qquad f_{fi} \equiv \frac{2m\omega_{fi}}{3\hbar e^2}|\hat{d}_{fi}|^2$$ oscillator strength (adimensional)

in reality lines have a finite with!



ⓑ Bound-free transition

- Account for number of possible free e^- states (shell in momentum space)

$$n_{e^-_{free}} = \frac{4\pi p_f^2\, dp_f}{(2\pi\hbar)^3}\cdot V = \frac{k_f^2\, dk_f}{2\pi^2} V = \frac{k_f\, m\, d\omega}{2\pi^2\hbar} V$$

free

$\bar p = \hbar \bar k$

(also free states are quantized)

- Energy conservation:

(bound electron) ignore because we assume "strong ionization"

$$(\text{incoming photon})\ \boxed{\hbar\omega} = \boxed{\frac{p_f^2}{2m} + E_1} = \frac{\hbar^2 k_f^2}{2m} + E_1 \Rightarrow \hbar\, d\omega = \frac{\hbar^2 2 k_f\, dk_f}{2m}$$

binding energy

$$\Rightarrow \boxed{\sigma_{bf}} = n_{ef}\cdot\frac{P_{bf}}{n_\gamma} = \left(\frac{k_f\, m\, d\omega V}{\cancel{2\pi^2}\cancel{\hbar}}\right)\cdot\left(\frac{\overset{4}{\cancel{8\pi^2}} e^2 \cancel{I_{\omega_{bf}}}}{m^2 c\, \omega_{bf}^{\cancel{2}}}\left|\langle f| e^{i\bar k\bar x}\bar e\bar\nabla|i\rangle\right|^2\right)\cdot\left(\frac{\cancel{\hbar\omega_{bf}}}{\cancel{I_{\omega_{bf}}d\omega}}\right)$$

$$= \frac{4e^2 k_f V}{m c\, \omega_{bf}}\left|\langle f| \underset{\simeq 1}{e^{i\bar k\bar x}}\hat e\bar\nabla|i\rangle\right|^2$$

dipole approx

ⓑ2 Photoionization, e.g. Hydrogen

($r = \sqrt{\bar x^2}$) (a_0 = Bohr rad.)

Wave functions: $\psi_i(\bar x) = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}$ bound $\qquad \psi_f(\bar x) = \frac{e^{i\bar k_f\bar x}}{\sqrt V}$ free

assumed to be in a finite volume

$$|\langle f|\hat e\bar\nabla|i\rangle|^2 = |\langle i|\hat e\bar\nabla|f\rangle|^2 \text{ because } \bar\nabla \to \hat p \text{ is Hermitian } (H = H^+)$$

$$\langle i|\hat e\bar\nabla|f\rangle = \frac{1}{\sqrt{\pi a_0^3 V}}\int d^3x\,(e^{-r/a_0})\,\hat e\bar\nabla\,(e^{i\bar k_f\bar x}) = \frac{1}{\sqrt{\pi a_0^3 V}}\, i\hat e k_f \frac{8\pi a_0^3}{(1+k_f^2 a_0^2)^2}$$

$$\approx \frac{i\hat e\bar k_f 8\pi}{\sqrt{\pi a_0^3 V}\; k_f^4 a_0}$$

Solve in polar coordinates

Strong photoionizat. $E_f >> E_i$

$$\Rightarrow \sigma_{bf} = \frac{4e^2 k_f V}{m c \omega_{bf}} \cdot \left| \frac{i\,\hat{e}\bar{k}_f\, 8\pi}{\sqrt{\pi a_0^3 V}\; k_f^4 a_0} \right|^2$$

$$|\hat{e}\bar{k}_f|^2 = \frac{k_f^2}{3}$$

average over all polarization angles (unpolarized light)

$$= \frac{256\pi e^2}{3 m c \omega_{bf} (k_f a_0)^5}$$

$$\hbar\omega = \frac{\hbar^2 k_f^2}{2m} + \cancel{E_1}$$

neglect: strong ionization

$$\hbar\omega \gg R_y$$

$$\alpha = \frac{e^2}{c\hbar}$$

$$= \frac{64\pi\alpha}{3\sqrt{2}\,a_0^5} \left(\frac{\hbar}{m\omega}\right)^{7/2}$$

$$= \left(\frac{4}{\alpha}\right)^3 \sigma_T \left(\frac{R_y}{\hbar\omega}\right)^{7/2}$$

$$R_y = \frac{m e^4}{2\hbar^2} = \frac{\alpha^2 m c^2}{2} = 13.6\,\mathrm{eV}$$

Rydberg energy

Shape of spectral lines

- Consider 2 states (transition)

$$\begin{array}{l} \rule{2cm}{0.4pt}\ |n\rangle \\ \rule{2cm}{0.4pt}\ |m\rangle \end{array} \ \updownarrow \ \Delta E_{nm} = E_n - E_m > 0$$

indetermination principle on E_n E_m

$\Rightarrow$ energy distribution of transition is around ΔE

- Use perturbative approach: $\hat{H} = \hat{H}^0 + \hat{H}^1(t)$

- State as linear combination of eigen-states: $|\Psi(t)\rangle = \sum_k c_k |k\rangle$

- Schrödinger eq.: (evolution of the state is given by the coefficients c_k)

1) idealization $c_k = \{c_n, c_m\}$: transition between any 2 states $|n\rangle, |m\rangle$ only i.e. no other states are allowed

2) we start with state $|n\rangle$ $\Rightarrow$ at $t=0$ $a_i = 1$ $a_f = 0$

$$\dot{c}_{mn} = -\frac{i}{\hbar}\langle m|\hat{H}^1(t)|n\rangle e^{i\omega_{mn}t}$$

$$= -\frac{i}{\hbar}\langle m|\hat{H}^1|n\rangle e^{-i(\omega-\omega_{mn})t}$$

$\leftarrow$ $\hat{H}^1(t) = \hat{H}^1 e^{-i\omega t}$ ω of incoming radiation

monochromatic plane wave

(for shape of line we care about $\omega-\omega_{mn}$)

3) integrate $\dot{c}$

$$c_m^{(1)}(t) = -\frac{i}{\hbar}\int_0^t \langle m|\hat{H}^1|n\rangle e^{-i(\omega-\omega_{mn})t'} dt'$$

$$= \cancel{-}\frac{\cancel{i}}{\hbar}\frac{\langle m|H^1|n\rangle}{\cancel{-i}(\omega-\omega_{mn})} e^{-i(\omega-\omega_{mn})t'}\Big|_0^t$$

$$= \frac{\langle m|H^1|n\rangle}{\hbar(\omega-\omega_{mn})}\left(e^{-i(\omega-\omega_{mn})t} - 1\right) \qquad [2]$$

- Account for spontaneous transitions rate Γ

$$\boxed{\dot{c}_m^{corr} = \dot{c}_m - \frac{\Gamma_s}{2} c_m} \leftarrow \cdot e^{\Gamma/2 t}$$

$[\Gamma] = 1/s$

$$\dot{c}_m^{corr} e^{\Gamma/2 t} = \dot{c}_m e^{\Gamma/2 t} - \frac{\Gamma}{2} c_m e^{\frac{\Gamma}{2} t}$$

plug eq. [1]

$$\dot{c}_m^{corr} e^{\Gamma/2 t} + \frac{\Gamma}{2} c_m e^{\frac{\Gamma}{2} t} = \partial_t \left(c_m^{corr} e^{\frac{\Gamma}{2} t} \right) = \dot{c}_m e^{\Gamma/2 t} = -\frac{i}{\hbar} \langle m | \hat{H}^1 | n \rangle e^{-i(\omega - \omega_{mn}) t} \, e^{\Gamma/2 t}$$

Integrate to get c_m^{corr} $\Rightarrow$ $$c_m^{corr} e^{\frac{\Gamma}{2} t} = -\frac{i}{\hbar} \langle m | \hat{H}^1 | n \rangle \frac{1 - e^{-i(\omega - \omega_{mn}) t + \frac{\Gamma}{2} t}}{-i(\omega - \omega_{mn}) - \Gamma/2}$$

$$\boxed{c_m^{corr} = \frac{\langle m | \hat{H}^1 | n \rangle}{\hbar} \frac{\overset{\star}{e^{-\frac{\Gamma}{2} t}} - e^{-i(\omega - \omega_{mn}) t}}{(\omega - \omega_{mn}) + i\Gamma/2}}$$

for $t \gg \Gamma^{-1}$ ☆ drops to zero $\Rightarrow$ neglect

- Probability to be in state $|m\rangle$

$$P_{mn} = |c_m^{corr}|^2 = |c_m c_m^*| = \frac{|\langle m | \hat{H}^1 | n \rangle|^2}{\hbar^2} \frac{1}{(\omega - \omega_{mn}) + \Gamma/4}$$

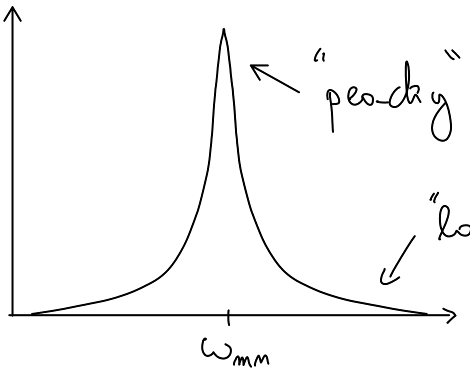
$$= \frac{|\langle m | \hat{H}^1 | n \rangle|^2}{\hbar^2} \frac{\pi}{\Gamma/2} \phi_\Gamma (\omega - \omega_{mn})$$

$$\boxed{\phi_\Gamma (\omega - \omega_{mn}) \equiv \frac{1}{\pi} \frac{\Gamma/2}{(\omega - \omega_{mn}) + \Gamma/4}}$$

Lorentz profile

natural width of spectral lines

$\frac{\Gamma}{2\pi}$ introduced to have $\int \phi \, d\omega \overset{!}{=} 1$



- The shape of lines can be affected by external factors ...

Collisional broadening

- Collision between quantum systems => • random phase shift $\delta\phi$
  - • flat distribution $\delta\phi \in [-\pi, \pi]$
  - • for N collisions $\delta\phi \in [-N\pi, N\pi]$ also flat
  - => • on average, phase factor $\langle e^{i\delta\phi} \rangle = 0$ becaus of flat distribution

- # of collisions within time t : $N = \Gamma_c t$ Γ_c = collision rate

- Collision are discreate events => Poissonian statistics $p_k = \frac{(\Gamma_c t)^k}{k!} e^{-\Gamma_c t}$

  probability to have k collisions

  $\langle e^{i\delta\phi} \rangle = \sum_{k=0}^{\infty} e^{i\delta\phi} p^k$ within time t

- For no collisions within t : $k = 0$, $\delta\phi = 0$ => $\langle e^{i\delta\phi} \rangle = \sum_{k=0}^{0} e^{i\delta\phi} p^k = p^0 = e^{-\Gamma_c t}$

- Transition probability $= |a_2|^2$ => add $\dot{a}_2 += -\frac{\Gamma_c}{2} a_2$

- Same as for spontaneous decay => add $\dot{a}^{cor} \longrightarrow \dot{a}^{cor} + \left(-\frac{\Gamma_c}{2} a_2\right)$

$\models\!> \Gamma \longrightarrow \Gamma + \Gamma_c$ => still Lorentzian profile but broader

Thermal broadening

parallel to line-of-sight

- Thermal motion of atoms ⇒ Doppler shift $\omega = \omega_0 \left(1 + \frac{\overset{\downarrow}{v_r}}{c}\right)$

- Thermal gas: $\frac{m \sigma_v^2}{2} = \frac{KT}{2}$ ⇒ $\sigma_v^2 = \frac{KT}{m}$

- Integrate over the velocities: Maxwell-Boltzman distribution

⇒ Gaussian distribution $\phi(\omega - \omega_0) = \frac{c}{\omega_0 \sqrt{2\pi \sigma_v^2}} \exp\left[-\frac{c^2}{2\sigma_v^2}\left(\frac{\omega - \tilde{\omega}}{\omega_0}\right)^2\right]$

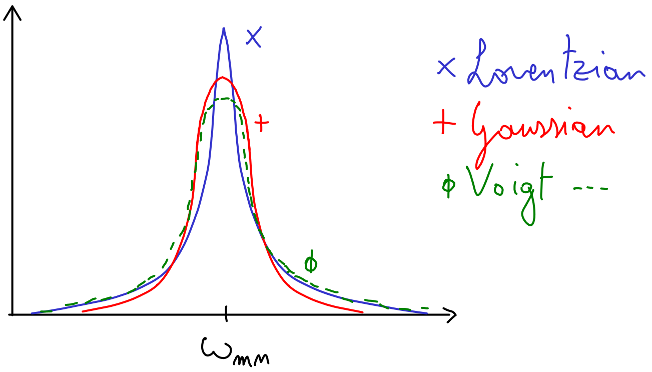
—— o ——

- All contributions combined: convolution of "enlarged" Lorentzian

⇓

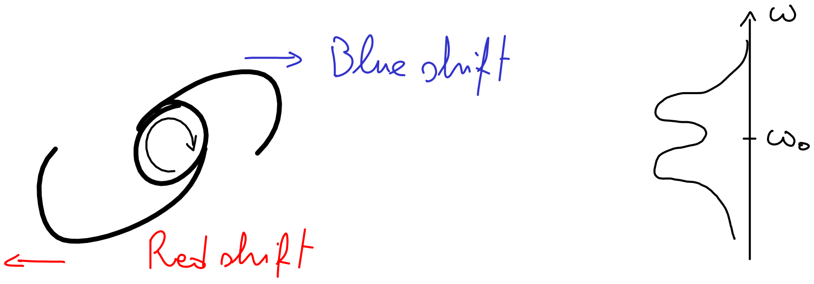
Voigt profile

(Gaussian with broad wings)



- Wider lines:

disk supported galaxy



Stimulated emission

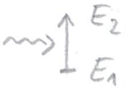
- Simplifyed atoms -> just $E_2 > E_1$ 2 levels
- In equilibrium T with radiation field

Rates ↓

⇒ atom - atom : transition

spontaneous emission → $\boxed{\propto A_{21}}$

field-atom : "



absorbtion rate → $\boxed{\propto I_\omega B_{12}}$

I_ω field-atom : stimulated emission

emission rate → $\boxed{\propto I_\omega B_{21}}$

I_ω

Photons -> Boson ⇒ if quantum state occupied by photon ⇒ increase of n_α is more likely

- A, B : Einstein coefficients
- Equilibrium ⇒

$$\boxed{\tilde{n}_1 I_\omega B_{12} \overset{\text{equilibrium}}{=} \tilde{n}_2 (A_{21} + I_\omega B_{21})}$$

(absorbtions) = (emission)

$\tilde{n}_i$ mean occupation number

E_i energy level

$$\Rightarrow I_\omega = \frac{\tilde{n}_2 A_{21}}{\tilde{n}_1 B_{12} - \tilde{n}_2 B_{21}} \qquad \frac{N_2}{N_1} = e^{-\frac{E_2 - E_1}{K_B T}} = e^{-\omega\hbar}$$

$\left(\text{equilibrium } N_2 e^{\frac{E_2}{kT}} = N_1 e^{\frac{E_1}{kT}}\right)$

$$= \frac{A_{21}}{B_{12} e^{\frac{\hbar\omega}{kT}} - B_{21}} \overset{!}{=} \text{Planck spectrum} \Rightarrow \boxed{B_{12} = B_{21}} \qquad \boxed{A_{21} = \frac{\hbar \omega^3}{4\pi^3 c^2} B_{21}}$$

$$= \frac{4\pi^3 c^2}{\hbar}$$

we started with
the radiation field in equilibrium

$$I_\omega \equiv B_\omega(T) = \frac{\hbar}{4\pi^3 c^2} \frac{\omega^3}{e^{\frac{\hbar\omega}{kT}} - 1}$$

$A_{21} \cdot B_{21}$

B_{12} B_{21}

⇒ you need stimulated emission (B_{21}) to have the Planck spectrum

Absorbtion / emission coefficients

- Energy absorbed/emitted per unit of V, t, ω from Ω : $\left[\frac{dE}{dV\,dt\,d\omega\,d\Omega}\right]$
  - absorbtion coefficent : $\alpha_\omega I_\omega$
  - induced emission coeff. : $\alpha^{ind}_\omega I_\omega$

  $\Big\}$ $\alpha^{net}_\omega = \alpha_\omega - \alpha^{ind}_\omega$ Net absorbtion
  - sponataneous emission : j_ω (emissivity)

- Meaning of $\alpha_\omega, \alpha^{net}_\omega, \alpha^{ind}_\omega$

$$[I_\omega] = \left[\frac{dE}{dA\,dt\,d\omega\,d\Omega}\right] \Rightarrow \alpha^{i}_\omega \text{ must be } [\text{length}^{-1}] \mapsto \frac{1}{\alpha_\omega} = \text{mean free path}$$

- Relate mean free path to properties of the absorbing/emitting medium

$$\alpha_\omega = n \cdot \sigma_\omega$$
$$= \rho\, K$$

n : n° density of absorbers $\left[\frac{1}{m^3}\right]$ σ_ω : absorbers cross section $[m^2]$

ρ : density of medium $\left[\frac{Kg}{m^3}\right]$ K : opacity $\left[\frac{m^2}{Kg}\right]$ (absorbtion per unit mas

- At equilibrium: $\boxed{\underset{\text{(absorbed)}}{\alpha_\omega I_\omega} = \underset{\text{(emitted)}}{j_\omega + \alpha^{ind}_\omega I_\omega}}$ $\Rightarrow$ $\boxed{I_\omega = \frac{j_\omega}{\alpha^{net}_\omega}}$ Kirkoff's law

$$\longrightarrow I_\omega = B_\omega(T) \text{ Planck spectrum} \Rightarrow \boxed{\alpha^{net}_\omega = \frac{j_\omega}{B_\omega(T)}}$$

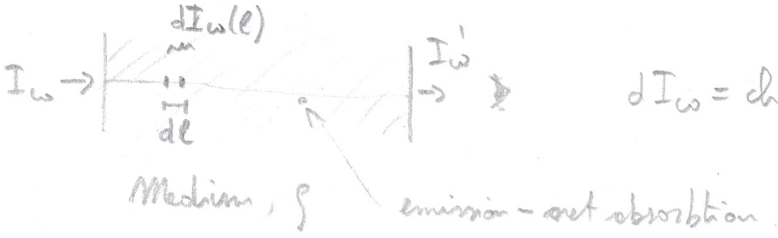
- Relation with Einstein's coefficents

$$I_\omega = \frac{\tilde{n}_2 A_{21}}{\tilde{n}_1 B_{12} - \hat{n}_2 B_{21}} = \frac{A_{21}}{B_{12}}\left(\frac{\tilde{n}_1}{\tilde{n}_2} - 1\right)^{-1} \Rightarrow \boxed{\frac{j_\omega}{\alpha^{net}_\omega} = \frac{\hbar\omega^3}{4\pi^3 c^2}\left(\frac{\tilde{n}_1}{\tilde{n}_2} - 1\right)^{-1}}$$

(again ... simplified atom with only 2 levels) ⟶ their ratio is related to the ratio of the mean occupations numbers of the 2 energy levels

Radiation transport

- How the intensity I_ω changes across the medium



dI_ω = change in intensity after length dl

$$\begin{cases} dI_\omega^{abs} = \alpha_\omega^{net} I_\omega \cdot dl & \text{absorbed intensity} \\ dI_\omega^{em} = j_\omega dl & \text{emitted "} \end{cases} \Rightarrow \boxed{dI_\omega = j_\omega dl - \alpha_\omega^{net} I_\omega dl} \quad (1)$$

(neglect scattering)

- Solve diff. equation

1) Homogeneous eq. $j=0$ $\Rightarrow$ $I_\omega = I_{\omega,0} \exp\left(-\int \alpha_\omega^{net}(l)\, dl\right)$ ↑ Plug

2) Plug in (1) $\Rightarrow$ $I_{\omega,0}(l) = \int dl \left[j_\omega \exp\left(\int \alpha_\omega^{net}(l)\, dl\right)\right] + C$
(variation of constants)

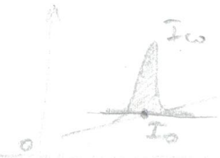
3) Assume $\alpha_\omega^{net}(l) = \text{const}$ (i.e. $\rho(l) = \text{const}$)
and $I_\omega(0) = 0$

$$\Rightarrow \boxed{I_\omega(l) = \frac{j_\omega}{\alpha_\omega^{net}}\left(1 - e^{-\alpha_\omega^{net} \cdot l}\right)}$$

Equivalent width

$$\downarrow$$

$$W \equiv \int \frac{I_0 - I(\omega)}{I_0} d\omega$$



I continuum

- intensity removed / add to the underlying spectrum

$$\tau \equiv N L \sigma(\omega) \leftarrow \underline{\text{Optical depth}}$$
$\left[\frac{1}{m^3} \cdot m \cdot m^3\right]$ (absorbtion)

N n° of absorbers
L geometrical extent of medium
$\sigma(\omega)$ cross section

$$I(\omega) = I_0 e^{-\tau(\omega)} \leftarrow \underline{\text{Intensity within the line}}$$

I_0 = continuum emission

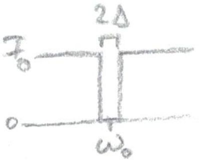
$$W = \int 1 - e^{-\tau(\omega)} d\omega \qquad \sigma \propto \phi(\omega) \quad (\text{line profile})$$

$$= \int 1 - e^{-c\phi(\omega)} d\omega \qquad c = 2\pi N L r_e c f_{12} = \text{const}$$ From transition cross section

$$\underline{\tau \ll 1} \Rightarrow \boxed{W} = \int \cancel{1} - \cancel{1} + \tau(\omega)\, d\omega = \int N L \sigma(\omega) d\omega = \boxed{N} \cdot \overbrace{L \int \sigma(\omega) d\omega}^{\text{const}}$$

$W \propto N$

$$\underline{\tau \gg 1} \Rightarrow$$



$$\boxed{W} = \int_{\omega_0 - \Delta}^{\omega_0 + \Delta} 1\, d\omega = \boxed{2\Delta} \Rightarrow W \approx 2\Delta$$

$\underline{\tau \approx 1}$ a) Optically thick in the core, not in the wing

- Gaussian core because of redshift broadening (for simplicity)

$$\tau = N L \sigma(\omega) = N L (\text{const}) \exp\left(-\frac{c^2(\omega - \omega_{12})^2}{2\sigma_v^2}\right) \qquad \sigma_v^2 = \frac{KT}{m}$$

$$\text{FWHM}: \Delta = \omega - \omega_{12} \Rightarrow \Delta \propto \sqrt{\ln N} \Rightarrow \boxed{W \propto \sqrt{\ln N}}$$

b) O. thick already in the Lorentzian wing

pg. 102

$$\phi(\omega) \approx \frac{\Gamma}{2\pi(\omega - \omega_{12})^2} \quad (\text{for simplicity } \Gamma \ll \omega - \omega_{12}$$

$$\Rightarrow \tau = N L \phi(\omega) = N(\text{const}) \frac{\Gamma}{\quad)^2} \ : \text{FWHM} \Rightarrow \boxed{W \propto \sqrt{N}}$$

Part IV

Gas/fluids: hydrodynamics

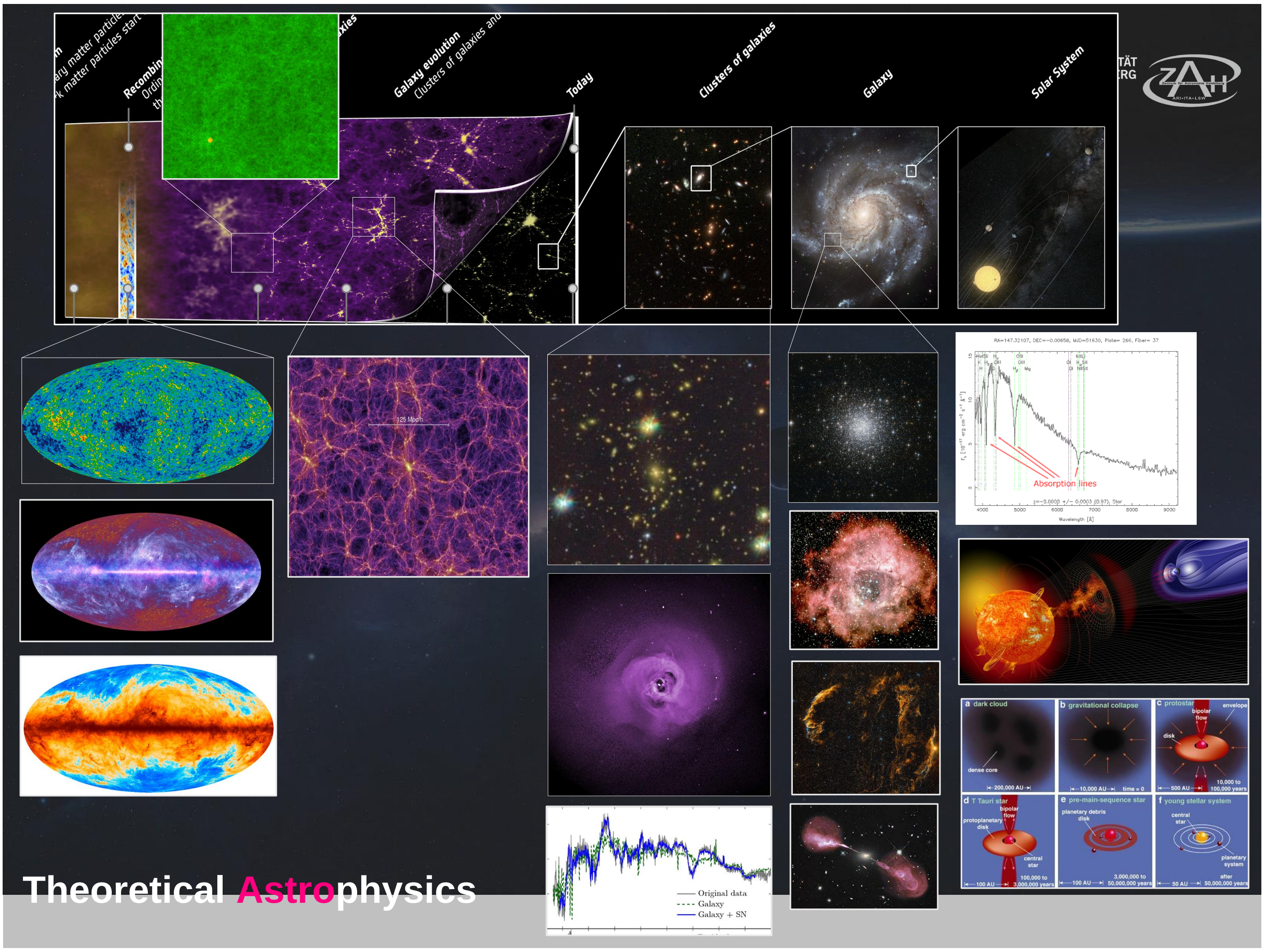
From theory to observations:

- charged particles in E.M. field
- E.M. fields
- energy-momentum tensor of E.M. field
- emission from accelerated charged particles
- back reaction

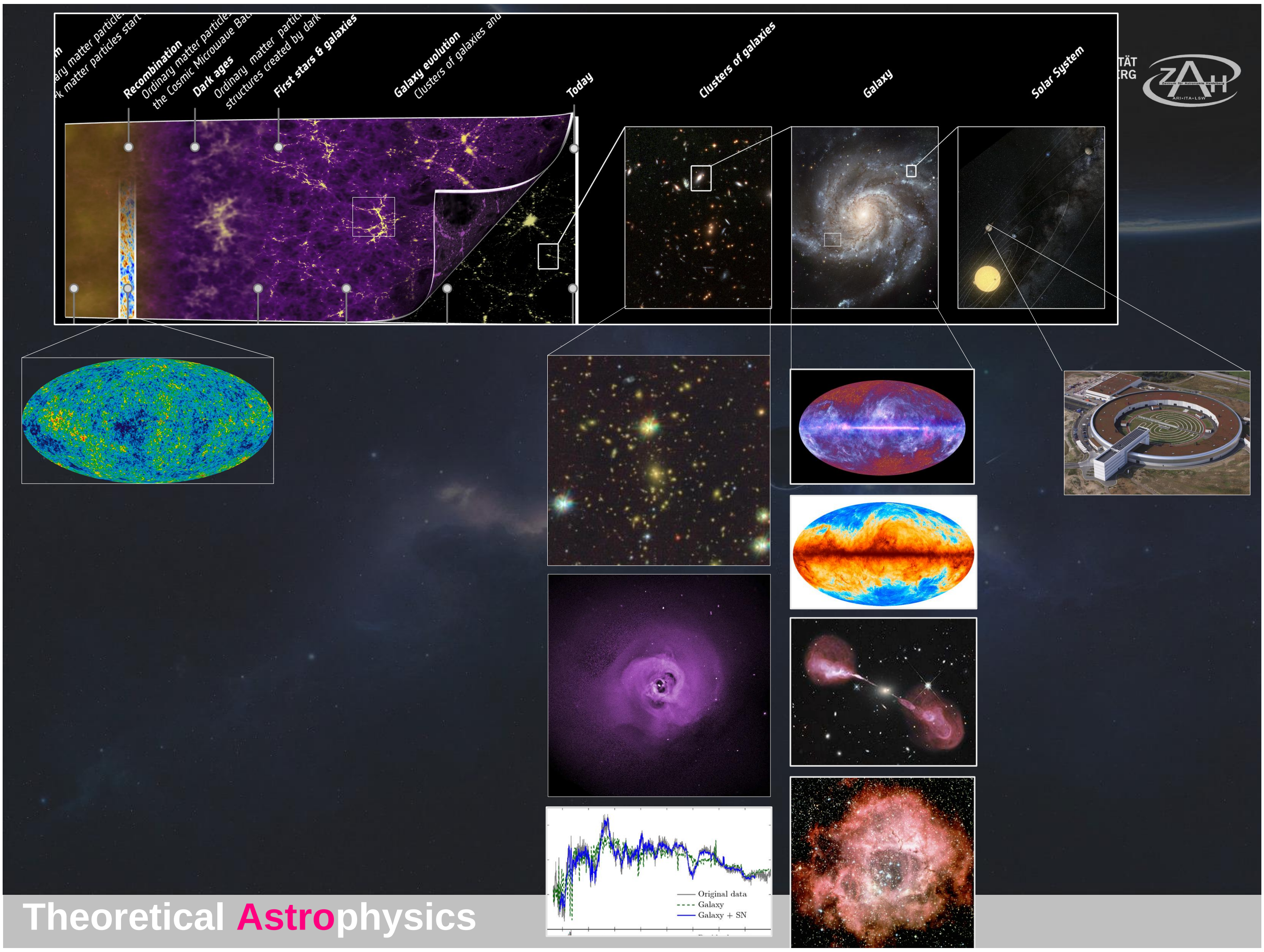
- Thompson and Compton scattering
- Synchrotron
- Bremsstrahlung (free-free)
- Planck spectrum
- Quantum transitions

Matteo Maturi

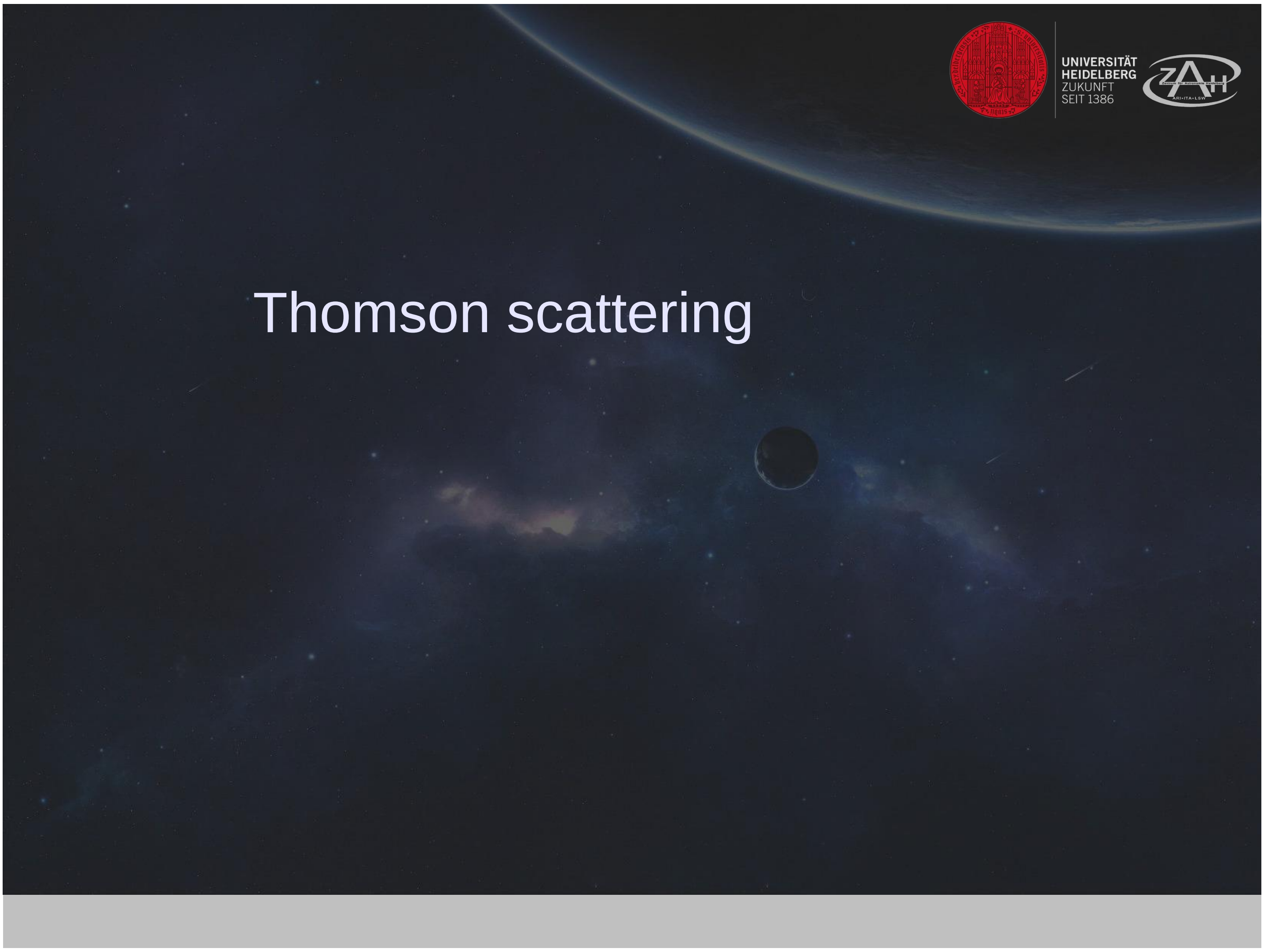
Center for Astronomy & Institute for Theoretical Physics (Heidelberg University)



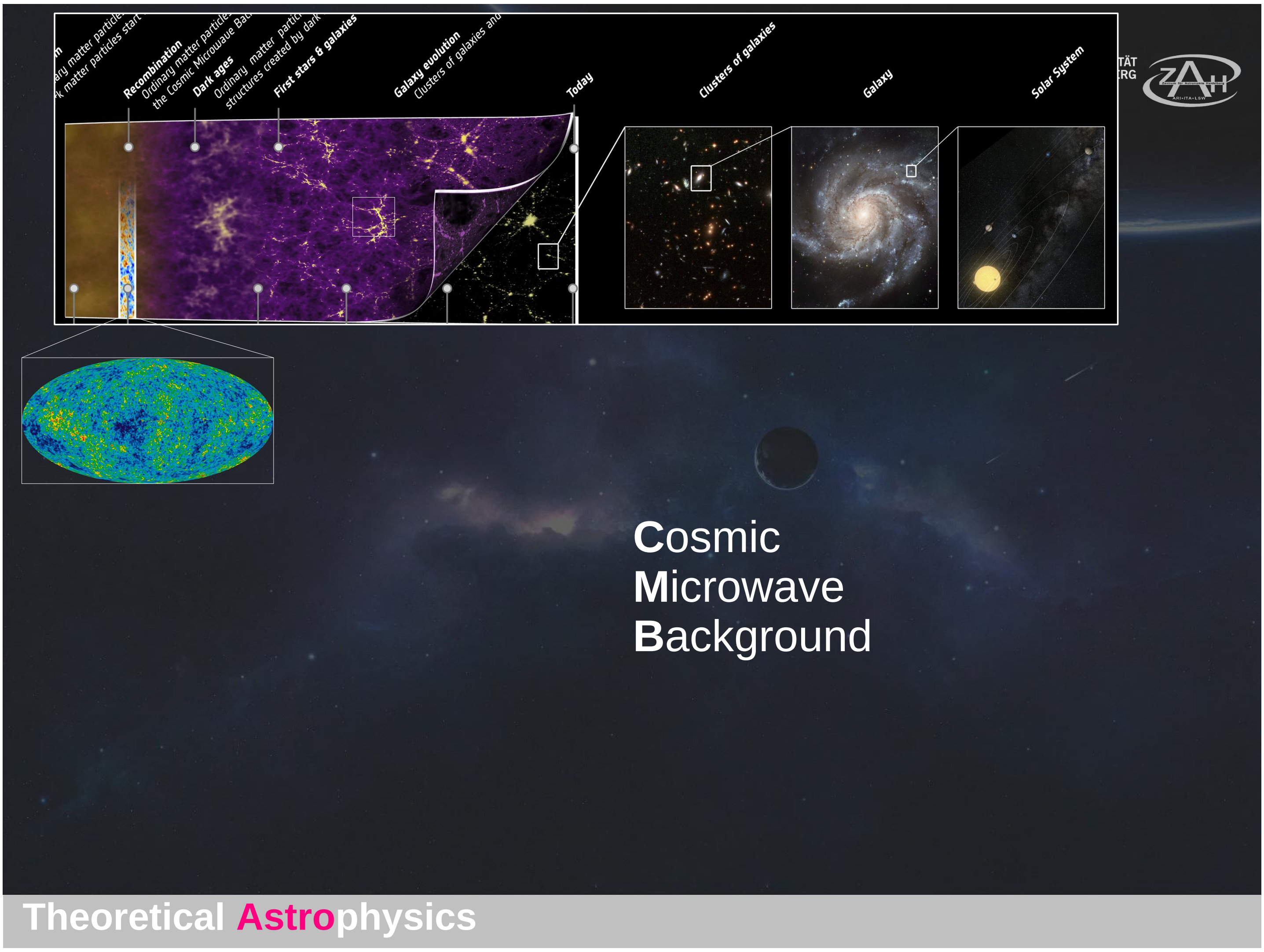
Theoretical Astrophysics



Theoretical Astrophysics

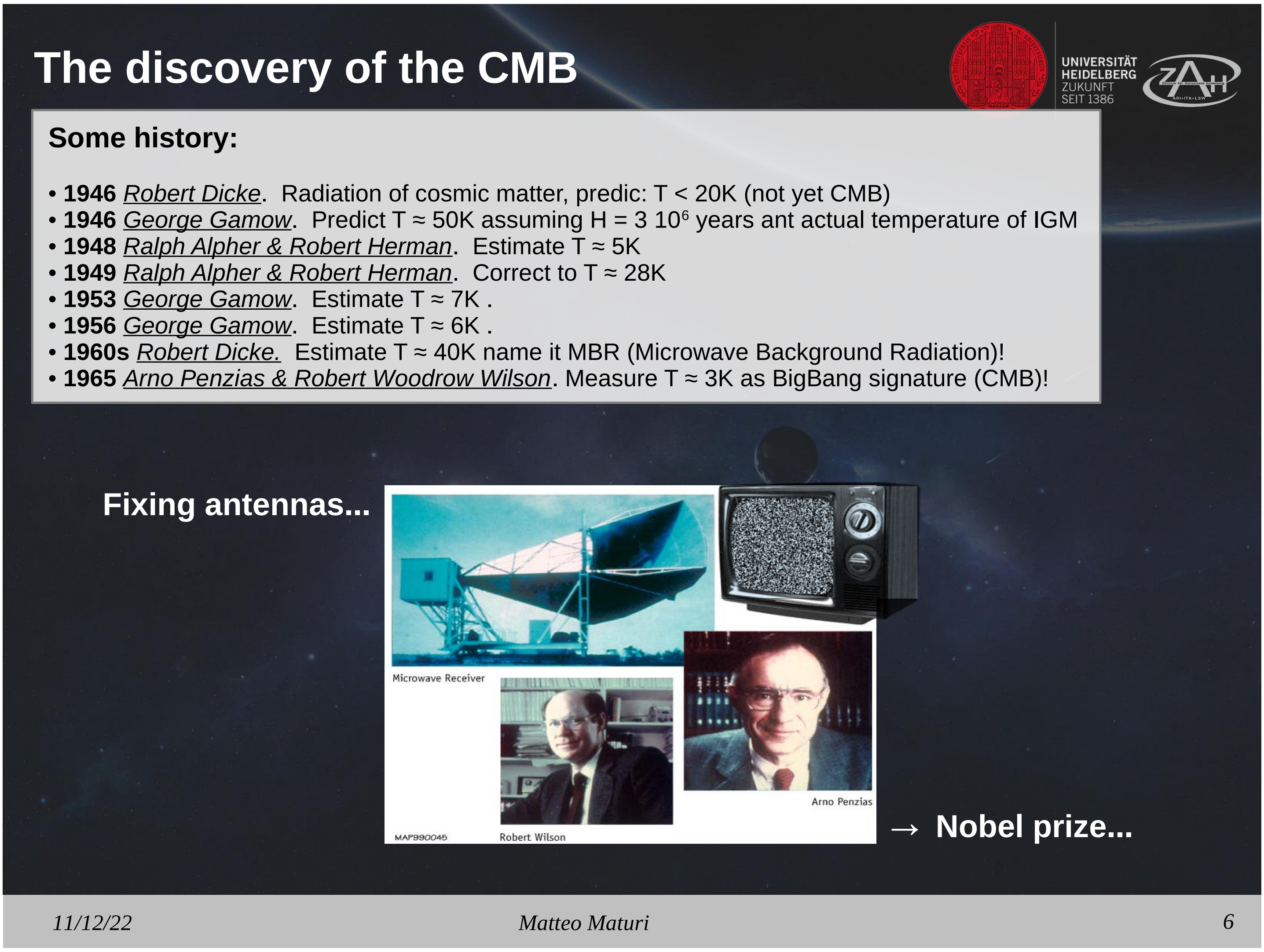


Thomson scattering



Cosmic
Microwave
Background

Theoretical Astrophysics



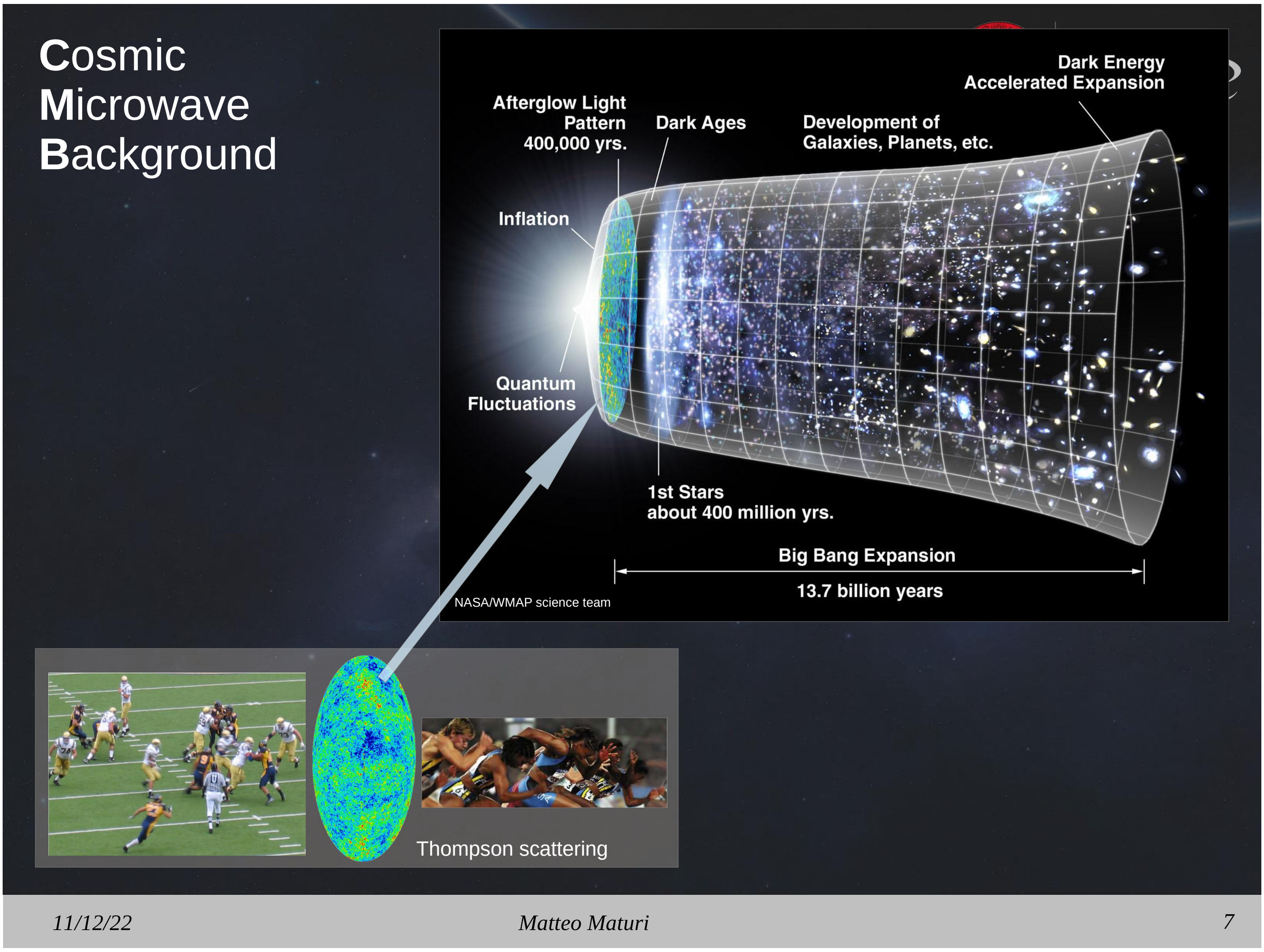
The discovery of the CMB

Some history:

- **1946** *Robert Dicke*. Radiation of cosmic matter, predic: T < 20K (not yet CMB)
- **1946** *George Gamow*. Predict T ≈ 50K assuming H = 3 10^6 years ant actual temperature of IGM
- **1948** *Ralph Alpher & Robert Herman*. Estimate T ≈ 5K
- **1949** *Ralph Alpher & Robert Herman*. Correct to T ≈ 28K
- **1953** *George Gamow*. Estimate T ≈ 7K .
- **1956** *George Gamow*. Estimate T ≈ 6K .
- **1960s** *Robert Dicke.* Estimate T ≈ 40K name it MBR (Microwave Background Radiation)!
- **1965** *Arno Penzias & Robert Woodrow Wilson*. Measure T ≈ 3K as BigBang signature (CMB)!

Fixing antennas...

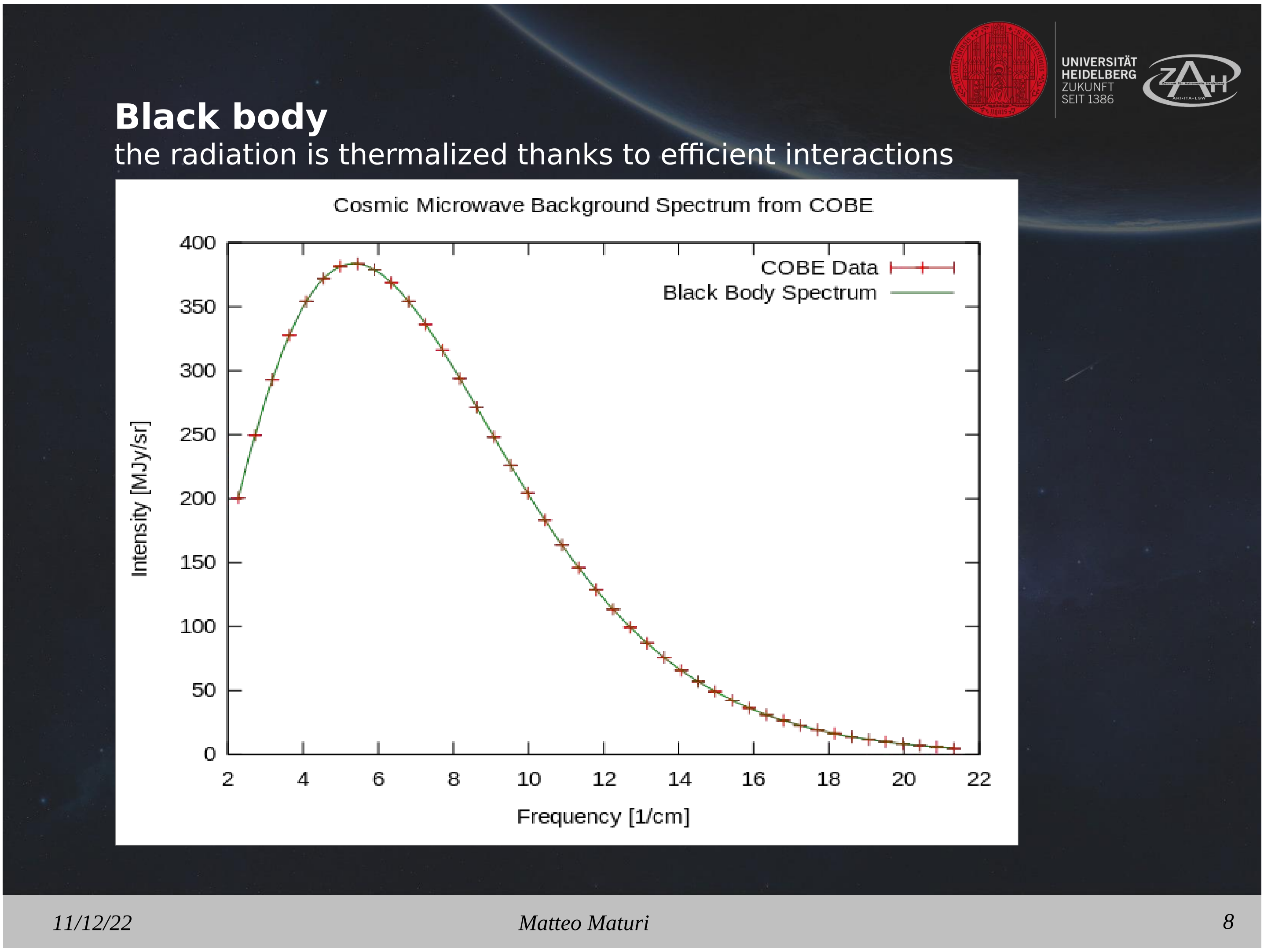
→ Nobel prize...



Cosmic Microwave Background

Thompson scattering

11/12/22 Matteo Maturi 7



Black body

the radiation is thermalized thanks to efficient interactions

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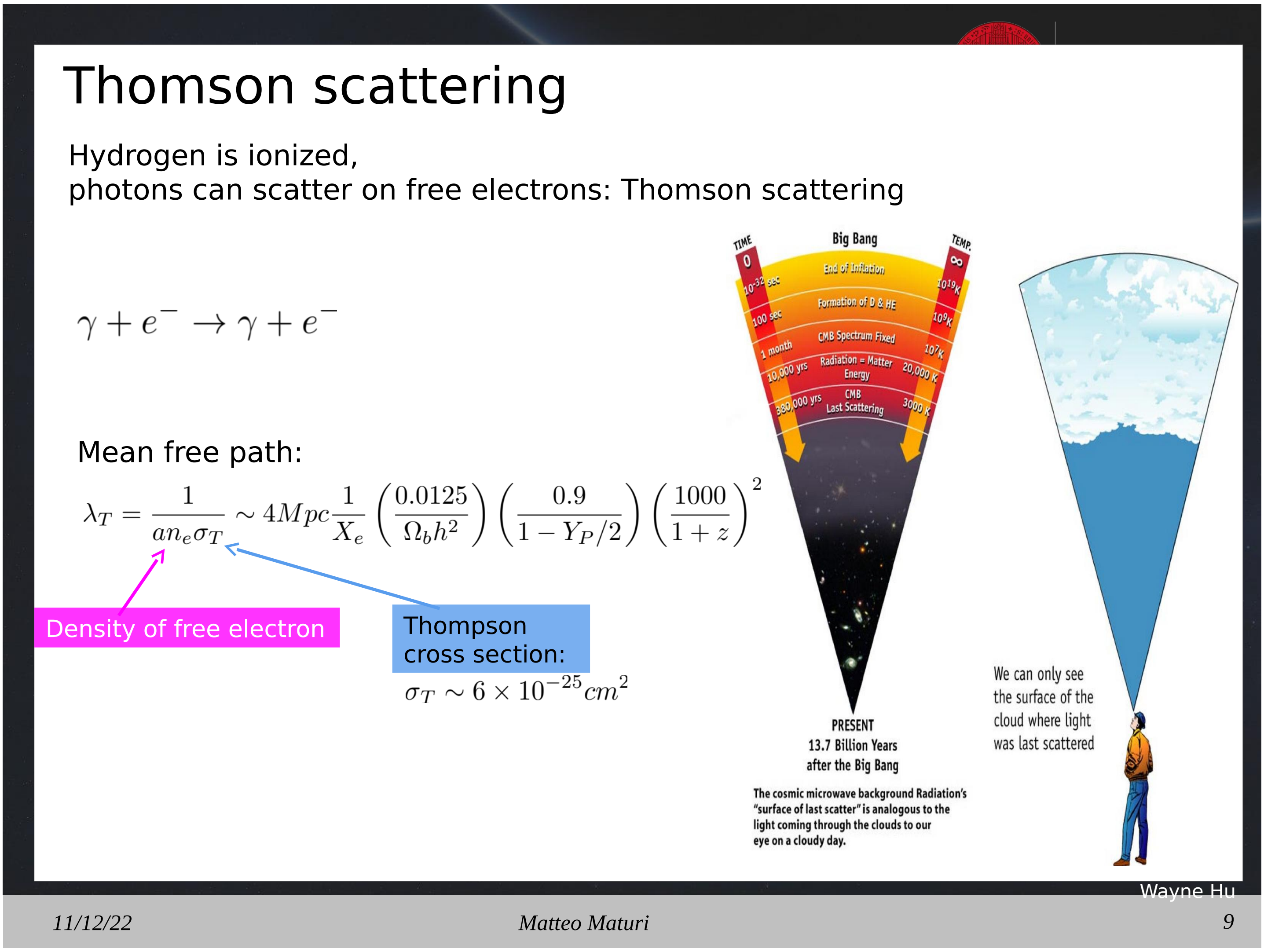


Thomson scattering

Hydrogen is ionized,
photons can scatter on free electrons: Thomson scattering

$$\gamma + e^- \rightarrow \gamma + e^-$$

Mean free path:

$$\lambda_T = \frac{1}{a n_e \sigma_T} \sim 4Mpc \frac{1}{X_e}\left(\frac{0.0125}{\Omega_b h^2}\right)\left(\frac{0.9}{1-Y_P/2}\right)\left(\frac{1000}{1+z}\right)^2$$

Density of free electron

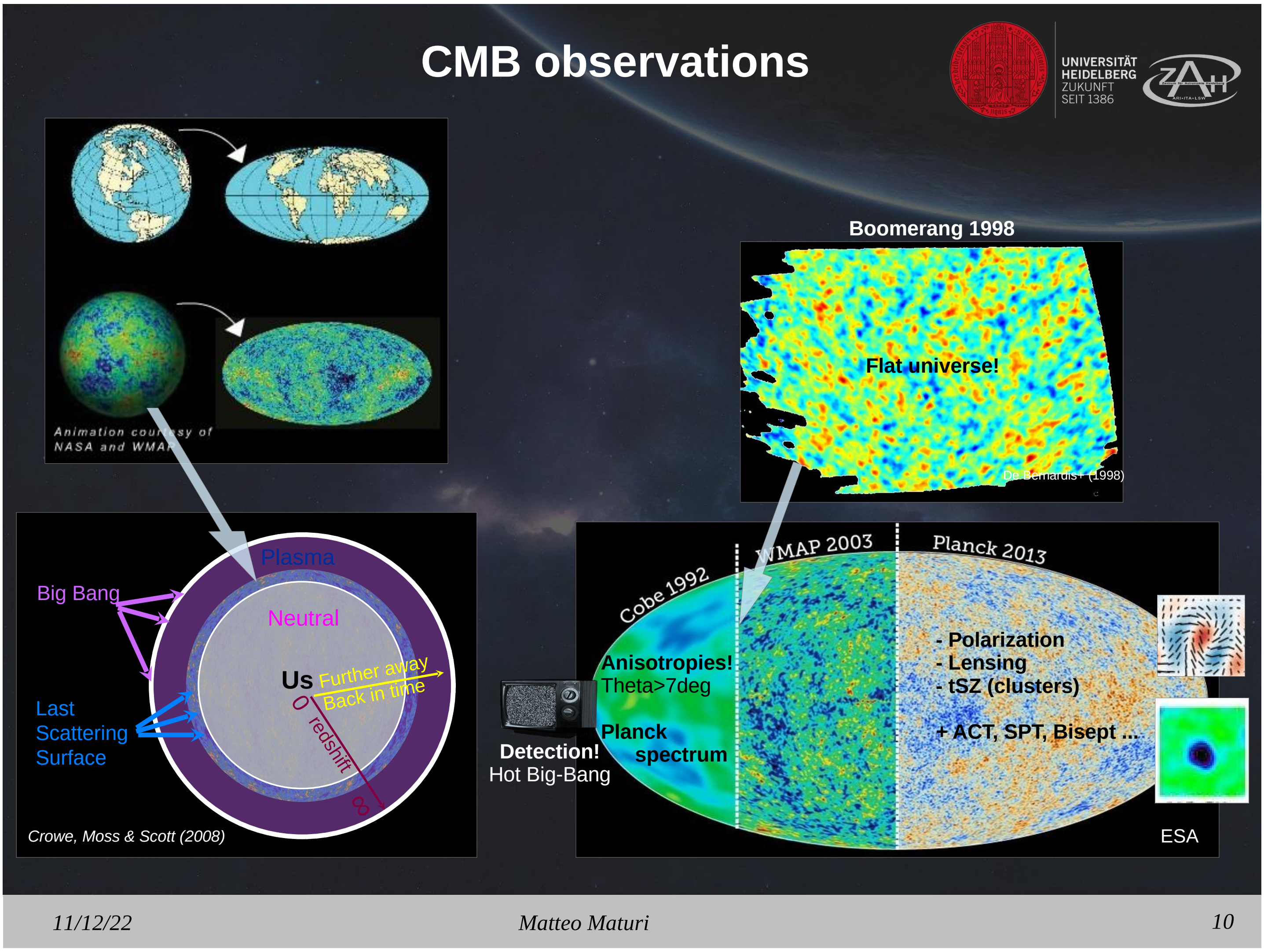
Thompson cross section:

$$\sigma_T \sim 6 \times 10^{-25} cm^2$$

The cosmic microwave background Radiation's "surface of last scatter" is analogous to the light coming through the clouds to our eye on a cloudy day.

We can only see the surface of the cloud where light was last scattered

Wayne Hu



CMB observations

Animation courtesy of NASA and WMAP

Crowe, Moss & Scott (2008)

Boomerang 1998

Flat universe!

De Bernardis+ (1998)

Anisotropies!
Theta>7deg

Planck spectrum

Detection!
Hot Big-Bang

- **Polarization**
- **Lensing**
- **tSZ (clusters)**

+ ACT, SPT, Bisept ...

ESA

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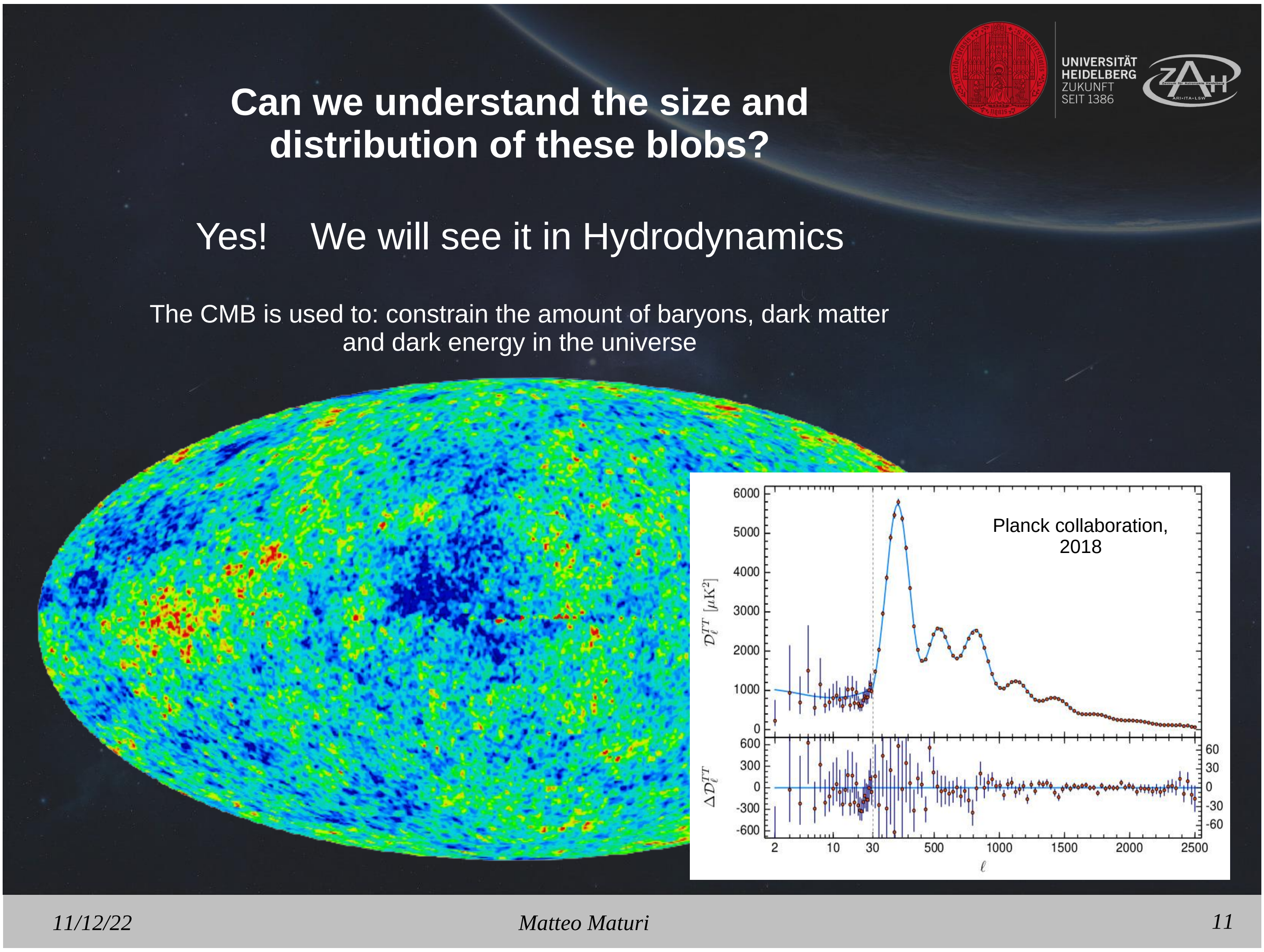
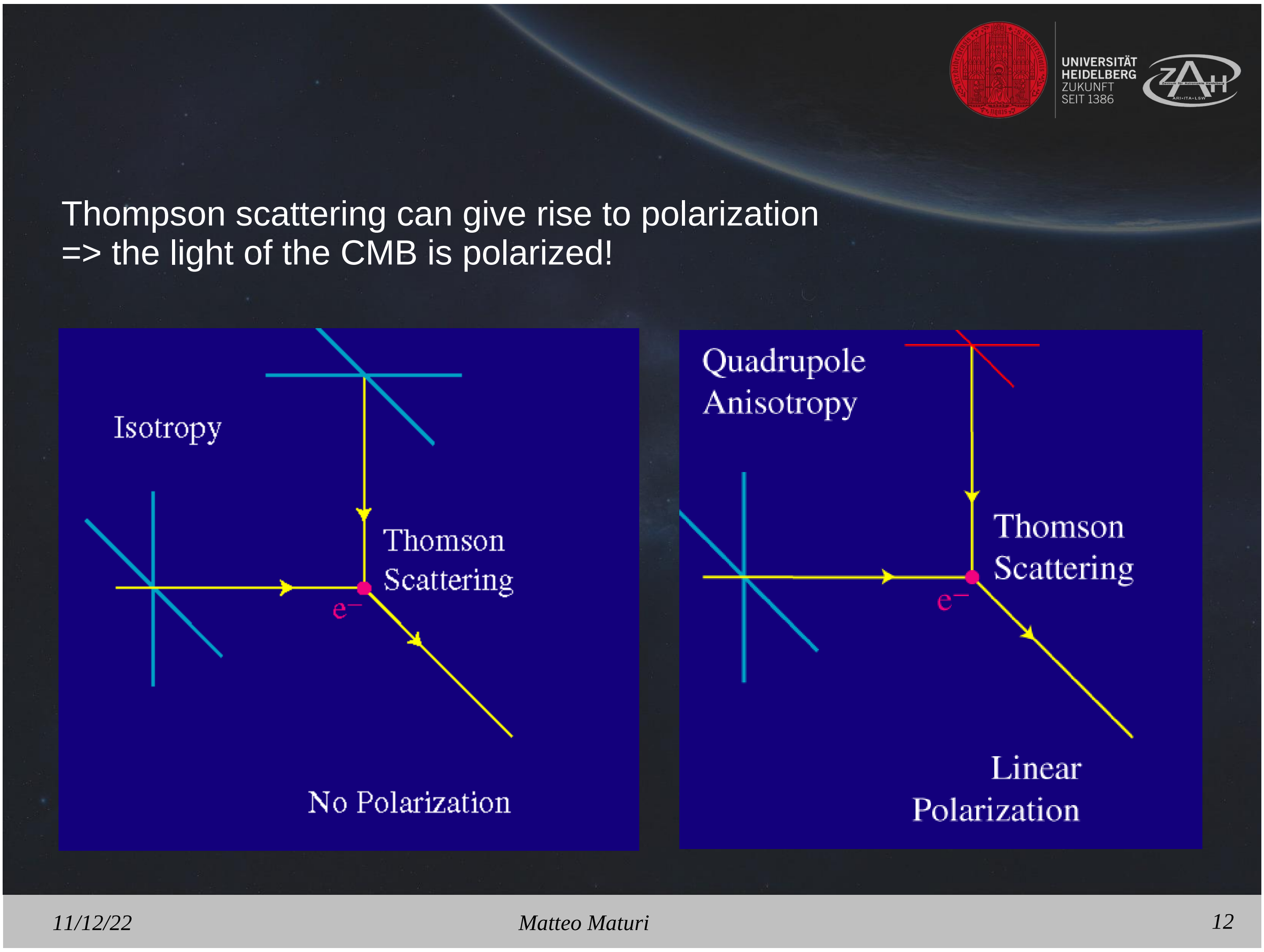


Can we understand the size and distribution of these blobs?

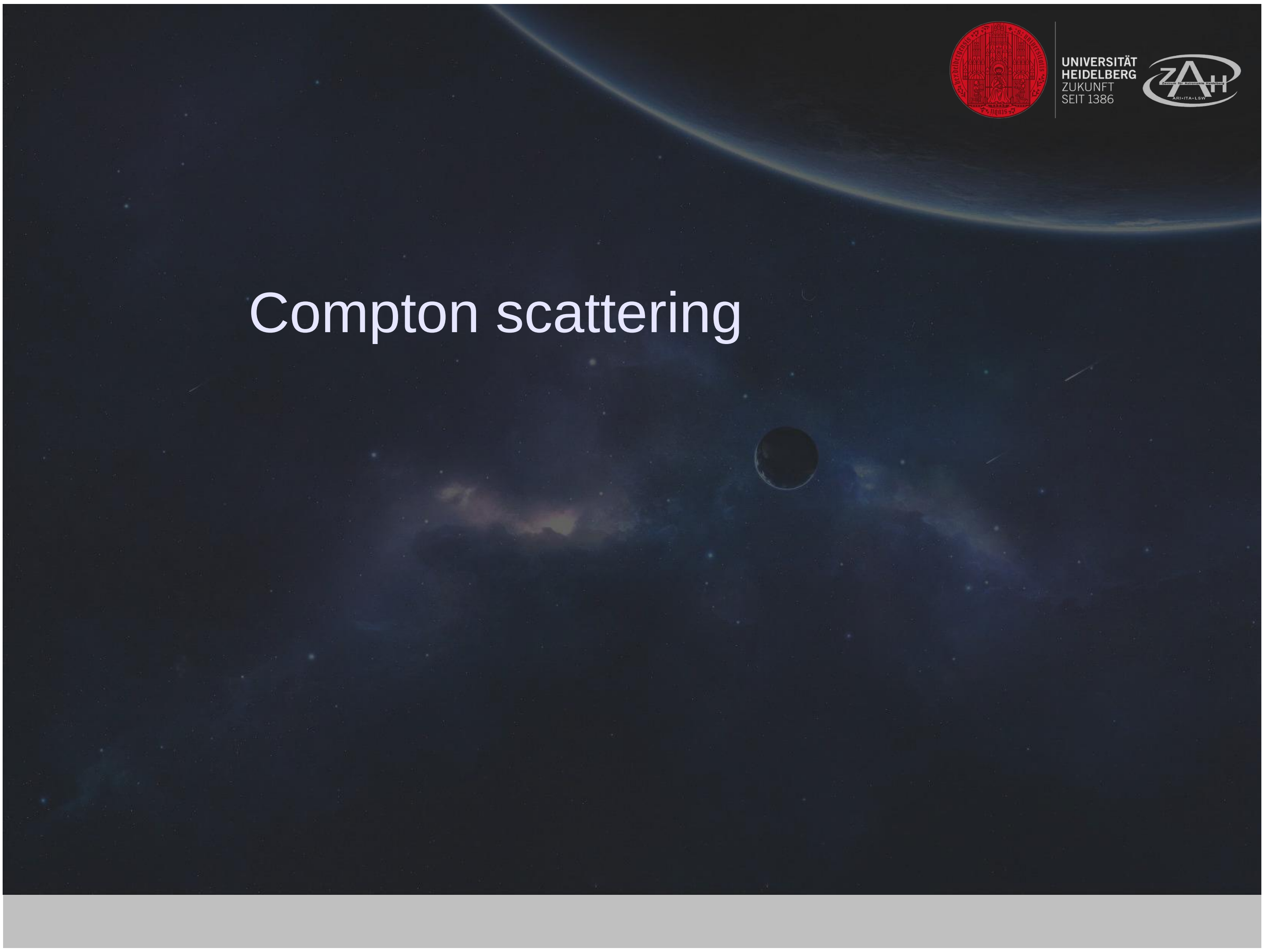
Yes! We will see it in Hydrodynamics

The CMB is used to: constrain the amount of baryons, dark matter and dark energy in the universe

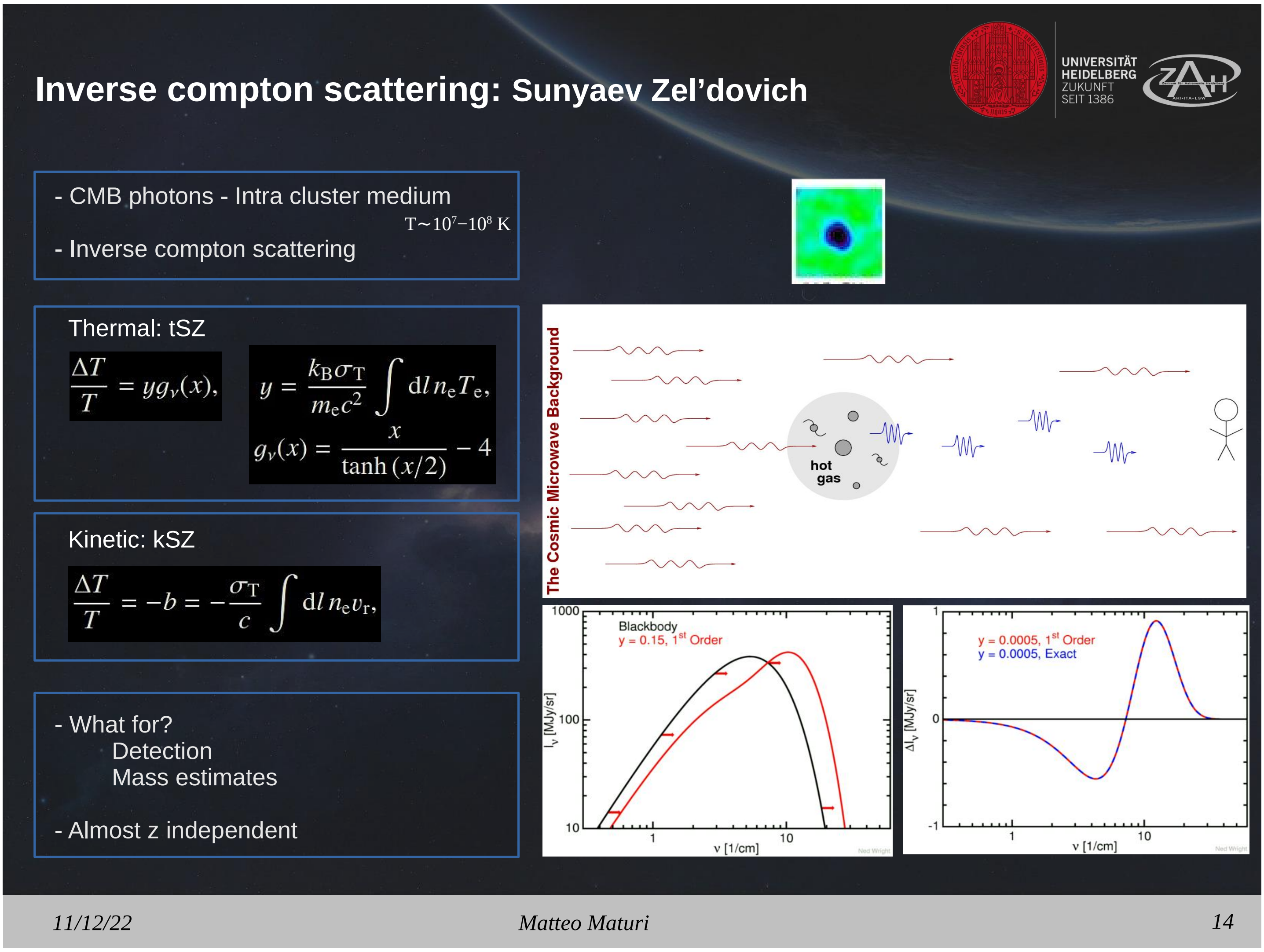
Planck collaboration, 2018



Thompson scattering can give rise to polarization
=> the light of the CMB is polarized!



Compton scattering



Inverse compton scattering: Sunyaev Zel'dovich

- CMB photons - Intra cluster medium

T~10^7–10^8 K

- Inverse compton scattering

Thermal: tSZ

$$\frac{\Delta T}{T} = y g_\nu(x),$$

$$y = \frac{k_\mathrm{B}\sigma_\mathrm{T}}{m_\mathrm{e}c^2}\int \mathrm{d}l\, n_\mathrm{e} T_\mathrm{e},$$

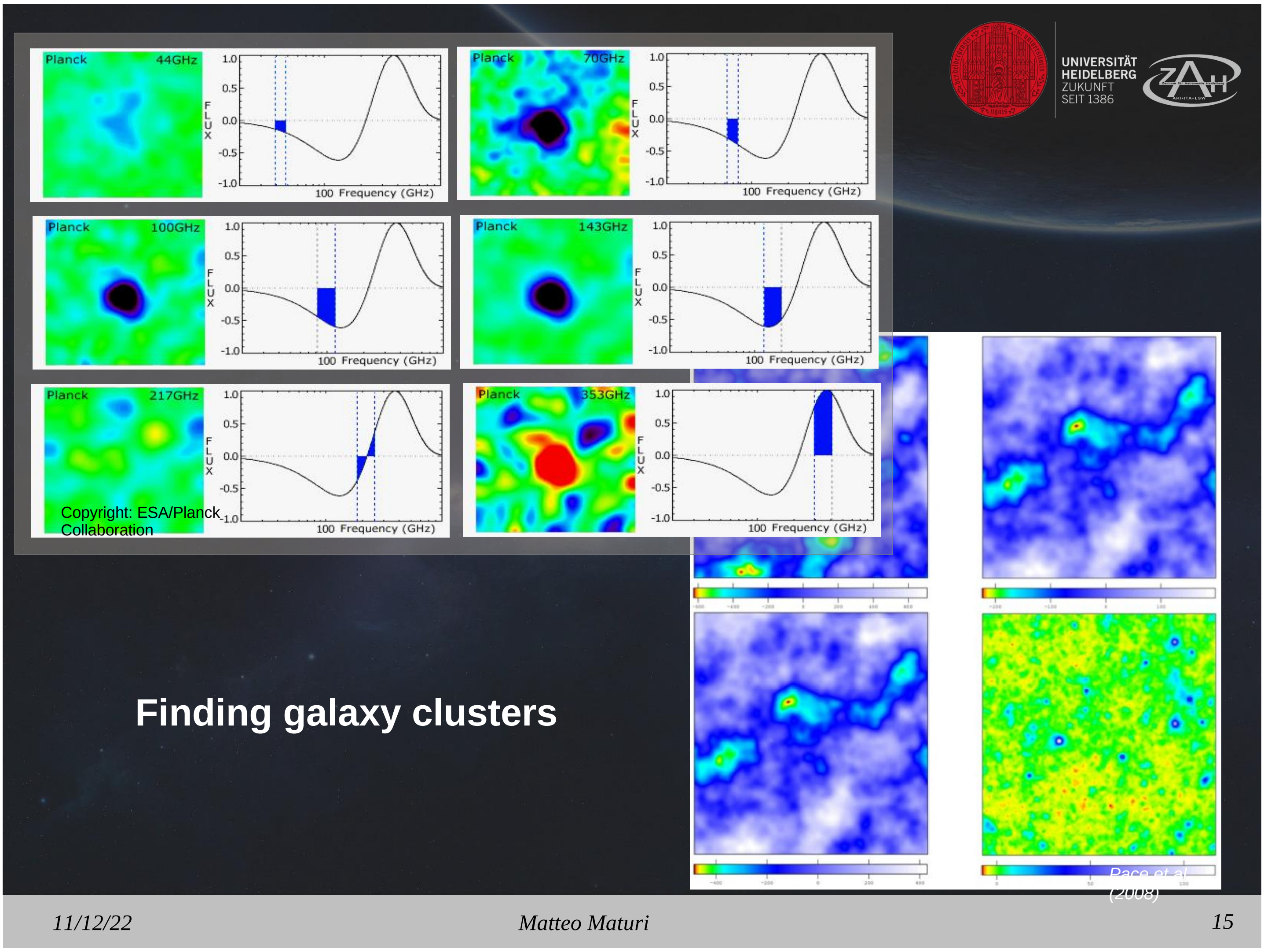
$$g_\nu(x) = \frac{x}{\tanh(x/2)} - 4$$

Kinetic: kSZ

$$\frac{\Delta T}{T} = -b = -\frac{\sigma_\mathrm{T}}{c}\int \mathrm{d}l\, n_\mathrm{e} v_\mathrm{r},$$

- What for?
  - Detection
  - Mass estimates

- Almost z independent



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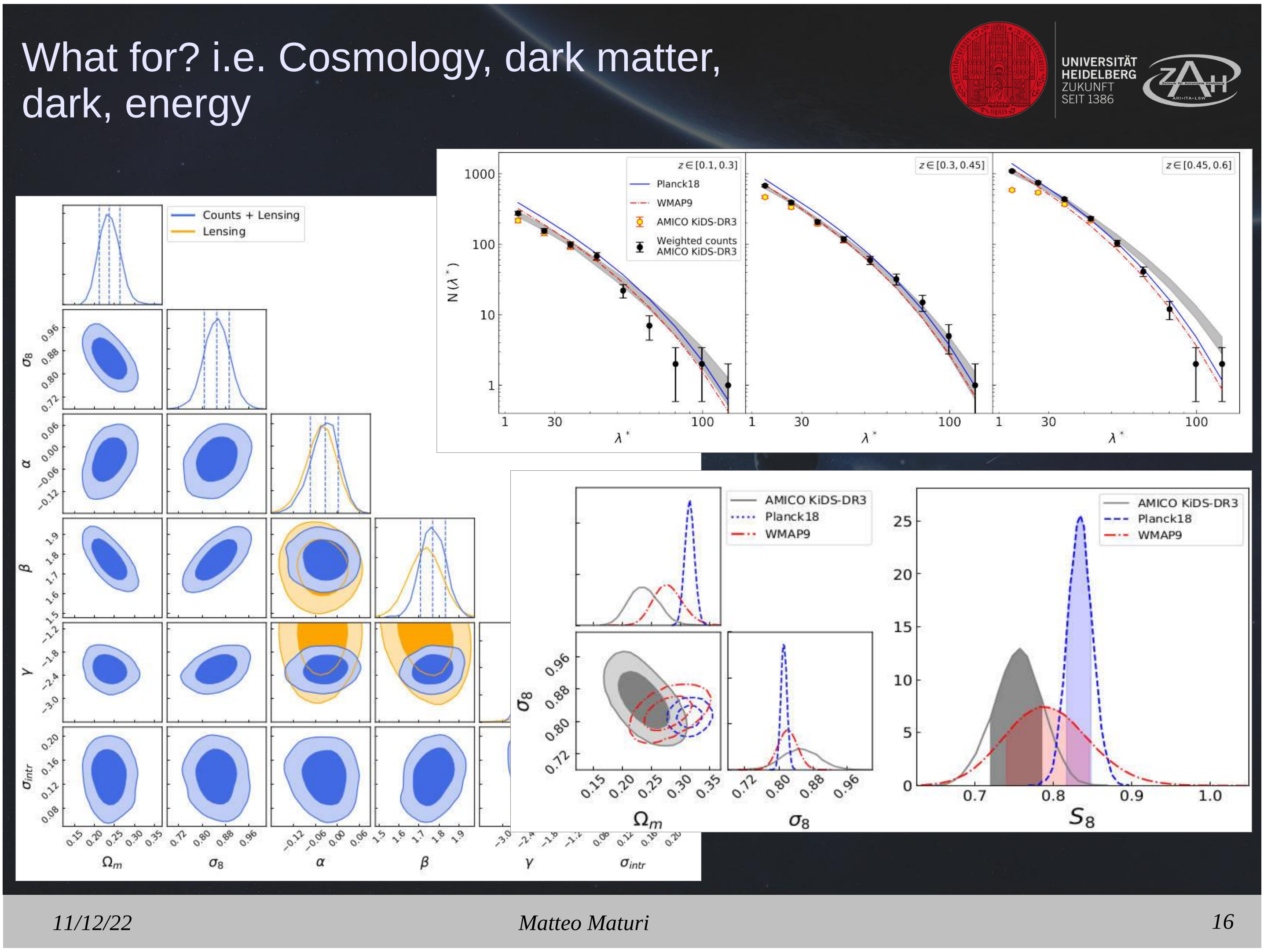
Finding galaxy clusters

Pace et al. (2008)



What for? i.e. Cosmology, dark matter, dark, energy

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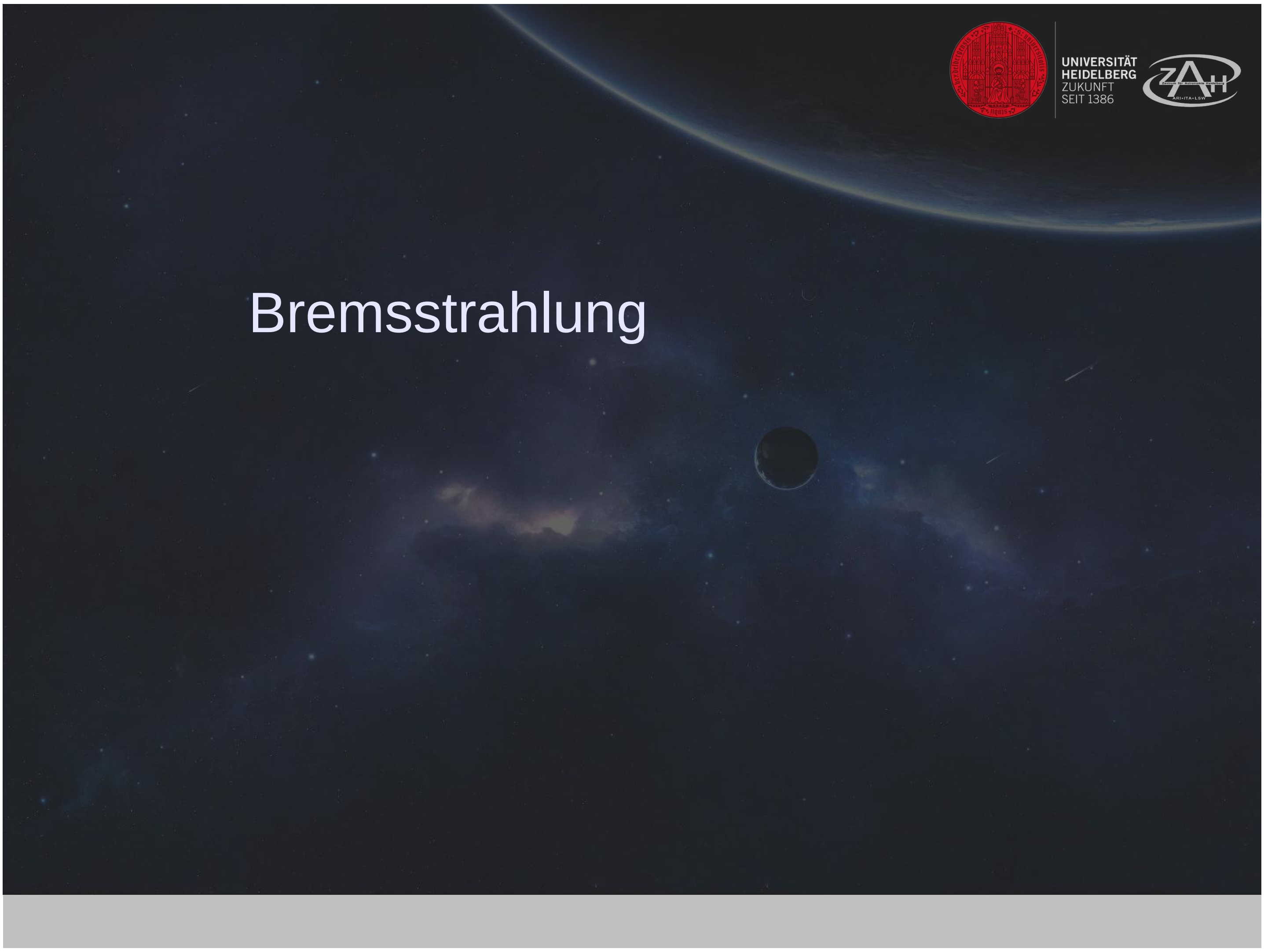
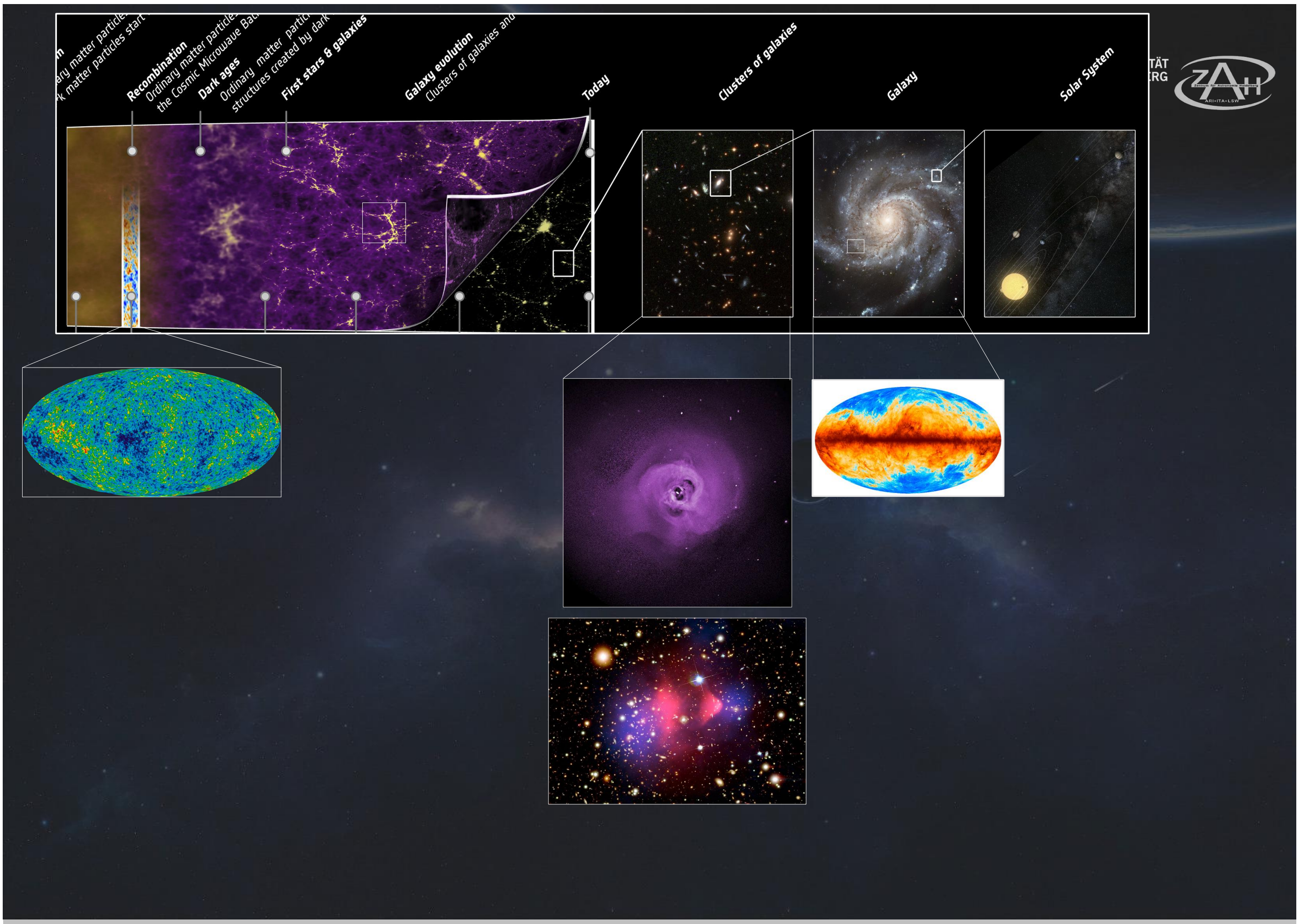
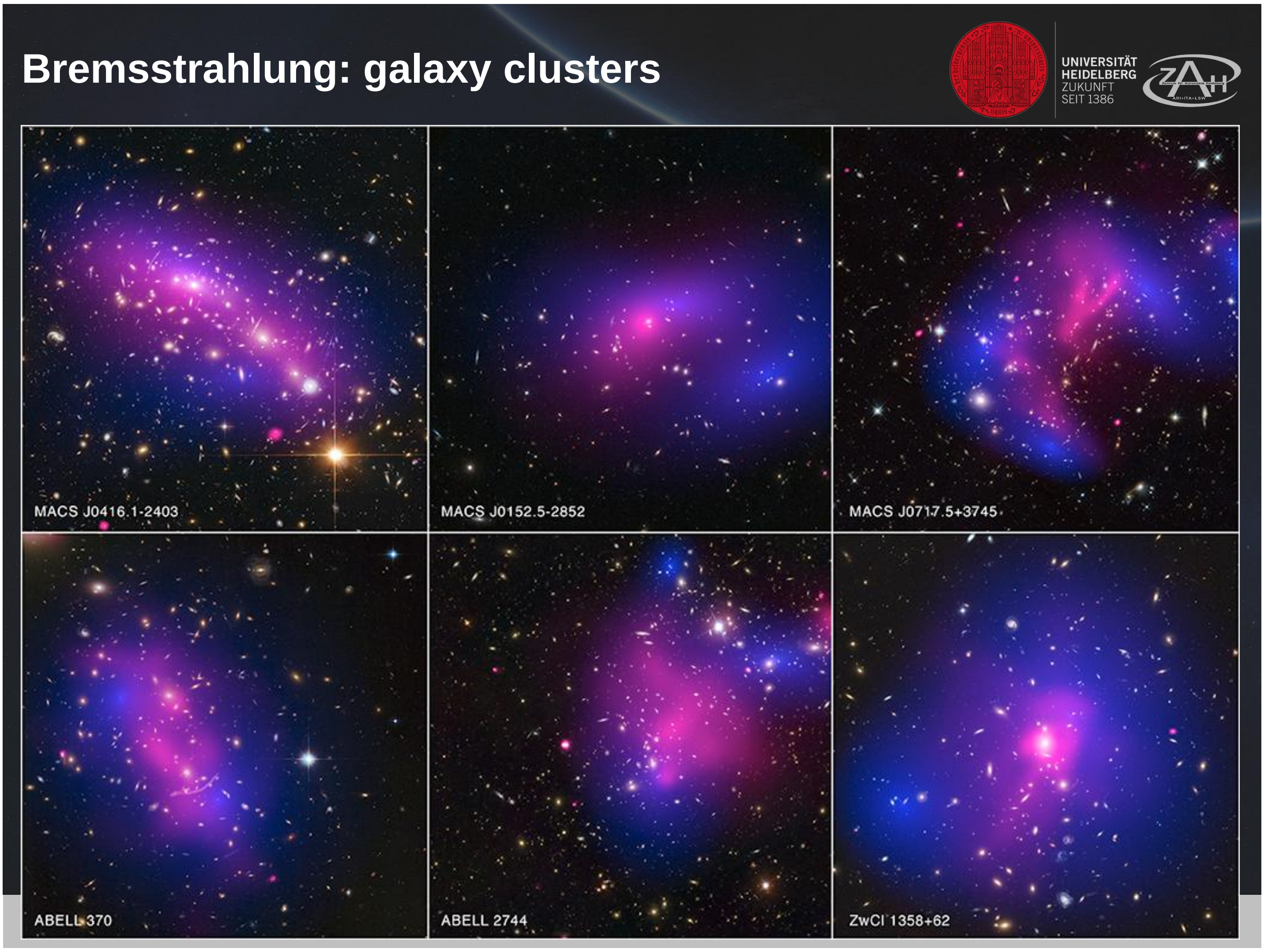


Bremsstrahlung



Theoretical **Astro**physics



Bremsstrahlung: galaxy clusters

Bremsstrahlung: galaxy clusters

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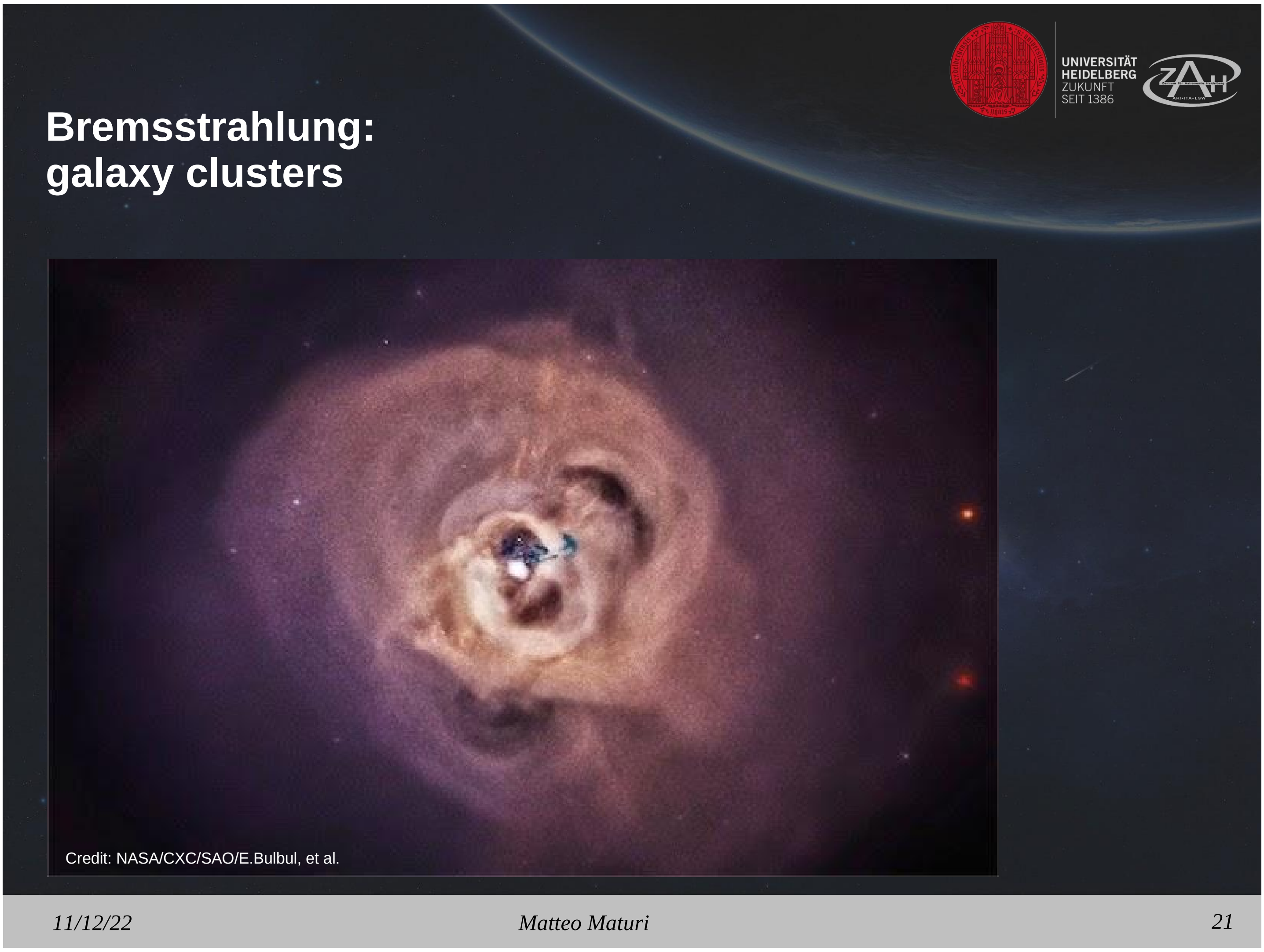

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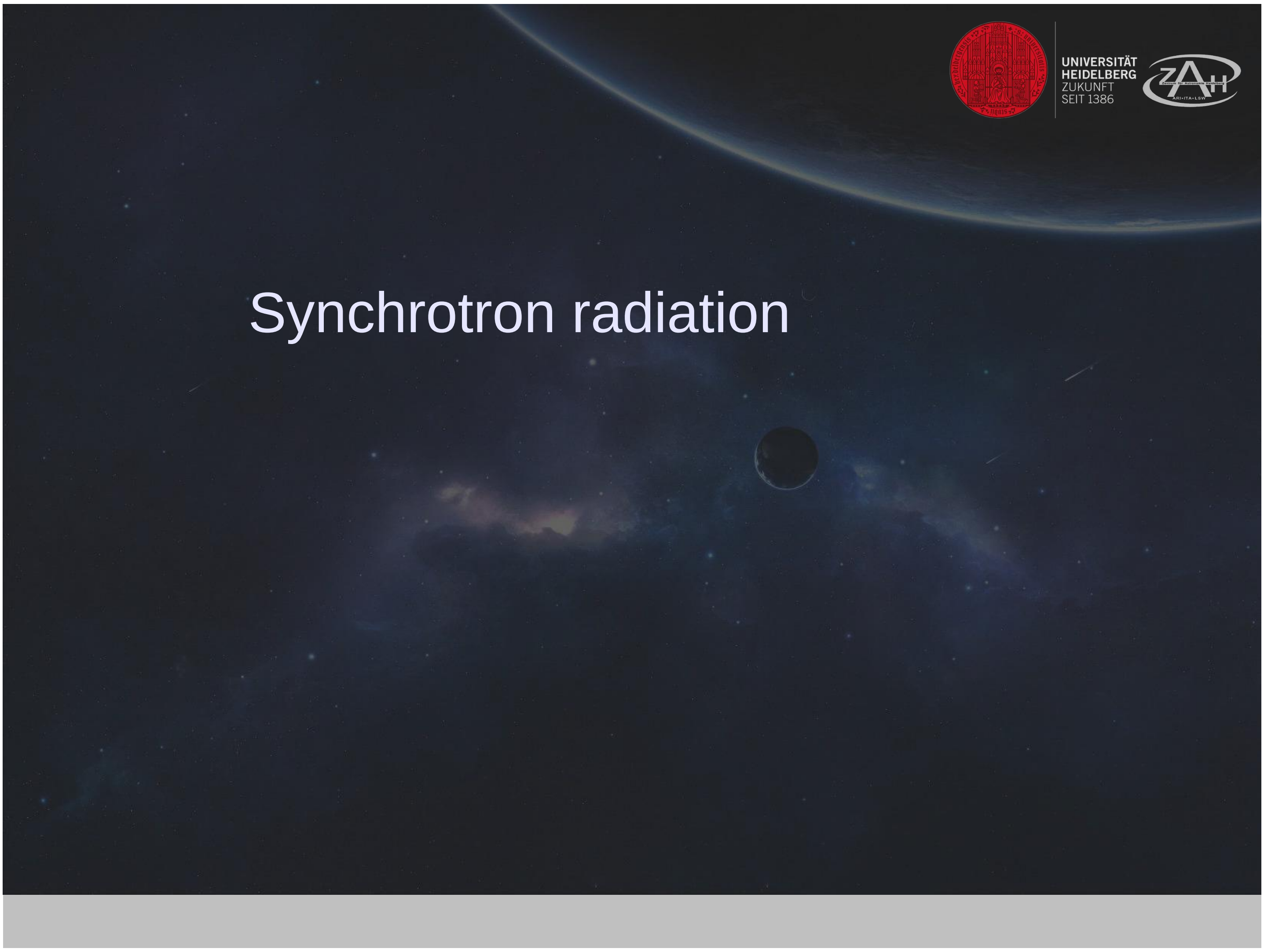
Bremsstrahlung: galaxy clusters

Credit: NASA/CXC/SAO/E.Bulbul, et al.



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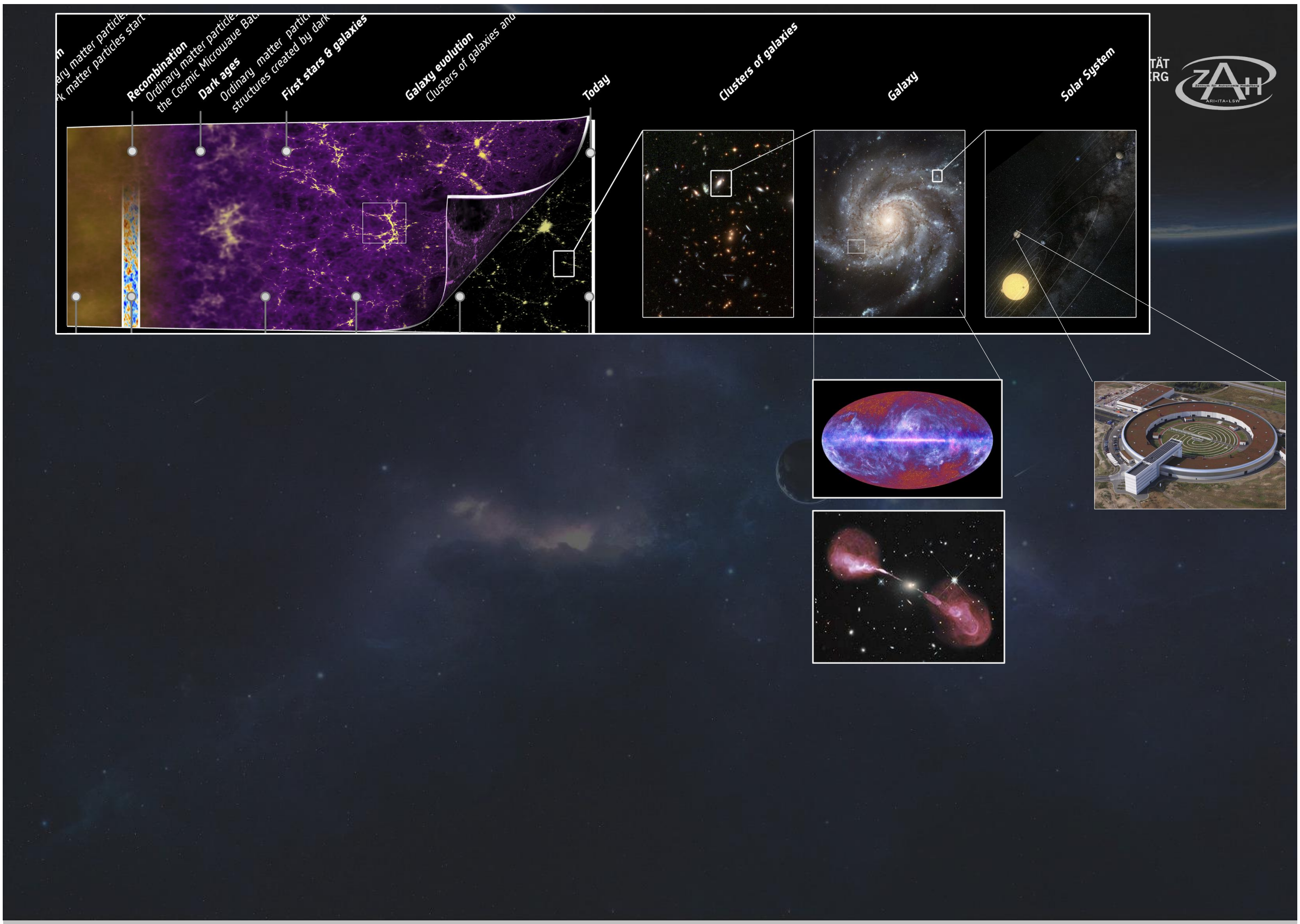
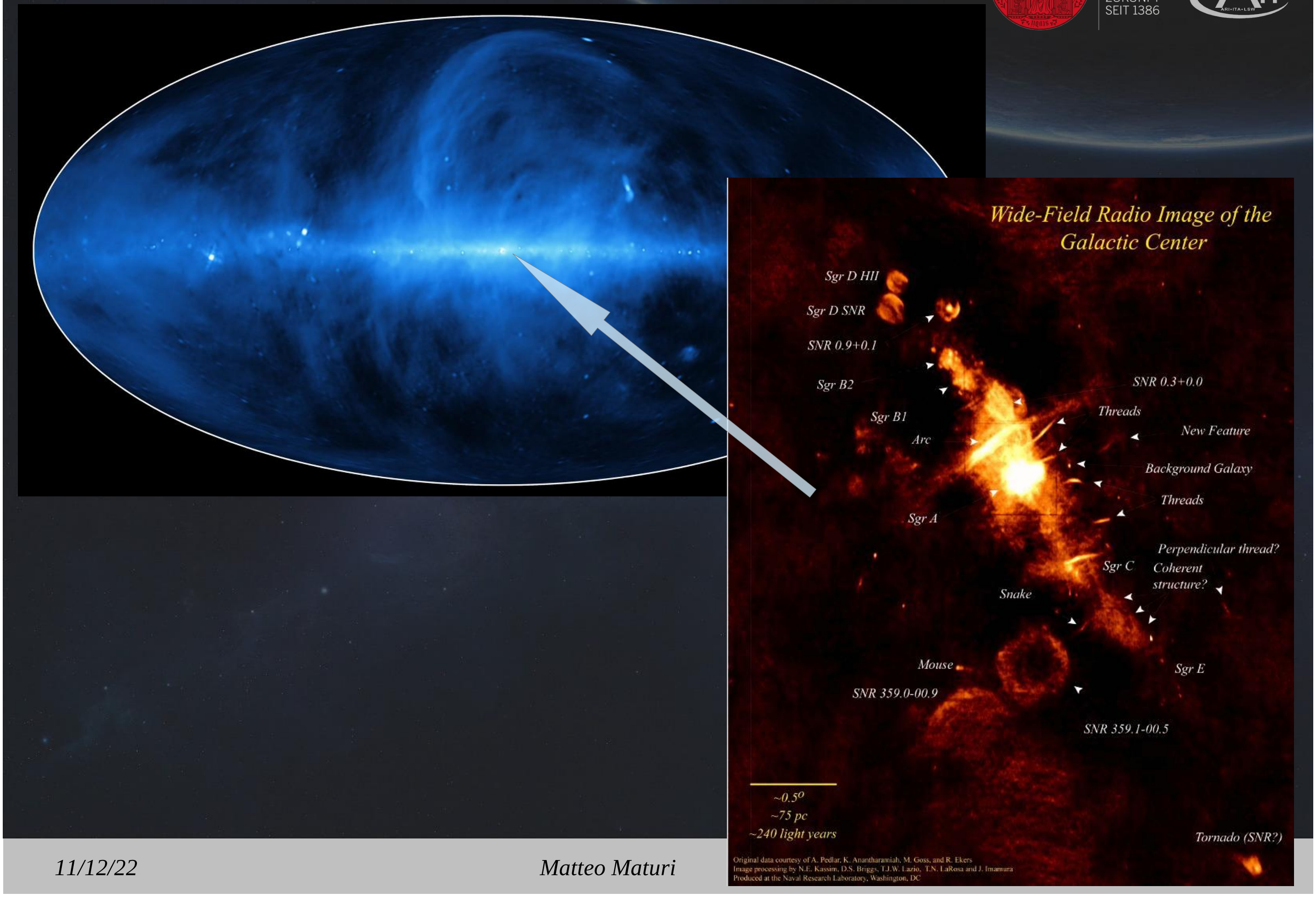
Synchrotron radiation



Theoretical Astrophysics

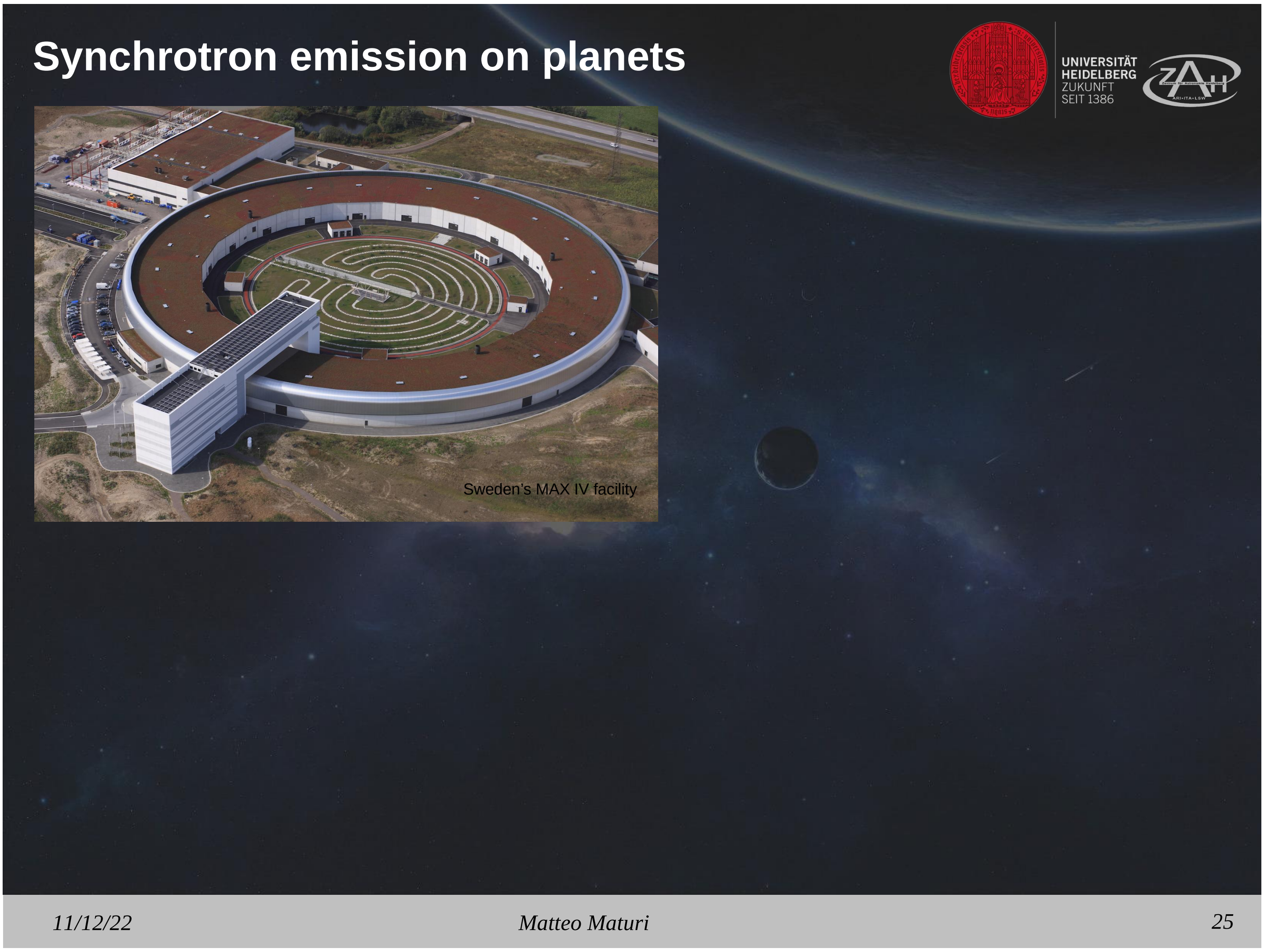
Synchrotron emission from the Galaxy



Wide-Field Radio Image of the Galactic Center

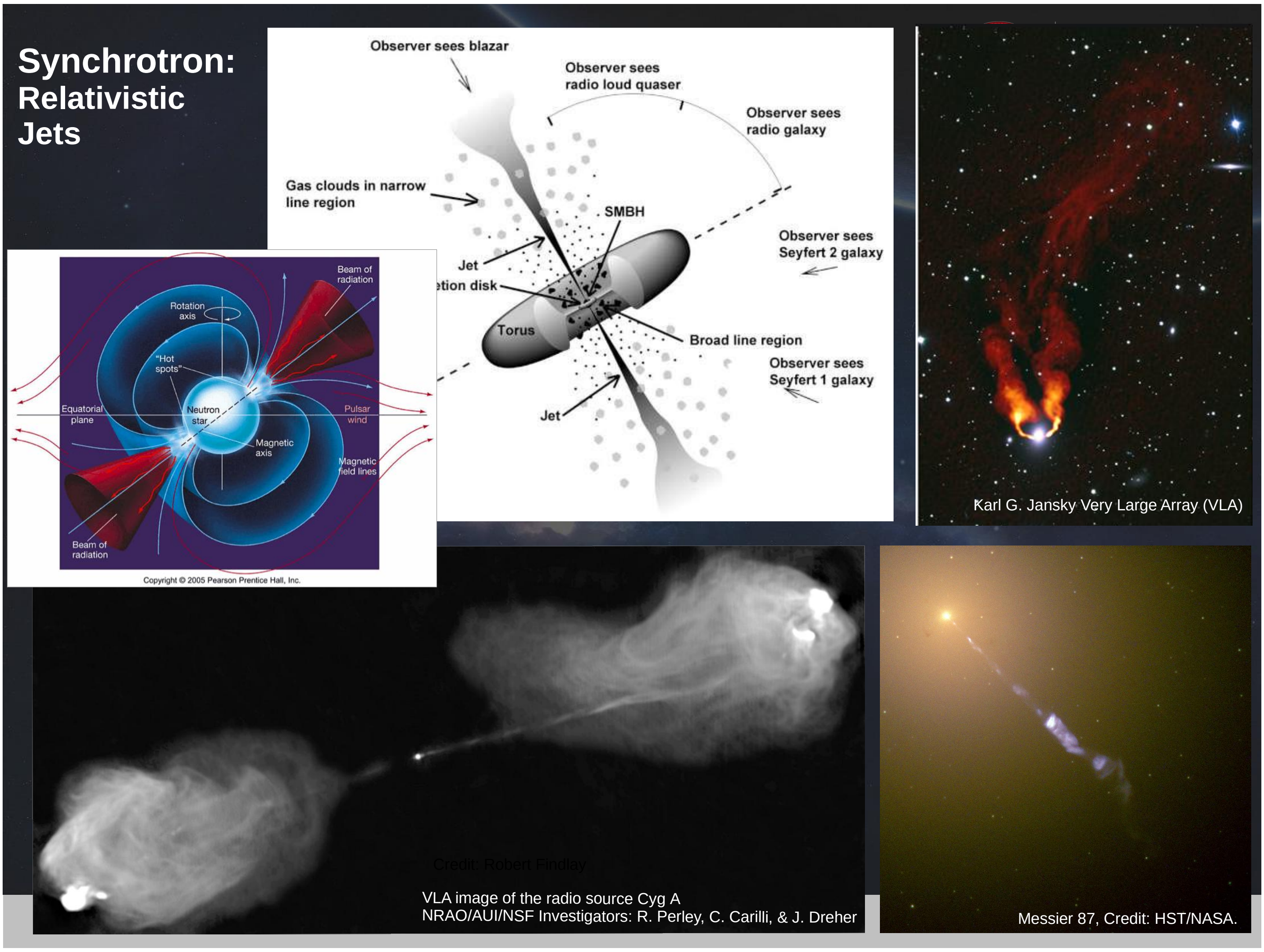
Original data courtesy of A. Pedlar, K. Anantharamiah, M. Goss, and R. Ekers
Image processing by N.E. Kassim, D.S. Briggs, T.J.W. Lazio, T.N. LaRosa and J. Imamura
Produced at the Naval Research Laboratory, Washington, DC

11/12/22 Matteo Maturi



Synchrotron emission on planets

Sweden's MAX IV facility



Synchrotron: Relativistic Jets

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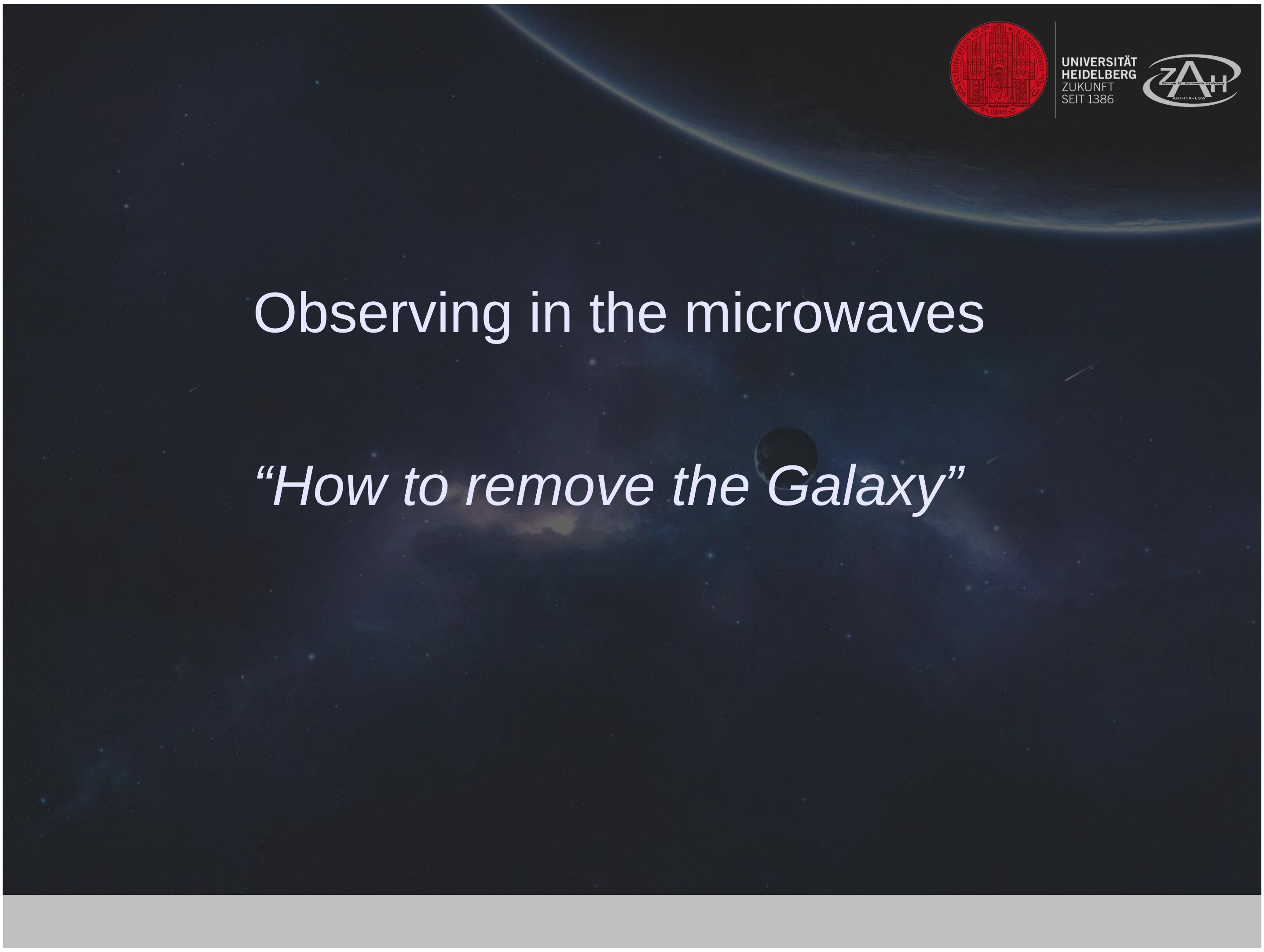
Karl G. Jansky Very Large Array (VLA)

Credit: Robert Findlay

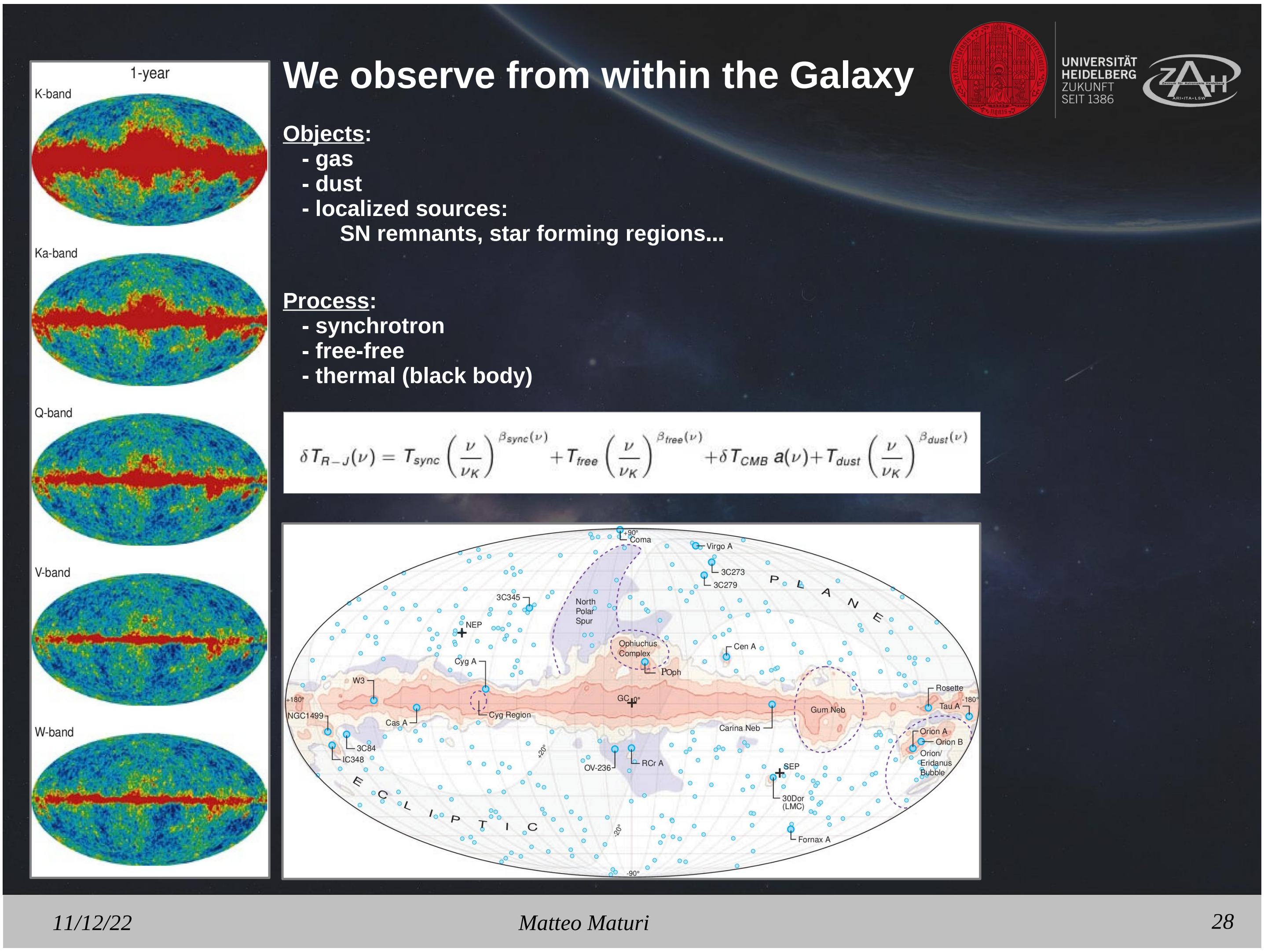
VLA image of the radio source Cyg A
NRAO/AUI/NSF Investigators: R. Perley, C. Carilli, & J. Dreher

Messier 87, Credit: HST/NASA.



Observing in the microwaves

“How to remove the Galaxy”

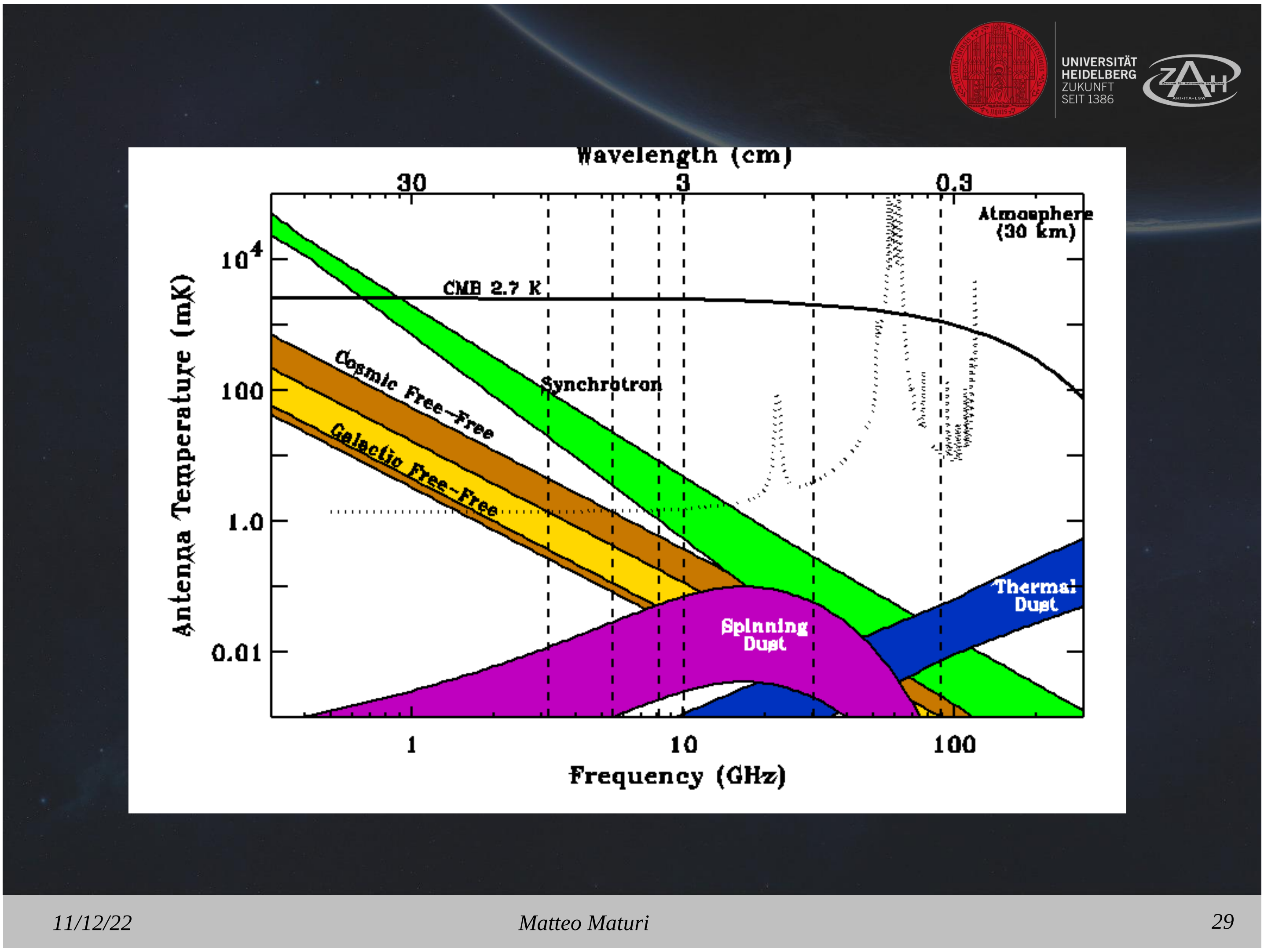


We observe from within the Galaxy

Objects:
- gas
- dust
- localized sources:
  SN remnants, star forming regions...

Process:
- synchrotron
- free-free
- thermal (black body)

$$\delta T_{R-J}(\nu) = T_{sync}\left(\frac{\nu}{\nu_K}\right)^{\beta_{sync}(\nu)} + T_{free}\left(\frac{\nu}{\nu_K}\right)^{\beta_{free}(\nu)} + \delta T_{CMB}\, a(\nu) + T_{dust}\left(\frac{\nu}{\nu_K}\right)^{\beta_{dust}(\nu)}$$



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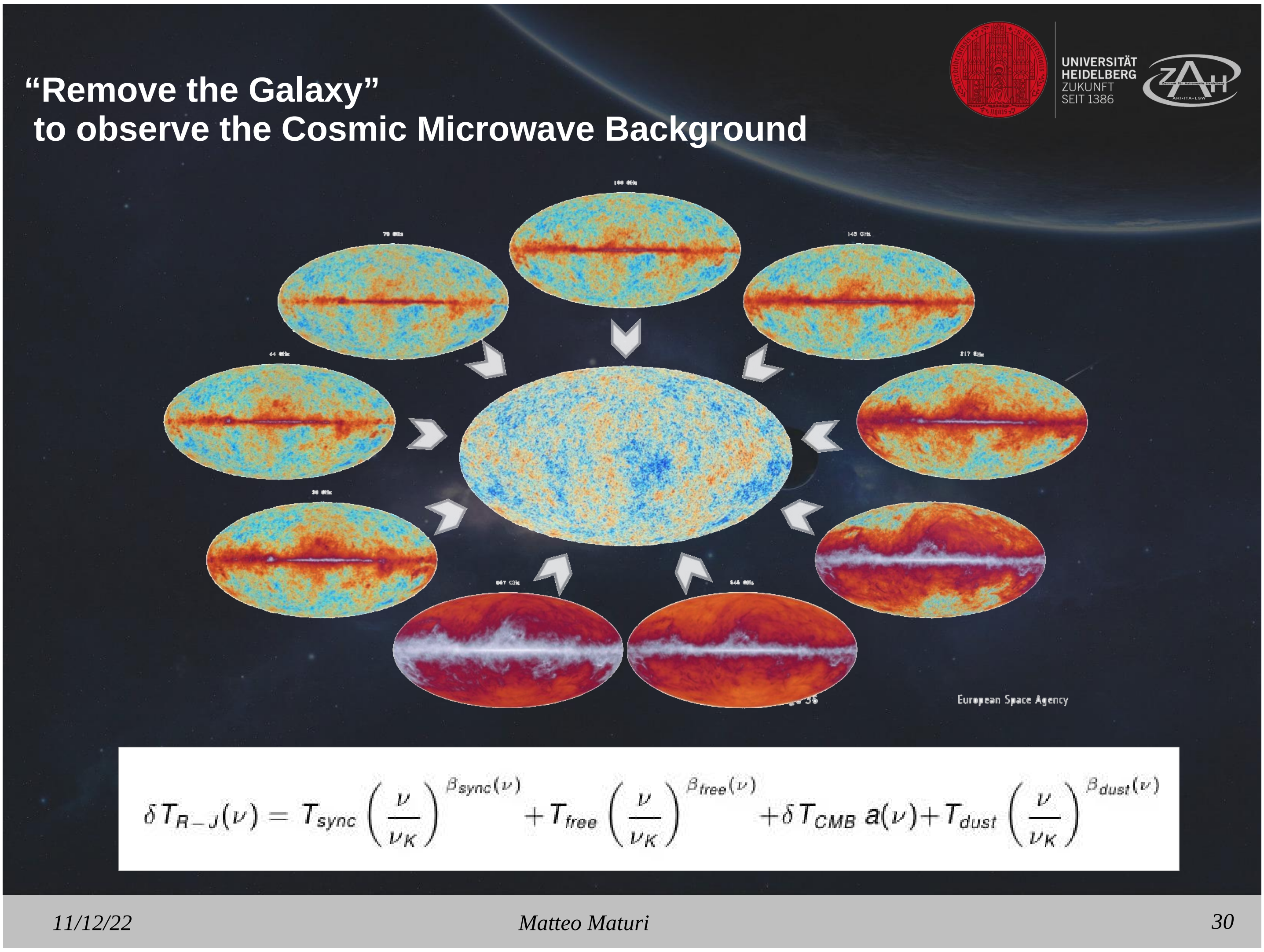
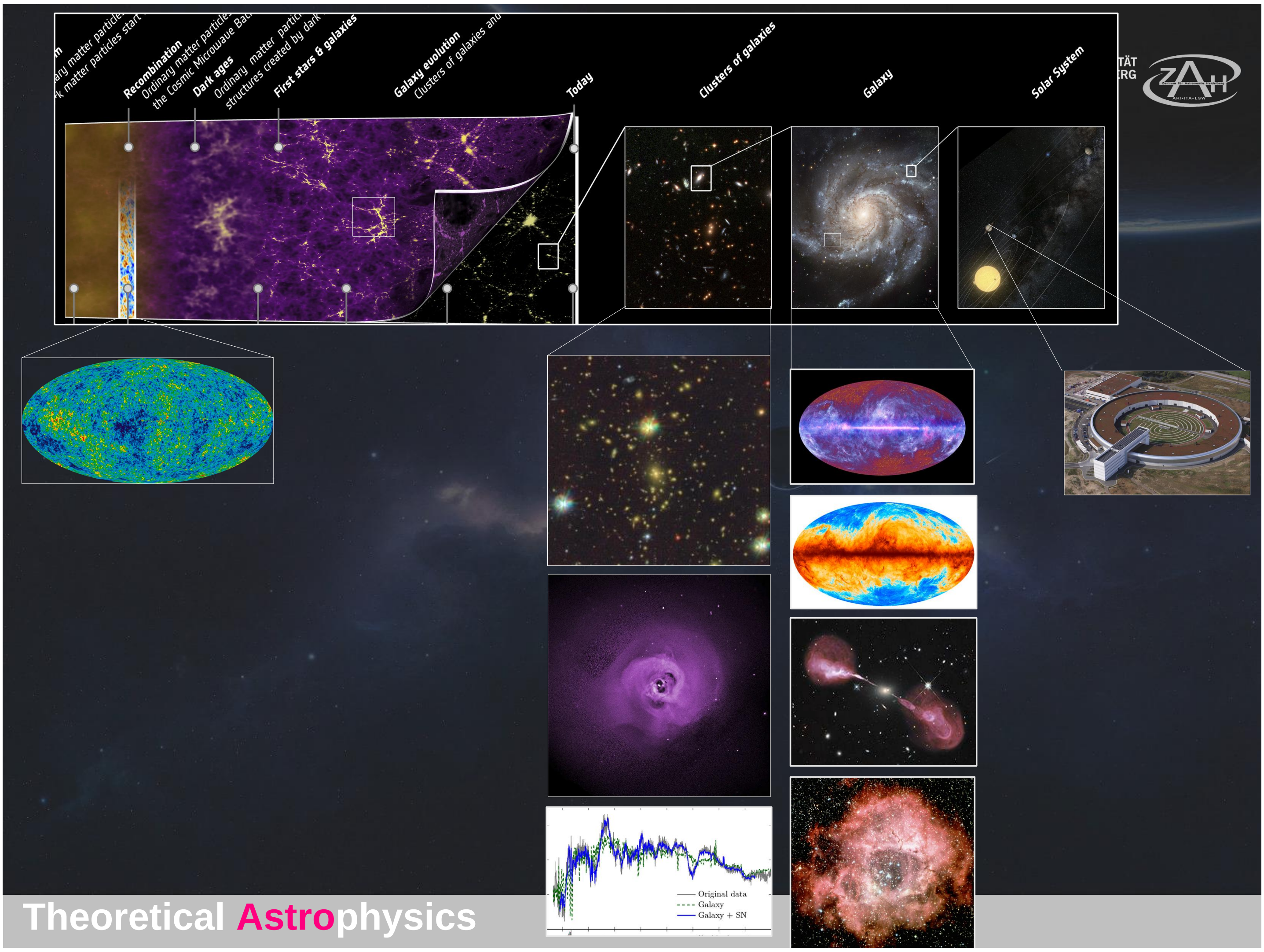
11/12/22 *Matteo Maturi* *29*



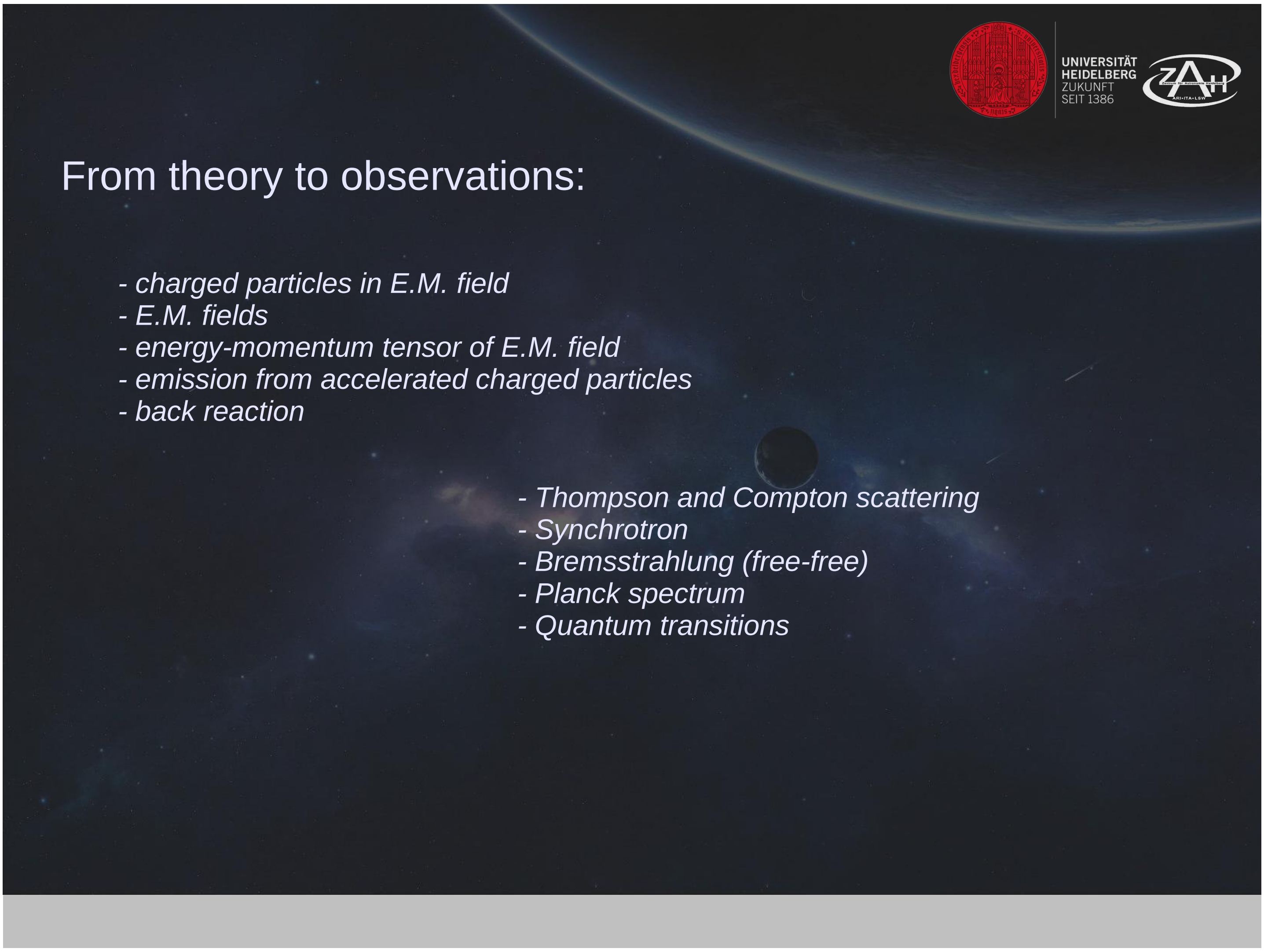
“Remove the Galaxy” to observe the Cosmic Microwave Background

European Space Agency

$$\delta T_{R-J}(\nu) = T_{sync}\left(\frac{\nu}{\nu_K}\right)^{\beta_{sync}(\nu)} + T_{free}\left(\frac{\nu}{\nu_K}\right)^{\beta_{free}(\nu)} + \delta T_{CMB}\, a(\nu) + T_{dust}\left(\frac{\nu}{\nu_K}\right)^{\beta_{dust}(\nu)}$$



Theoretical Astrophysics



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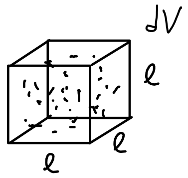
From theory to observations:

- *charged particles in E.M. field*
- *E.M. fields*
- *energy-momentum tensor of E.M. field*
- *emission from accelerated charged particles*
- *back reaction*

- *Thompson and Compton scattering*
- *Synchrotron*
- *Bremsstrahlung (free-free)*
- *Planck spectrum*
- *Quantum transitions*

Hydrodynamics: perfect fluid

- What is a fluid?



- System composed by many free moving particles
- Very frequent collisions => efficient energy-momentum exchange between part.

  mean free path $\boxed{\lambda = n^{-1/3} \ll \ell \ll L}$ size of the system

- Use statistical approach: 1) ensamble properties, e.g. mean velocities, ...

  2) phase-space distribution of particles $\boxed{f(t,\bar{x},\bar{p})}$

- Constrain fluid with conservation laws: $\partial_\mu j^\mu = 0 \quad \partial_\mu T^{\mu\nu} = 0$

$n(t,\bar{x}) = \int_{-\infty}^{\infty} d^3p\, f(t,\bar{x},\bar{p})$ assume $n(t,\bar{x}) = \text{const}$ $\forall t$ no particle creation/disruption

$N \overset{!}{=} \int_{-\infty}^{\infty} dx^4 \int_{-\infty}^{\infty} d^4p\, \tilde{f}(x^\mu, p^\mu)$ "go covariant" $\tilde{f}(x^\mu,p^\mu) = \tilde{f}(t,\bar{x},\bar{p},p^0)$

$$= \int_{-\infty}^{\infty} dx^4 \int_{-\infty}^{\infty} d^4p\, \tilde{f}(\underline{ct,\bar{x}}, p^0, \underline{\bar{p}})$$

$$= \int_{-\infty}^{\infty} dx^4 \int_{-\infty}^{\infty} d^4p\, \tilde{f}(\underline{ct,\bar{x}}, p^0, \underline{\bar{p}})\, \underset{(1)}{\delta_D\left[(p^0)^2 - \bar{p}^2 - m^2c^4\right]}\, \underset{(2)}{\Theta(p^0)}$$

particles obey: (1): $E^2 = (cp^0)^2 = c^2\bar{p}^2 + m^2c^4 \Rightarrow p^{0\,2} - \bar{p}^2 - m^2c^2 = 0$

(2): $p^0 \geq 0 \quad \Theta = \begin{cases} 1 & p^0 \geq 0 \\ 0 & p^0 < 0 \end{cases}$

dx^4, dp^4 invariant (Lorentz transf have $\det(\Lambda) = 1$)

δ_D, Θ "

$\Big\} \Rightarrow \underline{\tilde{f} \text{ invariant}}$ (!)

$$= \int_{-\infty}^{\infty} dx^4 \int_{-\infty}^{\infty} d^3p \int_0^{\infty} dp^0\, \tilde{f}(x^\mu, p^0, \bar{p}) \frac{c}{2E} \delta_D\left(p^0 - \sqrt{\bar{p}^2 - m^2c^4}\right)$$

δ_D acts

because $\delta_D(g(\bar{x})) \overset{(*)}{=} \sum_i \frac{(x-\tilde{x}_i)}{g'(x)}$ $\tilde{x}$ = root of g

$$= c \int_{-\infty}^{\infty} dx^4 \int_{-\infty}^{\infty} \frac{d^3p}{2E}\, \tilde{f}\left(x^\mu, p^0 = \frac{E}{c}, \bar{p}\right) \Rightarrow \boxed{\frac{d^3p}{E} = \text{invariant}}$$

because dx^4 and $\tilde{f}$ are

$\downarrow$

with constrain on $p^0 \rightarrow \tilde{f}$ identified with $f(t,\bar{x},\bar{p})$

One of the properties of $\delta_D(x)$

$$\circledast \quad \delta_D(g(x)) = \sum_i \frac{(x-\hat{x}_i)}{(g'(\tilde{x}_i))} \qquad \hat{x}_i = \text{roots of } g(x) \text{ i.e. } g(\tilde{x}_i)=0$$

- Here $g(x) \rightarrow g(p^0) = (p^0)^2 - \bar{p}^2 - m^2c^2$

$$\Rightarrow g(\tilde{p}^0) = 0 \quad (\tilde{p}^0)^2 - \bar{p}^2 - m^2c^2 = 0 \quad \Rightarrow (\tilde{p}_{1,2}) = \pm\sqrt{\bar{p}^2 + m^2c^2}$$

$$= \pm E/c \text{ (roots)}$$

$$\Rightarrow \frac{\delta}{\delta p^0} g(p^0) = 2p^0 \qquad |\delta'(g(\hat{p}^0_i))| = |2\tilde{p}^0_i| = 2\frac{E}{c}$$

$$\Rightarrow \delta_D(\overset{(x)}{(p^0)^2} - \bar{p}^2 - m^2c^2) = \frac{\delta_D(\overset{(x)}{p^0} - \overbrace{\sqrt{\bar{p}^2 - m^2c^2}}^{(\tilde{x}_1)})}{\underset{(g')\rightarrow}{2E/c}} + \underbrace{\frac{\delta_D(p^0 + \overbrace{\sqrt{\bar{p}^2 - m^2c^2}}^{(\hat{x}_2)})}{2E/c}}_{=0 \text{ because argument of } \delta_D \text{ can not be } 0}$$

$$= \frac{c\,\delta_D(p^0 - \sqrt{\bar{p}^2 - m^2c^2})}{2E}$$

- Construct the 4-matter density current

$$j^0 \equiv n(\bar{x},t) = \int d^3p\, f(t,\bar{x},\bar{p})\,\underbrace{\frac{cp^0}{cp^0}} = c\int \frac{d^3p}{E} f(t,\bar{x},\bar{p})\; p^0$$

generalize for i-th compo

$$j^i \equiv c\int \frac{d^3p}{E} f(t,\bar{x},\bar{p})\; p^i$$

$p^i = E\frac{\dot{x}^i}{c^2}$

$$= \cancel{c}\int \frac{d^3p}{\cancel{E}} f(t,\bar{x},\bar{p})\,\cancel{E}\,\frac{\dot{x}^i}{c^{\cancel{2}}}$$

$$= \int d^3p\, f(t,\bar{x},\bar{p})\,\frac{\dot{x}^i}{c}\cdot\frac{\int d^3p\, f(t,\bar{x},\bar{p})}{\int d^3p\, f(t,\bar{x},\bar{p})}$$

1) $= n(t,\bar{x})$

2) $= \frac{1}{c}\langle\dot{x}^i\rangle$

express everything in terms of ensamble properties

$$= \frac{1}{c}\, n(\bar{x},t)\,\langle\dot{x}^i\rangle \qquad \langle\dot{x}^i\rangle \equiv v^i$$

$\dot{\bar{x}}$ = particles

$\bar{v}$ = fluid ("bulk motion of dV")

$$\Rightarrow \boxed{(j^\mu)} = \left(n, \frac{n}{c}\bar{v}\right) = \boxed{\frac{n}{c}(c,\bar{v})}$$

$m\cdot n(\bar{x},t)$

↑ mass to get densities of dV

- Continuity equation

$$\partial_\mu j^\mu = \frac{\partial}{\cancel{c}\partial t} n(\bar{x},t) + \bar{\nabla}\left(n\frac{\bar{v}}{\cancel{c}}\right) = 0 \quad \xrightarrow{\cdot m} \quad \boxed{\partial_t\rho + \bar{\nabla}(\rho\bar{v}) = 0}$$

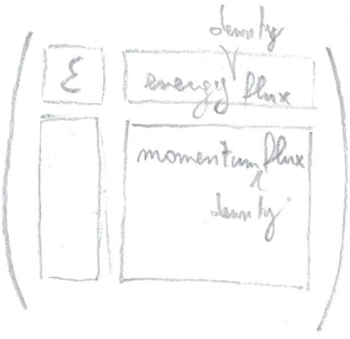
$$\partial_\mu = \left(-\frac{\partial}{c\partial t}, \bar{\nabla}\right)$$

$\partial_\mu j^\mu = 0$ expresses matter conservation!

i.e. $n(\bar{x},t)$ or equivalently $\rho(\bar{x},t)$ change as a func. of time because of a matter flux (particles move around)

③

Construct the momentum-energy tensor



$$\boxed{T^{00}} = \int d^3p\, f(\bar{x},\bar{p},t)\, \overset{E=\gamma mc^2}{E}$$

$$= \int d^3p\, f(\bar{x},\bar{p},t)\, \gamma mc^2 \cdot \frac{\int d^3p\, f(\bar{x},\bar{p},t)}{\int d^3p\, f(\bar{x},\bar{p},t)}$$

$$= n(\bar{x},t)\, mc^2 \langle\gamma\rangle(\bar{x},t)$$

$$= \boxed{\rho c^2 \langle\gamma\rangle}$$

Generalize it

$$\left(p^0 = \frac{E}{c}\right)$$

$$T^{00} = \int d^3p\, f(\bar{x},\bar{p},t)\, E \cdot \frac{E}{E} = c^2 \int \frac{d^3p}{E} f(\bar{x},\bar{p},t)\, p^0 p^0$$

$$\boxed{T^{0i}} = c^2 \int \frac{d^3p}{E} f(\bar{x},\bar{p},t)\, p^0 p^i = c^{\cancel{2}} \int \frac{d^3p}{\cancel{E}} f(\bar{x},\bar{p},t)\, \frac{\cancel{E}}{\cancel{c}} \gamma m \dot{x}^i \qquad \left(p^i = \gamma m \dot{x}^i\right)$$

$$= c\, n(\bar{x},t)\, m \langle\gamma\dot{x}^i\rangle = \boxed{\rho c \langle\gamma\dot{x}^i\rangle}$$

$$\boxed{T^{ij}} = c^2 \int \frac{d^3p}{E} f(\bar{x},\bar{p},t)\, p^i p^j = \cancel{c^2} \int \frac{d^3p}{\cancel{\gamma mc^2}} f(\bar{x},\bar{p},t)\, \gamma^{\cancel{2}} m^{\cancel{2}} \dot{x}^i \dot{x}^j$$

$$= n(\bar{x},t)\, m \langle\gamma\dot{x}^i\dot{x}^j\rangle = \boxed{\rho \langle\gamma\dot{x}^i\dot{x}^j\rangle}$$

Non-relativistic $\gamma \ll 1$: meaning of $T^{\alpha\beta}$

$$\gamma = (1-\beta^2)^{-1/2} \approx 1 + \frac{\dot{x}^2}{2c^2} \quad \Rightarrow \quad \langle\gamma\rangle \approx 1 + \frac{\langle\dot{x}^2\rangle}{2c^2}$$

$$\langle\gamma\dot{x}^i\rangle \approx \langle \dot{x}^i + \frac{\dot{x}^2}{2c^2}\dot{x}^i\rangle = \langle\dot{x}^i\rangle + \frac{1}{2c^2}\langle\dot{x}^i\dot{x}^2\rangle$$

$$\langle\gamma\dot{x}^i\dot{x}^j\rangle \approx \langle\dot{x}^i\dot{x}^j\rangle + \frac{1}{2c^2}\langle\dot{x}^i\dot{x}^j\dot{x}^2\rangle \approx \langle\dot{x}^i\dot{x}^j\rangle$$

2° ordine

$$\Rightarrow \begin{cases} \boxed{T^{00}} \approx \rho c^2 \langle\gamma\rangle = \boxed{\rho c^2 + \rho \frac{1}{2}\langle\dot{x}^2\rangle} \\ \boxed{T^{0i}} \approx \rho c \langle\dot{x}^i\rangle + \rho \cancel{c} \frac{1}{2c^{\cancel{2}}}\langle\dot{x}^i\dot{x}^2\rangle = \boxed{\rho c v^i + \frac{\rho}{2c}\langle\dot{x}^i\dot{x}^2\rangle} \\ \boxed{T^{ij}} \approx \boxed{\rho \langle\dot{x}^i\dot{x}^j\rangle} \end{cases}$$

(energy rest mass + kinetic energy) densities

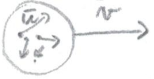
$$\left(\frac{\rho}{2}\langle\dot{x}^2\dot{x}^i\rangle \equiv q^i\right.$$

flow of kinetic energy)

④

$\boxed{\text{To better understand } T^{\mu\nu}}$

- Split $\dot{\vec{x}} = \vec{v} + \vec{u}$



$\vec{u}$ ← microscopic: (random velocities around the mean)

$\vec{v}$ ← macroscopic: flow of fluid

- $$\boxed{T^{00}} = \rho c^2 + \frac{1}{2}\rho\langle\dot{x}^2\rangle$$

$$\langle\dot{x}^2\rangle = \vec{v}^2 + 2\langle\vec{v}\vec{u}\rangle + \langle\vec{u}^2\rangle = \vec{v}^2 + \langle\vec{u}^2\rangle$$

$\langle\vec{v}\overset{\text{random}}{\vec{u}}\rangle = 0$ (not an approximation

$$\Rightarrow \quad = \rho c^2 + \frac{1}{2}\rho\vec{v}^2 + \underbrace{\overbrace{\frac{1}{2}\rho\langle\vec{u}^2\rangle}^{\varepsilon}}_{\text{microscopic: thermal interpretation}} \qquad \left[\varepsilon = \frac{1}{2}\rho\langle\vec{u}^2\rangle \overset{\text{ideal gas}}{=} \frac{3}{2}nK_BT\right] = \text{thermal energy density}$$

- $$\boxed{T^{0i}} \simeq \rho c v^i + \frac{\rho}{2c}\langle\dot{x}^i\dot{x}^2\rangle = \rho c v^i + \frac{q^i}{c}$$

$\Rightarrow$ Kinetic-energy density current

$v_j u^j \qquad u_j u^j$

$$q^i \equiv \frac{\rho}{2}\langle\dot{x}^2\dot{x}^i\rangle = \frac{\rho}{2}\langle(\vec{v}^2 + 2\vec{v}\vec{u} + \vec{u}^2)(v^i + u^i)\rangle$$

all terms $\propto\langle u^i\rangle$ vanish because $\langle u^i\rangle = 0$ (random motions)

$$= \frac{\rho}{2}\left[\langle\vec{v}^2 v^i\rangle + 2\langle v_j u^j u^i\rangle + \langle\vec{u}^2 v^i\rangle\right]$$

$$= \frac{\rho}{2}\left[\vec{v}^2 v^i + 2v_j\langle u^j u^i\rangle + v^i\langle\vec{u}^2\rangle\right] = \frac{\rho}{2}(\vec{v}^2 + \langle\vec{u}^2\rangle)v^i + \rho v_j\langle u^j u^i\rangle$$

$$= \left(\frac{\rho}{2}\vec{v}^2 + \varepsilon\right)v^i + \rho v_j\underbrace{\langle u^j u^i\rangle}_{\circledast}$$

- $$\boxed{T^{ij}} = \rho\langle\dot{x}^i\dot{x}^j\rangle = \rho\langle(v^i + u^i)(v^j + u^j)\rangle = \rho v^i v^j + \rho\underbrace{\overbrace{\langle u^i u^j\rangle}^{\neq 0 \text{ if } i=j}}_{\circledast}$$

⊛ • Look at $\langle u^i u^j\rangle$ only (in rest frame of flow ($\vec{v} = 0$) for an ideal gas

$$\rightarrow \boxed{tr(T^{ij})} = \rho\, tr(\langle u^i u^j\rangle) = \rho\langle(u^1)^2 + (u^2)^2 + (u^3)^2\rangle = \underbrace{\rho\langle\vec{u}^2\rangle}_{3nK_BT} = \boxed{3P} \leftarrow \text{Pressure}$$

ideal gas!

$$\Rightarrow \boxed{T^{ij} = \rho v^i v^j + P\delta^{ij}} \qquad \boxed{q^i = \left(\frac{\rho}{2}\vec{v}^2 + \varepsilon + P\right)v^i}$$

(0)

Hydrodynamics : summary

- **Fluid** = collection of particles
  frequent collisions $\lambda \ll l \ll L$

- **Density current:** $(j^\mu) = \frac{n(\bar{x},t)}{c}\begin{pmatrix} c \\ \bar{v} \end{pmatrix}$ $\qquad j^\nu = c\int \frac{d^3p}{E} f(\bar{x},\bar{p},t)\, p^i$

$$\boxed{\partial_\mu j^\mu = 0}$$ continuity equation (matter conservation)

$\partial \rho + \bar{\nabla}(\rho \bar{v}) = 0$

- **Momentum-energy tensor:** $T^{\mu\nu} = c^2 \int \frac{d^3p}{E} f(\bar{x},\bar{p},t)\, p^\mu p^\nu$

(1)- Non relativistic $\gamma \simeq 1 + \frac{\dot{x}^2}{2c^2}$ $(\beta \ll 1)$

(2)- Split velocity $\dot{\bar{x}} = \bar{v} + \bar{u}$ bulk flow + microscopic velocity

$\langle \dot{x}^2 \rangle = v^2 + \langle \bar{u}^2 \rangle$ Kinetic + thermal energy

$\mathcal{E} = \frac{1}{2}\rho\langle \bar{u}^2 \rangle = \frac{3}{2}k_BT$ ideal gas

$$T^{\mu\nu}\begin{cases} T^{00} = \rho c^2 \langle \gamma \rangle & \overset{(1)}{\approx} \rho c^2 + \frac{\rho}{2}\langle \dot{x}^2 \rangle & \overset{(2)\text{ + ideal gas}}{=} \rho c^2 + \frac{1}{2}\rho v^2 + \mathcal{E} \\ T^{0i} = \rho c \langle \gamma \dot{x}^i \rangle & \approx \rho c v^i + \overset{q^i}{\frac{\rho}{2c}\langle \dot{x}^i \dot{x}^2 \rangle} & = \rho c v^i + \left(\frac{\rho}{2}\bar{v}^2 + \mathcal{E} + P\right) v^i \\ T^{ij} = \rho \langle \gamma \dot{x}^i \dot{x}^j \rangle & \approx \rho \langle \dot{x}^i \dot{x}^j \rangle & = \rho v^i v^j + P\delta^{ij} \end{cases}$$

$(P = nk_BT$ ideal gas$)$

- **Energy-Momentum conservation:**

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$(\nu = 0)$: $\partial_t \mathcal{E} + \bar{\nabla}(\mathcal{E}\bar{v}) + P\bar{\nabla}\bar{v} = 0$ energy conservation

$(\nu = i)$: $(\partial_t + \bar{v}\bar{\nabla})\bar{v} + \frac{\bar{\nabla}P}{\rho} = 0$ momentum conservation

$\uparrow$ $[-\bar{\nabla}\phi$ external forces$]$

- **Equation of state** $P = P(\rho)$ $\Rightarrow$ $1 + 4 + 1 = 6$ equations

$\rho, P, \bar{v}, \mathcal{E} = 6$ variables

(5)

$\boxed{\text{Energy-momentum conservation}}$ (non relativistic $T^{\mu\nu}$)

$$\boxed{\partial_\mu T^{\mu\nu} \overset{!}{=} 0}$$

4 equations: $\nu = 0$ energy conservation

$\nu = i$ momentum "

$$\partial_0 = \frac{\partial}{c\,\partial t} = c^{-1}\partial_t$$

$$\boxed{\nu = 1}: \quad \partial_0 T^{0i} + \partial_j T^{ij} = c^{-1}\partial_t\left(\rho c v^i + \frac{q^i}{c}\right) + \partial_j\left(\rho v^j v^i + P\delta^{ji}\right) = 0 \quad \text{[ideal gas]}$$

$$\simeq \partial_t(\rho v^i) + \partial_j(\rho v^j v^i + P\delta^{ji}) \qquad \frac{q^i}{c^2} = \text{negligible}$$

$$= (\partial_t\rho)\underset{\sim}{v^i} + \underline{\rho}\,\partial_t v^i + \partial_j(\rho v^j)\underset{\sim}{v^i} + \underline{\rho}\,v^j\partial_j v^i + \partial_j P\delta^{ji}$$

$$= \underbrace{\left(\partial_t\rho + \partial_j(\rho v^j)\right)}_{=0 \text{ continuity eq.}} v^i + \rho\left(\partial_t v^i + v^j\partial_j v^i\right) + \partial^i P$$

$$= \rho\left(\partial_t + v^j\partial_j\right)v^i + \partial^i P$$

$$\Rightarrow \boxed{\cancel{\rho}\left(\frac{\partial}{\partial t} + \bar{v}\bar{\nabla}\right)\bar{v} + \frac{\bar{\nabla}\bar{P}}{\rho} = 0} \longleftarrow \underline{\text{Euler's equation}}$$

$$= \left(\frac{\partial}{\partial t} + \frac{\partial\bar{x}}{\partial t}\frac{\partial}{\partial\bar{x}}\right) = \frac{d}{dt} \quad \to \underline{\text{Meaning:}} \quad \rho\frac{d\bar{v}}{dt} = -\bar{\nabla}P$$

$\underset{\sim}{\bar{\nabla}P < 0}$

$P_1 > P_2$

$\xrightarrow{\partial}$

Fluids are accelerated by pressure gradients

- <u>External forces</u>

  - Add a source term

  - e.g. gravity: $(\partial_t + \bar{v}\bar{\nabla})\bar{v} + \frac{\bar{\nabla}\bar{P}}{\rho} = \underline{-\bar{\nabla}\phi}$

⑥

$$\boxed{\nu=0}: \quad \partial_0 T^{00} + \partial_j T^{0j} = c^{-1}\partial_t\left(\rho c^2 + \frac{1}{2}\rho\bar{v}^2 + \varepsilon\right) + \partial_j\left(\rho c v^j + \frac{q^j}{c}\right) \quad \text{(ideal gas)}$$

$$= \underbrace{c\,\partial_t\rho} + c^{-1}\partial_t\left(\frac{1}{2}\rho\bar{v}^2 + \varepsilon\right) + \underbrace{c\,\partial_j(\rho v^j)} + \partial_j\frac{q^j}{c}$$

$=0$ continuity

$$= \cancel{c^{-1}}\partial_t\left(\frac{1}{2}\rho\bar{v}^2 + \varepsilon\right) + \partial_j\left[\left(\frac{\rho}{2}\bar{v}^2 + \varepsilon + P\right)\frac{v^j}{\cancel{c}}\right] = 0$$

$$= \frac{1}{2}(\partial_t\rho)\bar{v}^2 + \frac{1}{2}\rho\partial_t\bar{v}^2 + \partial_t\varepsilon + \frac{1}{2}\bar{\nabla}(\rho\bar{v})\bar{v}^2 + \frac{1}{2}\rho\bar{v}\bar{\nabla}\bar{v}^2 + \bar{\nabla}\left[(\varepsilon+P)\bar{v}\right] = 0$$

$$= \frac{1}{2}\underbrace{\left[\partial_t\rho + \bar{\nabla}(\rho\bar{v})\right]}_{=0\ \text{continuity}}\bar{v}^2 + \underbrace{\frac{\rho}{2}(\partial_t + \bar{v}\bar{\nabla})\bar{v}^2} + \partial_t\varepsilon + \bar{\nabla}\left[(\varepsilon+P)\bar{v}\right]$$

$$\frac{\rho}{2}2\bar{v}(\partial_t + \bar{v}\bar{\nabla})\bar{v} = (-\bar{\nabla}P)\cdot\bar{v} \quad \text{momentum conservation}$$

$\frac{1}{2}$ Faktor?

$$= \cancel{(-\bar{\nabla}P)\bar{v}} + \partial_t\varepsilon + \bar{\nabla}(\varepsilon\bar{v}) + \cancel{\bar{v}\bar{\nabla}P} + P\bar{\nabla}\bar{v} = 0$$

$$\Rightarrow \boxed{\underset{(0)}{\partial_t\varepsilon} + \underset{(1)}{\bar{\nabla}(\varepsilon\bar{v})} + \underset{(2)}{P\bar{\nabla}\bar{v}} = 0} \quad \circledast \qquad \underline{\text{Energy conservation}}$$

(0): time variation of internal energy ε because of:

(1) current energy density (heat transfer)

(2) work done by pressure (pressure volume work)

↳ if $\underline{\bar{\nabla}\bar{v} > 0}$ ⇒ expansion ⇒ $\partial_t\varepsilon < 0$ cools down ↙ work done by the system

if $\bar{\nabla}\bar{v} < 0$ ⇒ compression ⇒ $\partial_t\varepsilon > 0$ heats up ↖ work done on the system

__Note:__

$$\circledast = \left(\partial_t\varepsilon + \bar{v}\bar{\nabla}\varepsilon\right) + \varepsilon\bar{\nabla}\bar{v} + P\bar{\nabla}\bar{v} = \boxed{\frac{d\varepsilon}{dt} + (\varepsilon + P)\bar{\nabla}\bar{v} = 0}$$

__1° law of thermodynamic at given pressure__

$\varepsilon + P = h$ enthalpy

⑦

Equation of state

5 equations : mass conservation (1) $\partial_\mu j^\mu = 0$

momentum " (2) $(\partial_t + \bar{v}\bar{\nabla})\bar{v} + \frac{\bar{\nabla}\bar{P}}{\rho} = 0$

energy " (3) $\partial_t \varepsilon + \bar{\nabla}(\varepsilon\bar{v}) + P\bar{\nabla}\bar{v} = 0$

6 variables : $\rho, P, \bar{v}, \varepsilon$

$\Rightarrow$ To close the system you need another constraint

$\rightarrow$ Equation of state $P = P(\rho)$

↳ eg. ideal gas...

polytropic...

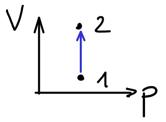
$$PV = nRT$$

Equation of state

- Equation linking P to other thermodynamical properties of a fluid e.g. $P(\rho, T)$

- <u>Ideal gas</u>: $\boxed{PV = NRT}$ ⇒ $\frac{P_1 V_1}{\cancel{N}RT_1} = \frac{P_2 V_2}{\cancel{N}RT_2}$ ⇒ $\frac{T_2}{T_1} = \frac{P_2 V_2}{P_1 V_1}$ (1)

  ↑ thermodyn. state 1 ↑ state, eg.



isobaric proces

- <u>Polytropic eq. of state</u> (parametric)

$\boxed{PV^n = \text{const}}$ ≠ parameter n ⇒ ≠ termodynamic proces

$n = 0$ ⇒ $P \cdot 1 = \text{const}$ adiepatic

$n = 1$ ⇒ $PV = NRT = \text{const}$ $T = \text{const}$ isotermal

$n = \infty$ ⇒ $V = \text{const}$ isochoric

any adiabatic gas → $n = \gamma = \frac{c_P}{c_V}$ entropy $= \text{const}$ isenchoric

$P_1 V_1^n = P_2 V_2^n$ ⇒ $\frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^n$ (2) or $\frac{V_2}{V_1} = \left(\frac{P_1}{P_2}\right)^{1/n}$ (3) P, V relations

(2) → (1) ⇒ $\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{n-1}$ (3) → (1) ⇒ $\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}}$ T relations

- <u>Link to density</u> $\rho \equiv \frac{M}{V}$ ⇒ $P_1 \left(\frac{\cancel{M}}{\rho_1}\right)^n = P_2 \left(\frac{\cancel{M}}{\rho_2}\right)^n$ ⇒ $\boxed{P_2 = P_1 \left(\frac{\rho_2}{\rho_1}\right)^n}$

- A closer look on a more "fundamental" level

$\boxed{P(\rho) = P_0 \left(\frac{\rho}{\rho_0}\right)^\gamma}$ is valid for any fluid under adiabatic conditions

γ = adiabatic index, e.g. in astrophysics : $\gamma = 5/3$ white dwarfs, $\gamma = 4/3$ other stars

Prove:

- 1° law of thermodynamic $\boxed{dQ = dE + PdV}$ change of heat = change internal energy + work

- Assume no exchange with external environment $\boxed{dQ = 0}$

$\Rightarrow dE = \boxed{C_v dT = -PdV}$ (1) C_v = heat capacity at const. volume

- Define Enthalpy $H \equiv E + PV$ Legendre transf.

$\Rightarrow dH = \cancel{dE} + dP \cdot V + \cancel{PdV} = VdP$ $\quad dH = \boxed{C_p dT = VdP}$ (2)

$\Rightarrow \frac{(2)}{(1)} = \frac{C_p \cancel{dT}}{C_v \cancel{dT}} = \frac{VdP}{-PdV}$ $\quad \frac{C_p}{C_v} \equiv \gamma$ $\quad \frac{dP}{P} = -\gamma \frac{dV}{V}$ $\Rightarrow \boxed{P} = P_0 \left(\frac{V_0}{V}\right)^\gamma = \boxed{P_0 \left(\frac{\rho}{\rho_0}\right)^\gamma}$

No assumpion on ideal gas, only adiabatic $dQ = 0$!

- Further assume ideal gas $PV = NRT$ * further relations with T

$$PV^\gamma = \overset{*}{PV} \cdot V^{\gamma-1} = NRTV^{\gamma-1} = \text{const} \Rightarrow \boxed{TV^{\gamma-1} = \text{const}}$$

$$= P_0 V_0^\gamma \Rightarrow \boxed{T} = \frac{P_0 V_0^\gamma}{NR\, V^{\gamma-1}} = \overset{*}{\left(\frac{P_0 V_0}{NR}\right)} \left(\frac{V_0}{V}\right)^{\gamma-1} = \boxed{T_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}}$$

↳ for adiabatic + ideal

- Some considerations about entolpy and polytropic gas

from

- 1° principle of thermodynamics $dQ = dE + PdV$
- adiabatic condition $dQ = 0$

$\Rightarrow$ Change of Entholpy: $dH = VdP$

specific entholpy: $\frac{1}{M_{ass}}$ $\frac{dH}{M} \equiv d\tilde{h} = \frac{V}{M} dP = \frac{dP}{\rho}$ $\boxed{d\tilde{h} = \frac{dP}{\rho}}$

- $\boxed{\tilde{h}} = \int \frac{dP}{\rho} = \frac{1}{\rho}\int dP = \boxed{\frac{P}{\rho}}$ for $\rho = \text{const.}$ (incompressible fluid)

- $\boxed{\tilde{h}} = \int \frac{1}{\rho} \frac{P_0 \gamma}{\rho_0^\gamma} \rho^{\gamma-1} d\rho = \frac{P_0\gamma}{\rho_0^\gamma} \int \rho^{\gamma-2} d\rho = \frac{P_0 \gamma\, \rho^{\gamma-1}}{\rho_0^\gamma (\gamma-1)} = \frac{\gamma}{(\gamma-1)} \frac{1}{\rho} \overbrace{P_0 \left(\frac{\rho}{\rho_0}\right)^\gamma}^{=P}$

$= \boxed{\frac{\gamma}{\gamma-1}\frac{P}{\rho}}$ for polytropic eq. of state (ρ not const)

- Now assume

$P = P_0\left(\frac{\rho}{\rho_0}\right)^\gamma$ and $\rho = \text{const} \Rightarrow P = \text{const} \Rightarrow dP = 0 \Rightarrow \tilde{h} = 0$

- This does not make sense, the variation of entholpy (specific) should be given by a change of T
- But the polytropic eq. of state has no T dependency (it ignores T)

$\Rightarrow$ The polytropic eq. of state is not a "real" eq. of state, a full eq. of state has $P(\rho, T)$

- You can not use $P = P_0\left(\frac{\rho}{\rho_0}\right)^\gamma$ and assume $\rho = \text{const}$!!

Application: Sound Waves

- Ideal fluid (no viscosity)
- Use perturbative approach :

time dependent perturbation (not a derivative!!)

$\rho = \rho_0 + \delta\rho \qquad \delta\rho \ll \rho_0$

$\vec v = \vec v_0 + \delta\vec v \qquad \delta\vec v \ll \vec v_0$

$P = P_0 + \delta P \qquad \delta P \ll P_0$

- Assume to already know background solution : $\rho_0, \vec v_0, P_0$ fixed
  (here we focus on the perturbation)
- Move to rest frame of background solution ⇒ $\vec v_0 = 0$

- Continuity eq.

$\partial_t\rho_0 = 0$ $\qquad$ $\delta\rho\,\delta\vec v$ is at 2° order ⇒ neglect

$$\partial_t(\rho_0+\delta\rho) + \vec\nabla[(\rho_0+\delta\rho)\delta\vec v] \simeq \partial_t\delta\rho + \vec\nabla(\rho_0\delta\vec v) = \partial_t\delta\rho + \rho_0\vec\nabla\delta\vec v = 0 \qquad (1)$$

- Euler eq.

$$\boxed{P(\rho): \quad \vec\nabla P = \frac{\delta P}{\delta\rho}\vec\nabla\rho = c_s^2\vec\nabla(\rho_0+\delta\rho) \qquad \frac{\delta P}{\delta\rho} \simeq \frac{\delta P_0}{\delta\rho} \equiv c_s^2}$$

$$(\partial_t + \delta\vec v\vec\nabla)\delta\vec v + \frac{\vec\nabla P}{\rho_0+\underset{\ll\rho_0}{\cancel{\delta\rho}}} = \partial_t\delta\vec v + \underset{2^\circ\text{ order}}{\cancel{\delta\vec v\vec\nabla\delta\vec v}} + c_s^2\frac{\vec\nabla\delta\rho}{\rho_0} = 0 \qquad (2)$$

- Combine (1) and (2)

$\partial_t(1)$: $\quad \partial_t^2\delta\rho + \rho_0\partial_t\vec\nabla\delta\vec v = 0$

$\vec\nabla(2)$: $\quad \vec\nabla\partial_t\delta\vec v + c_s^2\frac{\vec\nabla^2\delta\rho}{\rho_0} = 0$

$$\boxed{\partial_t^2\delta\rho - c_s^2\vec\nabla^2\delta\rho = 0} \qquad \Box\delta\rho = 0 \qquad \Box = \partial_t^2 - c_s^2\vec\nabla^2$$

D'Alambert eq.

- Solution : decompose in plane waves

linear density waves → sound waves

c_s = characteristic velocity: $[c_s] = \left[\frac{\delta P}{\delta\rho}\right] = \left(\frac{g}{cm^2}\frac{cm}{s^2}\right)\left(\frac{cm^3}{g}\right) = \left(\frac{cm}{s}\right)^2$

(sound speed)

$\boxed{\delta\rho = a e^{i(\vec k\vec x - \omega t)}}$ continuity eq. → $\delta\vec v = \vec b e^{i(\vec k\vec x - \omega t)}$ Euler eq. → P

$\rho, P, \vec v$ are coupled : eg. compression $\uparrow\rho \Rightarrow \overset{\downarrow}{\uparrow}\vec v \Rightarrow P\uparrow$

- <u>Dispersion relation</u> : plug solution in d'Alembert eq.

$$\partial_t^2\left(\cancel{a}\, e^{i(\bar{k}\bar{x}-\omega t)}\right) - c_s^2\, \bar{\nabla}^2\left(\cancel{a}\, e^{i(\bar{k}\bar{x}-\omega t)}\right) = 0$$

$$-\omega^2 \cancel{e^{i(\bar{k}\bar{x}-\omega t)}} + c_s^2\, \bar{k}^2 \cancel{e^{i(\bar{k}\bar{x}-\omega t)}} = 0 \qquad \Rightarrow \qquad \boxed{\bar{k}^2 = \frac{\omega^2}{c_s^2}}$$

like E.M. waves (!)

- <u>Direction of propagation</u> : plug in continuity eq. $\partial_t \overset{\downarrow}{\delta\rho} + \rho_0 \bar{\nabla}\, \overset{\downarrow}{\delta\bar{v}} = 0$

$$\partial_t\left(a\, e^{i(\bar{k}\bar{x}-\omega t)}\right) + \rho_0 \bar{\nabla}\left(\bar{b}\, e^{i(\bar{k}\bar{x}-\omega t)}\right) = 0$$

$$-\cancel{i}\,\omega a \cancel{e^{i(\bar{k}\bar{x}-\omega t)}} + \rho_0\, \cancel{i}\, \bar{k}\bar{b} \cancel{e^{i(\bar{k}\bar{x}-\omega t)}} = 0 \qquad \Rightarrow \qquad \boxed{\bar{k}\bar{b} = \frac{\omega a}{\rho_0}}$$

≠ E.M. waves (!)

i.e. $\bar{k}\bar{b} \neq 0$ => not orthogonal
$\bar{k}\bar{b} \neq |\bar{k}\bar{b}|$ => not parallel

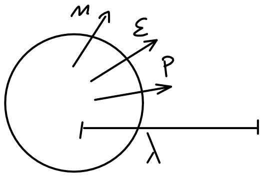
} => <u>longitudinal component</u> along the direction $\bar{k}$

Viscosity

- Ideal fluid : $\lambda \ll 1$ negligible mean free path (infinitesimal)
- Now : λ small but finite

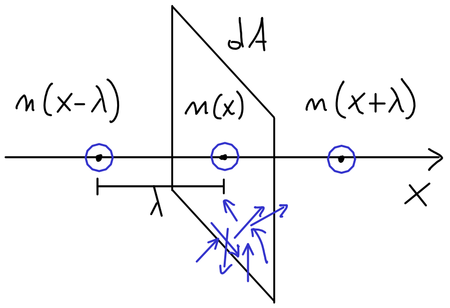
⇒ Particles can move with respect to fluid flow:

- transport of momentum ⇒ friction $\frac{d\bar{p}}{dt} < 0$ (they remove momentum from one volume)
- " " energy ⇒ heat conduction



⇒ Need to modify j^{μ} and $T^{\mu\nu}$ to account for it

Particles diffusion



$\bar{v}$ = bulk velocity of fluid

$\bar{u}$ = proper motion of particles with respect to $\bar{v}$

$\langle \bar{u} \rangle = 0$

use $\langle \bar{u}^2 \rangle^{1/2} \equiv u$ as typical particle velocity

$u_x = \frac{u}{\sqrt{3}}$ because of isotropy (random direction)

- Heuristic approach

$$\frac{dN}{dt} = -u_x\, dA \left[\frac{n(x+\lambda)-n(x-\lambda)}{2}\right] \simeq -\frac{u}{\sqrt{3}}\, dA\, \frac{\partial n}{\partial x}\cdot\lambda$$ change of number of particles

- $\lambda = 0$ or $\bar{\nabla} n = 0$ ⇒ $\frac{dN}{dt} = 0$ i.e. no diffusion
- Diffusion if $\lambda > 0$ and $\bar{\nabla} n \neq 0$!

⇒ particles current density: $\bar{j}_p \equiv \frac{dN}{dt\,dA} = -\frac{u\lambda}{\sqrt{3}} \bar{\nabla} n$ $D \equiv \frac{u\lambda}{\sqrt{3}}$ diffusion coefficient

Continuity Equation for diffusion

$$\partial_t n + \bar{\nabla}\bar{j}_p = \partial_t n - \bar{\nabla}\left(\frac{u\lambda}{\sqrt{3}} \bar{\nabla} n\right) = 0 \Rightarrow \boxed{\partial_t n = \bar{\nabla}(D\bar{\nabla} n)}$$ Fick's 2° law

Energy-Momentum tensor, diffusion component

- $\boxed{T^{\mu\nu} \longrightarrow T^{\mu\nu} + T_d^{\mu\nu}}$ include diffusive contribution => derive T_d

- Diffusion of energy density : T_d^{0i}

in analogy to current density $\left(\bar j_p = -\frac{u\lambda}{\sqrt{3}}\bar\nabla n\right)$

$$\bar q_\varepsilon = \frac{nu\lambda}{\sqrt{3}}\bar\nabla \overset{\varepsilon(T)}{\varepsilon} = \underbrace{-\frac{nu\lambda}{\sqrt{3}}\overbrace{\frac{\delta\varepsilon}{\delta T}}^{c_v}}_{K}\bar\nabla T = -K\bar\nabla T$$

$c_v \equiv \frac{\delta\varepsilon}{\delta T}$ heat capacity at const. volume

$K \equiv \frac{nu\lambda}{\sqrt{3}}c_v$ heat conductivity

gradient of internal energy

proportional to number density

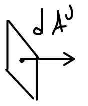
$$\Rightarrow T_{0i} = \rho c v^i + \frac{q^i}{c} + \frac{q^i_\varepsilon}{c} + \text{(work flux done by friction)}$$ see later

- Diffusion of momentum density : T_d^{ij}

$$(j_v)^i_j = \frac{nu\lambda}{\sqrt{3}}\frac{\delta v^i}{\delta x^j}$$

i-th velocity component $\bar v \nearrow$

across surface orthogonal to j-th direction



gradient of momentum density per unit of mass (nv^i)

1) $T^{ji} \propto \delta_j v^i$: - Rank-2 tensor ($3\times3$) : ok

- in general non symmetric but T_d^{ij} must be symmetric

$T_d^{ij} \propto -\frac{1}{2}(\delta^i v^j + \delta^j v^i)$ take symmetric part only $T_d \propto -[(\bar\nabla\otimes\bar v)+(\bar\nabla\otimes\bar v)^T]$

The symmetrization is reasonable

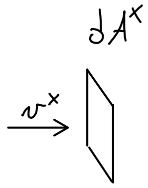
e.g. particles rotating as a rigid body: $\bar v = \bar\omega \times \overset{\bar r = (x^1,x^2,x^3)^T}{\bar r}$ $v^j = \varepsilon^j_{ke}\omega^k r^e$ ε = Levi-Civita symbol

=> No momentum transport, i.e. $T_d^{ij} = 0$, in fact:

$$\delta_i v^j = \varepsilon^j_{ke}\omega^k \underbrace{\delta_i x^e} = \varepsilon^j_{ke}\omega^k\delta^e_i = \varepsilon^j_{ki}\omega^k$$ anti-symmetric because ε is anti-sym.

$\Rightarrow T_d^{ij} = 0$ ✓

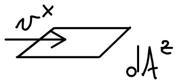
2) Convenient to split T_d in diagonal / off-diagonal parts:



- divergent and convergent flow : $\partial_i v^i = \bar{\nabla}\bar{v} = tr(\bar{\nabla}\otimes\bar{v})$

- shear flow : $\frac{1}{2}(\partial^i v^j + \partial^j v^i) - \frac{1}{3}\delta_3^{ij}\bar{\nabla}\bar{v}$ trace-free part

(all of it, without trace)



$\mapsto$ All together

$$\boxed{T_d^{ij} = -\eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta_3^{ij}\bar{\nabla}\bar{v}\right) - \xi\,\delta^{ij}\bar{\nabla}\bar{v}}$$

η, ξ viscosity parameters

↑ units of momentum density current

$$[T_d^{ij}] = \underbrace{\frac{g}{cm^{\cancel{3}}}\,\frac{\cancel{cm}}{s}}_{\substack{\uparrow\\ \text{momentum}\\ \text{density}}} \cdot \underbrace{\frac{\cancel{cm}}{s}}_{\substack{\uparrow\\ v}} = \frac{g}{cm\,s^2} = \frac{g}{cm\,s}\cdot\underbrace{\frac{1}{s}}_{\substack{\uparrow\\ \partial^i v^j}} \Rightarrow [\eta] = [\xi] = \frac{g}{cm\,s}$$

Viscous Hydrodynamics equations: $\partial_\mu T^{\mu\nu} = 0$

- Continuity eq. is unchanged (just matter conservation)
- Momentum conservation $\partial_\mu T^{\mu j} = 0$ is affected $(\nu = j)$

$$\partial_\mu (T^{\mu j} + T_d^{\mu j}) = \partial_\mu T^{\mu j} + \partial_\mu T_d^{\mu j}$$

as before ↗ ↖ compute now

$$\partial_i T_d^{ij} = -\eta\left(\partial_i \partial^i v^j + \partial_i \partial^j v^i - \frac{2}{3}\partial_i \delta^{ij} \bar\nabla \bar v\right) - \xi\, \partial_i \delta^{ij} \bar\nabla \bar v$$

$$= -\eta\left(\bar\nabla^2 v^j + \partial^j \bar\nabla \bar v - \frac{2}{3}\partial^j \bar\nabla \bar v\right) - \xi\, \partial^j \bar\nabla \bar v$$

$$= -\eta \bar\nabla^2 v^j - \left(\frac{\eta}{3} + \xi\right)\partial^j \bar\nabla \bar v$$

$$\Rightarrow \boxed{\underbrace{\rho(\partial_t + \bar v \bar\nabla) v^j + \partial^j P}_{\text{as Euler eq.}} = \underbrace{\eta \bar\nabla^2 v^j + \left(\frac{\eta}{3} + \xi\right)\partial^j \bar\nabla \bar v}_{\text{it acts as a source term}}}$$

Navier-Stokes equation

(viscosity acts as an "inner force")

change of momentum given by transfer of momentum $\bar F = \frac{d\bar P}{dt}$

- Energy conservation $\partial_\mu T^{\mu 0} = 0$

1) Include in current q^i diffusive internal energy transfer: $\bar{q}_\varepsilon = -K\bar{\nabla} T$

2) Includ work (density) done by friction, as a force, it makes work:

$dW = F ds = F \frac{ds}{dt} dt \quad \frac{dW}{dt} = F v \Rightarrow F = \frac{dP}{dt} =$ work flux $\Rightarrow q^i_{fR} = -v_j T_d^{ij}$

$$\Rightarrow \boxed{q^i \to q^i - K\bar{\nabla} T - v_j T_d^{ij}} \xrightarrow{\text{plug in}} \partial_\mu T^{\mu 0} = \partial_t\left(\frac{\rho}{2}\langle \dot{x}^2\rangle\right) + \bar{\nabla}\bar{q} = 0 \quad ⊛$$

$$\partial_i(K\bar{\nabla} T + v_j T^{ij}) = \partial_i(K\bar{\nabla} T) + T^{ij}\underset{(1)}{\partial_i v_j} + \underset{(2)}{v_j \partial_i T^{ij}}$$

(1) $= \frac{1}{2}(\partial_i v_j + \partial_j v_i) \equiv v_{ij}$ only the symmetric part is kept because $T^{\mu\nu}$ is symmetric

(2) drops out when pluggin in momentum conservation in ⊛

$$\Rightarrow \boxed{\partial_t \varepsilon + \bar{\nabla}(\varepsilon \bar{v}) + P\bar{\nabla}\bar{v} = \bar{\nabla}(K\bar{\nabla} T) + v_{ij} T_d^{ij}}$$

Energy conservation

↳ heat by friction (a)

↳ diffusive heat transfer (b)

↳ work done by pressure (2)

↳ current energy density (1)

↳ time change of internal energy because of (1), (2), (a), (b)

Evolution of specific entropy

- $\boxed{d\tilde{\varepsilon} + P\,d\tilde{V} = T\,d\tilde{s}}$ (1° law of thermodynamic)

$$\downarrow$$

$$\left(\frac{d\tilde{\varepsilon}}{dt} + P\frac{d\tilde{V}}{dt} = T\frac{d\tilde{s}}{dt}\right)$$

$\tilde{\varepsilon} \equiv \varepsilon/\varrho$

$\tilde{s} \equiv s/\varrho$

$\tilde{V} = \frac{V}{M} = \varrho^{-1} \rightarrow \frac{d\tilde{V}}{dt} = -\varrho^{-2}\frac{d\varrho}{dt} = -\varrho^{-2}\left(\partial_t\varrho + \bar{v}\bar{\nabla}\varrho\right)$

$$\underline{\partial_t\varrho} + \bar{\nabla}(\varrho\bar{v}) = \underline{\partial_t\varrho} + \varrho\bar{\nabla}\bar{v} + \bar{v}$$

$$\downarrow$$

$$= -\varrho^{-2}\left(-\varrho\bar{\nabla}\bar{v} - \cancel{\bar{v}\bar{\nabla}\varrho} + \cancel{\bar{v}\bar{\nabla}\varrho}\right)$$

$$= \varrho^{-1}\underline{\bar{\nabla}\bar{v}}$$

- Use energy conservation ($\partial_t\varepsilon + \ldots$)

$$\boxed{\partial_t\varepsilon + \bar{\nabla}(\varepsilon\bar{v}) + P\bar{\nabla}\bar{v}} = \partial_t(\varrho\tilde{\varepsilon}) + \bar{\nabla}(\varrho\tilde{\varepsilon}\bar{v}) + P\varrho\partial_t\tilde{V}$$

$$= \varrho\partial_t\tilde{\varepsilon} + \underbrace{\tilde{\varepsilon}\partial_t\varrho + \tilde{\varepsilon}\bar{\nabla}(\varrho\bar{v})}_{=0 \text{ continuity}} + \varrho\bar{v}\bar{\nabla}\tilde{\varepsilon} + P\varrho\partial_t\tilde{V}$$

$$= \varrho\left[(\partial_t + \bar{v}\bar{\nabla})\tilde{\varepsilon} + P\partial_t\tilde{V}\right] \qquad \partial_t\tilde{V} = \frac{d\tilde{V}}{dt} \text{ because } \overset{\text{only time}}{\tilde{V}(t)}$$

$$= \varrho\left(\frac{d\tilde{\varepsilon}}{dt} + P\frac{d\tilde{V}}{dt}\right) \qquad \leftarrow (\text{Plug } 1° \text{ law})$$

$$= \boxed{\varrho T\frac{d\tilde{s}}{dt} = \bar{\nabla}(\kappa\bar{\nabla}T) + v_{ij}T_d^{ij}}$$

→ Viscous friction (η, ζ)

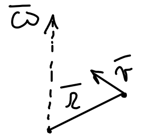
→ heat conduction

they make $\tilde{s}$ evolve!

- If $\kappa = 0$, $\eta = 0 = \zeta$: absence of transport process (ideal fluid)

$\Rightarrow \frac{d\tilde{s}}{dt} = 0$ i.e. specific entropy is conserved (isentropic fluid)

Vorticity and its evolution



- Vorticity : $\boxed{\bar{\Omega} \equiv 2\bar{\omega} = \bar{\nabla} \times \bar{v}}$

rotational velocity of a fluid element as it moves along the flow (vortex)

- Evolution of vorticity

- Navier-Stokes eq. :

$$\boxed{\partial_t \bar{v} + \underbrace{\bar{v}\bar{\nabla}\bar{v}} + \frac{\bar{\nabla}P}{\rho} + \overset{\text{external force}}{\bar{\nabla}\phi} = \left[\eta \bar{\nabla}^2 \bar{v} + \left(\frac{\eta}{3} + \zeta\right)\bar{\nabla}(\bar{\nabla}\bar{v})\right]\frac{1}{\rho}} \qquad (1)$$

- introduce vorticity : $\bar{v} \times (\underbrace{\bar{\nabla} \times \bar{v}}_{\bar{\Omega}}) = \bar{\nabla}\frac{\bar{v}^2}{2} - (\bar{v}\bar{\nabla})\bar{v}$ (property of double cross product)

- apply $\bar{\nabla} \times (1)$ to get time evolution of $\bar{\Omega}$ ($\partial_t \bar{v} \to \bar{\nabla} \times \partial_t \bar{v} \to \partial_t(\bar{\nabla} \times \bar{v}) = \partial_t \bar{\Omega}$)

$$\bar{\nabla} \times (1) = \partial_t \bar{\Omega} + \underbrace{\cancel{\bar{\nabla} \times \left(\frac{\bar{\nabla}\bar{v}^2}{2}\right)}}_{=0} - \bar{\nabla} \times (\bar{v} \times \bar{\Omega}) + \underbrace{\bar{\nabla} \times \left(\frac{\bar{\nabla}P}{\rho}\right)}_{= \frac{1}{\rho}\underbrace{\cancel{\bar{\nabla} \times \bar{\nabla}}P}_{=0} - \frac{1}{\rho^2}\bar{\nabla}\rho \times \bar{\nabla}P} + \underbrace{\cancel{\bar{\nabla} \times \bar{\nabla}\phi}}_{=0}$$

=0 the effect of external forces drops!

$$= \partial_t \Omega - \bar{\nabla} \times (\bar{v} \times \bar{\Omega}) - \frac{1}{\rho^2}\bar{\nabla}\rho \times \bar{\nabla}P$$

- Case (1) :

- assume ideal fluid → $\eta = 0 = \zeta$: right-hand side of (1) $= 0$
- assume barotropic fluid → P depends on ρ only $P(\rho)$:

$$\bar{\nabla}P(\rho) = \frac{\partial P}{\partial \rho}\bar{\nabla}\rho \Rightarrow \bar{\nabla}\rho \times \left(\frac{\partial P}{\partial \rho}\bar{\nabla}\rho\right) = 0$$

$$\Rightarrow \boxed{\partial_t \Omega = \bar{\nabla} \times (\bar{v} \times \bar{\Omega})}$$

Meaning:

- $$\frac{d\bar{v}}{dt} = \partial_t \bar{v} + \bar{v}\bar{\nabla}\bar{v} = \partial_t \bar{v} + \bar{\nabla}\frac{\bar{v}^2}{2} - \bar{v} \times \bar{\Omega} \qquad (2)$$

- $$\bar{\nabla} \times \frac{d\bar{v}}{dt} = \boxed{\frac{d\bar{\Omega}}{dt}} = \cancel{\partial_t \bar{\Omega}} + \underbrace{\cancel{\bar{\nabla} \times \bar{\nabla}}}_{=0}\frac{\bar{v}^2}{2} - \bar{\nabla} \times \cancel{(\bar{v} \times \bar{\Omega})} \; \boxed{= 0}$$

because of (2)

No evolution for an ideal ($\eta = 0 = \zeta$) barotropic fluid !

- <u>Case (2)</u>: viscous fluids ($\eta \neq 0$, $\zeta \neq 0$) and incompressible fluids ($\overline{\nabla}\rho = 0 = \partial_t \rho$)

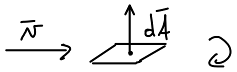
$$\boxed{\partial_t \overline{\Omega} - \overline{\nabla}\times(\overline{v}\times\overline{\Omega}) - \frac{1}{\rho^2}\overline{\nabla}\rho\times\overline{\nabla}P} = \frac{\eta}{\rho}\overline{\nabla}\times(\overline{\nabla}^2\overline{v}) + \cancel{\frac{1}{\rho}\left(\frac{\eta}{3}+\zeta\right)\underset{=0}{\underbrace{\overline{\nabla}\times\overline{\nabla}}}(\overline{\nabla}\overline{v})}$$

$$= \frac{\eta}{\rho}\overline{\nabla}^2(\overline{\nabla}\times\overline{v})$$

$$= \boxed{\nu\overline{\nabla}^2\overline{\Omega}}$$

$\nu \equiv \frac{\eta}{\rho}$ <u>kinetic viscosity</u> $\left[\frac{cm^2}{s}\right]$

shear flow only: (v^{ij})



<u>Meaning:</u>

$\partial_t \overline{\Omega}$: 1° order time derivative
$\overline{\nabla}\overline{\Omega}$: 2° order spacial derivative

$\Rightarrow$ <u>diffusion equation</u>

vorticity is diffused away by shear viscosity

The Reynolds number

- Look for self similar solutions of $\partial_t\vec\Omega - \vec\nabla\times(\vec v\times\vec\Omega) = \nu\,\vec\nabla^2\vec\Omega$

$\Rightarrow$ rescale length by L i.e $x \to \frac{x}{L}$
velocities by v i.e $\vec v \to \frac{\vec v}{v}$
$\Big\} \Rightarrow$ times by $\frac{L}{v}$ i.e. $t \to \frac{v}{L}t$

$$\Rightarrow \left(\partial_t \to \frac{L}{v}\partial_t\right) \quad \left(\partial_x \to L\partial_x\right) \quad \left(\vec\Omega \equiv \underset{\uparrow}{\vec\nabla}\times\underset{\uparrow}{\vec v} \to \frac{L}{v}\vec\Omega\right) \quad \left(\nu\left[\frac{cm^2}{s}\right] \to \frac{\nu}{Lv}\right)$$

$$\mapsto \frac{L}{v}\partial_t\left(\frac{L}{v}\vec\Omega\right) - L\vec\nabla\times\left(\frac{\vec v}{v}\times\frac{L}{v}\vec\Omega\right) = \boxed{\frac{\nu}{Lv}}\,L^2\vec\nabla^2\left(\frac{L}{v}\vec\Omega\right)$$

$$\cancel{\frac{L^2}{v^2}}\left[\partial_t\vec\Omega - \vec\nabla\times(\vec v\times\vec\Omega)\right] = \cancel{\frac{L^2}{v^2}}\left[\nu\,\vec\nabla^2\vec\Omega\right]$$

(adimensional)

$R \equiv \frac{Lv}{\nu}$ Raynolds number

$R \to \infty$ ideal fluid

↳ This means:

if you have one system with a certain size, velocity and kinetic viscosity ν and a similar one but rescaled by L, v and $\frac{\nu}{Lv}$
same geometry

$\Rightarrow$ they ~~have~~ have the same flow! (same solution)

$\Rightarrow$ self similar solutions characterized by R ⇐ distinguish ≠ solutions
the same

Meaning of R

$$R \equiv \frac{Lv}{\nu} = \frac{\rho v L}{\mu} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

← cause fluid to move (1)

← frictional shear forces between layers "sticky" (2)

viscous forces ⟷ inertial forces

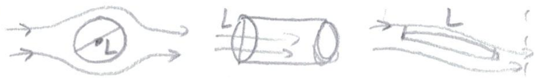
viscous forces ↓ push towards laminar flow

similar ↓ transition

inertial forces ↓ push towards tourbolent flow

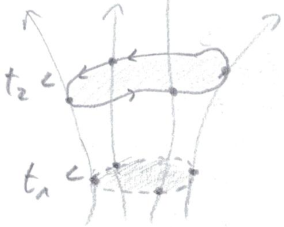
R ——{ transition / R critical }——→

what is L?



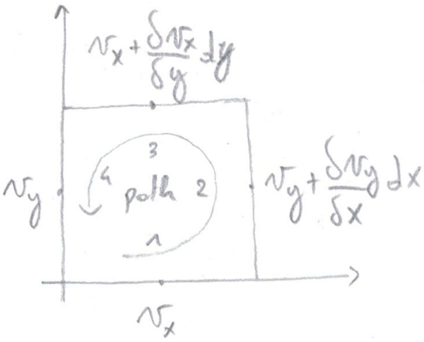
Circulation

- $\Gamma \equiv \oint_c \bar{v}\, d\bar{s}$



integral of velocity along a closed circuit c moving with the fluid flow

- Γ and $\bar{\Omega}$ relation



$$d\Gamma = \overset{\downarrow}{\bar{v}}\, d\overset{\downarrow}{\bar{s}}$$

$$= \underset{(1)}{\cancel{v_x dx}} + \underset{(2)}{\left(\cancel{v_y} + \frac{\delta v_y}{\delta x} dx\right) dy} - \underset{(3)}{\left(\cancel{v_x} + \frac{\delta v_x}{\delta y} dy\right) dx} - \underset{(4)}{\cancel{v_y dy}}$$

$$= \left(\frac{\delta v_y}{\delta x} - \frac{\delta v_x}{\delta y}\right) dx\, dy$$

$$= (\bar{\nabla} \times \bar{v})_z\, dx\, dy \underset{\substack{\uparrow \\ \text{in general}}}{=} \bar{\Omega}\, d\bar{A}$$

$$\Rightarrow \boxed{\Gamma = \oint_c \bar{v}\, d\bar{s} = \int_A \bar{\Omega}\, d\bar{A}}$$ Stokes theorem

The circulation along a closed circuit is the integral of the vorticity within it

Kelvin circulation theorem

For an ideal barotropic fluid $\frac{d\bar{\Omega}}{dt} = 0$ => Γ is conserved

$$\frac{d\Gamma}{dt} = \int_A \frac{d\bar{\Omega}}{dt}\, d\overset{\substack{\text{const} \\ \downarrow}}{\bar{A}} = 0 \quad ✓$$

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← in fact

Bernoulli's constant

- Non static : $\bar{v} \neq \bar{0}$
- Stationary : $\partial_t f = 0 \;\; \forall f$ variable of the system (e.g. ρ)

e.g. whater in a pipe : water flows but "nothing is changing"

- Ideal fluid , no energy dissipation, no heat flow $\Big\} \Rightarrow \dfrac{d\tilde{s}}{dt} = 0$ specific entropy

total derivative ↓

$$\frac{d\tilde{s}}{dt} = \underbrace{\partial_t \tilde{s}}_{\substack{=0 \\ \text{stationary}}} + (\bar{v}\bar{\nabla})\tilde{s} = (\bar{v}\bar{\nabla})\tilde{s} = 0 \quad \Rightarrow \tilde{s} = \text{const along flow lines}$$

- Apply to Euler's eq. these settings :

$$\cancel{\partial_t \bar{v}}_{=0} + \frac{1}{2}\bar{\nabla}(\bar{v}^2) - \bar{v} \times \bar{\Omega} + \frac{\bar{\nabla}P}{\rho} + \bar{\nabla}\phi = 0 \qquad \bar{\Omega} \equiv \bar{\nabla} \times \bar{v} \quad \text{vorticity} \equiv \text{curl of velocity}$$

- Plug Adiabatic condition : $d\tilde{h} = \dfrac{dP}{\rho} \Rightarrow \dfrac{\bar{\nabla}P}{\rho} = \bar{\nabla}\tilde{h}$

↑ (Specific enthalpy)

$$(\cdot\bar{v}) : \quad \frac{1}{2}\bar{v}\bar{\nabla}(\bar{v}^2) - \underbrace{\bar{v}\cdot(\bar{v} \times \bar{\Omega})}_{\substack{=0 \\ (\bar{v} \perp \bar{v}\times\bar{\Omega})}} + \bar{v}\left(\frac{\bar{\nabla}P}{\rho} + \bar{\nabla}\phi\right) = 0$$

$$\underbrace{(\bar{v}\bar{\nabla})}_{\circledast}\left(\frac{1}{2}\bar{v}^2 + \tilde{h} + \phi\right) = 0 \quad \leftarrow (\circledast \text{ variation along flow lines})$$

$$\Longrightarrow \boxed{\frac{1}{2}\bar{v}^2 + \tilde{h} + \phi = B} \quad \text{along flow lines} \quad \Rightarrow \text{expresses energy conservation}$$

$\tilde{h}$: (Pressure work) → (!)

B : Bernoully's const.

ϕ : Specific gravitational potential energy

$\frac{1}{2}\bar{v}^2$: Specific kinetic energy

Bernoulli's law for ....

- Irrotational flows: $\bar{\nabla}\times\bar{v} \equiv \bar{\Omega} = 0$
- Non stationary: $\partial_t f \neq 0$ (!)

Bernoulli had ...
$\bar{v} \neq 0$
$\partial_t f = 0$

Euler's eq:

$$\partial_t \bar{v} + \frac{1}{2}\bar{\nabla}\bar{v}^2 - \underset{=0}{\cancel{\bar{v}\times\bar{\Omega}}} + \frac{1}{2}\bar{\nabla}(\bar{v}^2) + \frac{\bar{\nabla}P}{\rho} + \bar{\nabla}\phi = 0$$

- Define $\bar{v} \equiv \bar{\nabla}\psi$ velocity potential ψ to have all gradients

$$\partial_t \overset{\equiv}{\bar{\nabla}}\psi + \frac{1}{2}\overset{\equiv}{\bar{\nabla}}(\bar{v}^2) + \frac{\overset{\equiv}{\bar{\nabla}}P}{\rho} + \overset{\equiv}{\bar{\nabla}}\phi = 0$$

- For an adiabatic flow: $\bar{\nabla}\hat{h} = \frac{\bar{\nabla}P}{\rho}$

$$\Rightarrow \bar{\nabla}\left(\partial_t\psi + \frac{1}{2}\bar{v}^2 + \hat{h} + \phi\right) = 0$$

$$\Rightarrow \partial_t\psi + \frac{1}{2}\bar{v}^2 + \hat{h} + \phi = B(t)$$

↑

ψ is not unique: you can add any constant to ψ because in the end what matters is $\bar{v} = \bar{\nabla}\psi$

$\Rightarrow$ choose: $\psi' = \psi + \int dt\, B(t)$ to get rid of $B(t)$

$$\models\!> \boxed{\partial_t\psi' + \frac{v^2}{2} + \hat{h} + \phi = 0}$$

↳ new term with respect to previous case

Classical, relativistic classical linear theory in Minkowski space-time

here we have one deg. of freedom

1) Lagrangian density: Similar to electrodynamics, use: a classical scalar field

$$\boxed{\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{8\pi G}\partial_\mu \varphi \partial^\mu \varphi + \phi \rho}$$

$\mu = 0, 1, 2, 3$ scalar classical field (adimensional)

$\phi\rho$ → field-matter interaction

$\partial_\mu \varphi \partial^\mu \varphi$ → Self interaction, free field (kinetic term)

$\frac{1}{8\pi G}$ → Const. to get the Newtonian limit

$\eta_{\mu\nu}$ is there! $\eta_{\mu\nu}\partial^\nu \varphi \partial^\mu \varphi$ Flat space-time

↳ quadratic to have a linear theory

2) Euler-Lagrange eq., equation of motion of the field

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} - \frac{\delta \mathcal{L}}{\delta \phi} = \frac{2}{8\pi G}\partial_\mu \partial^\mu \varphi - \rho = 0 \quad \Rightarrow \quad \boxed{\Box \varphi(x^\mu) = 4\pi G \rho(x^\mu)}$$

$\Box \varphi \equiv \partial_\mu \partial^\mu \varphi = (-c^{-2}\partial_t^2 \varphi + \vec{\nabla}^2 \varphi)$ D'Alambert operator

- Solution = plane waves : $\phi \propto e^{\pm i \eta_{\mu\nu} K^\mu x^\nu} = e^{\mp i(\vec{k}\vec{x} - \omega t)}$ $\quad K_\mu x^\nu = (\frac{\omega}{c}, \vec{k})\begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = -\omega t + \vec{k}\vec{x}$
- Propagation in vacuum : $\rho = 0$ $\quad \Box \phi = 0 \Rightarrow \eta_{\mu\nu} K^\mu K^\nu = 0$

$\Rightarrow K^\mu K_\mu = 0$ i.e. K^μ = null vector

i.e wave propagates along light-cone with velocity c

- Yes!

↳ Newtonian gravity : $-c^{-2}\partial_t^2 \phi \ll \vec{\nabla}^2 \phi$ i.e. slow time evolution of ϕ, retardation can be ignored

↳ retardation + $\phi \neq 0$ fluctuating in vacuum (gravitational waves!)

↑ d'Alampert operator $\Box$ give rise to retardation

$$\left(\Delta \equiv \partial_j \partial^j = \partial_{ij} \partial^i \partial^j = \vec{\nabla}^2\right) \xrightarrow{\text{change}} \left(\Box \equiv \partial_\mu \partial^\mu = \underset{\uparrow (!)}{\eta_{\mu\nu}} \partial^\nu \partial^\mu = -\frac{1}{c^2}\partial_t^2 + \vec{\nabla}^2\right)$$

- Energy-Momentum tensor of the field $\left(\mathcal{L}_f = \frac{1}{8\pi G}\partial_\alpha \varphi \partial^\alpha \varphi\right)$ field only

$$T_{\text{grav}}{}^\mu{}_\nu = \partial_\nu \varphi \frac{\delta \mathcal{L}_f}{\delta \varphi_{,\mu}} - \delta^\mu{}_\nu \mathcal{L}_f = \partial_\nu \varphi \frac{\overset{1}{\cancel{2}}}{\underset{4}{\cancel{8}}\pi G}\partial^\mu \varphi - \frac{1}{8\pi G}\delta^\mu{}_\nu \partial_\alpha \varphi \partial^\alpha \varphi = \frac{1}{4\pi G}\left(\partial_\nu \varphi \partial^\mu \varphi - \frac{1}{2}\delta^\mu{}_\nu \partial_\alpha \varphi \partial^\alpha \varphi\right)$$

- Join with $T^{\mu\nu}$ of fluids $\Rightarrow T^{ij} \to T^{ij}_{\text{fluid}} + T^{ij}_{\text{grav}}$ $\quad \boxed{\partial_j T^{ij}_{\text{fluid}} + \partial_j T^{ij}_{\text{grav}} = 0}$ $\quad \frac{\text{Navier-Stokes eq.}}{\text{with gravity}}$

In special relativity $T^{\mu\nu}_{\text{grav}}$ is a conserved quantity: $\boxed{\partial_j T^{ij}_{\text{grav}} = \rho \partial^i \varphi}$ gravitational force

Hydro + Poisson eq.s to solve for: $\rho, \vec{v}, \varepsilon, P, \varphi$

More about gravity

- Let's investigate T^{ij}_{grav} : potential energy

1) Re-express T^{ij} :

drop $\frac{1}{2c^2}\partial_t\varphi\partial^t\varphi$ $\qquad$ $\bar{\nabla}^2\varphi = 4\pi G\rho$ Poisson eq.

$$T^{ij}_{grav} = \frac{1}{4\pi G}\left(\partial^i\varphi\partial^j\varphi - \frac{1}{2}\partial^{ij}\partial_\alpha\varphi\partial^\alpha\varphi\right) \simeq \frac{\partial^i\varphi\partial^j\varphi}{4\pi G} - \frac{\delta^{ij}}{8\pi G}\left(\partial_k(\varphi\partial^k\varphi) - \varphi\underbrace{\partial_k\partial^k}\varphi\right)$$

$$= \frac{\partial^i\varphi\partial^j\varphi}{4\pi G} - \frac{\delta^{ij}}{8\pi G}\partial_k(\varphi\partial^k\varphi) + \frac{\delta^{ij}}{\cancel{8\pi G}\,2}\varphi\,\cancel{4\pi G}\rho = \frac{\partial^i\varphi\partial^j\varphi}{4\pi G} - A^{ij} + \frac{1}{2}\delta^{ij}\rho\varphi$$

2) Define $U^{ij} \equiv \int d^3x\, T^{ij}$ and study trace $U = Tr(U^{ij}) = \int d^3x\, Tr(T^{ij}) = \int d^3x\, T$

$$Tr(T^{ij}) = T = \frac{\partial^i\varphi\partial_i\varphi}{4\pi G} + \frac{1}{2}\overset{=3\rho\varphi}{\delta^i_i\rho\varphi} - A^i_i = \frac{\overbrace{\partial^i(\varphi\partial_i\varphi)}^{B} - \varphi\overbrace{\partial^i\partial_i}^{\bar{\nabla}^2\varphi=4\pi G\rho}\varphi}{4\pi G} + \frac{3\rho\varphi}{2} - \overset{Tr(A^{ij})}{A} = B + \frac{1}{2}\rho\varphi - A$$

$$\Rightarrow \boxed{U = \frac{1}{2}\int d^3x\,\rho\varphi}$$

gravitational potential energy

here $\int d^3x\, A \overset{\text{gauss these}}{=} 0 = \int d^3x\, B$ Because A, B = gradient of a function

in fact: Gauss theorem $\int_V d^3x\,\partial_k f = \int_{\delta V} dA\, f = 0$ $\quad\delta V$ surface enclosing V

because you want to integrate over the entire source body ρ

i.e. at ∞ where $f=0$

→ Phisical meaning : φ is not unique ⇒ you can add a const.

- Chandrasekhar expression for grav. potential energy

- remember: $\partial_j T_f^{ij} = \rho\partial^i\varphi$ gravitational force

gradient of a function ⇒ drops in integral

("drop it" = rescale T^{ij})

- note: $x^i\partial_k T^{jk}_{grav} = \overbrace{\partial_k(x^i T^{jk}_{grav})}^{C^{ij}} - T^{jk}_{grav}\overset{\delta^i_k}{\partial_k x^i}$ $\Rightarrow$

$$T^{ji}_{grav} = -x^i\rho\partial^j\varphi - \cancel{C^{ij}}$$

$$U = \int d^3x\, Tr(T^{ij}) = -\int d^3x\,\rho x^i\partial_i\varphi$$

- join to previous result:

$$\boxed{\frac{1}{2}\int d^3x\,\rho\varphi = -\int d^3x\,\rho x^i\partial_i\varphi}$$

Chandrasekhar expression for grav. potential energy

potential on shells, e.g. useful for stars ◎

Virial tensor theorem

- Generalization of the virial theorem : $T+\frac{1}{2}V=0 \Rightarrow$ equilibrium

  typically point mass in orbit $T=\frac{1}{2}m\bar{v}$

- Now express this concept in terms of momentum of inertia $\ddot{I}^{ij}$ of orbiting body

• $I^{ij}\equiv\int_V d^3x\,\rho\,x^i x^j$ Inertial tensor of a body x^i, x^j not time dependent

$V=$ fixed volume ↗

• Go for $\frac{d^2\dot{I}^{ij}}{dt^2}$:

plug continuity eq.

$$\frac{dI^{ij}}{dt}=(\partial_t+\bar{v}\bar{\nabla})\int_V d^3x\,\rho\,x^ix^j=\int_V d^3x\Big[x^ix^j\partial_t\rho+\underbrace{\cancel{\rho\,\partial_t(x^ix^j)}}_{=0}+\underbrace{\cancel{\bar{v}\bar{\nabla}(\rho x^ix^j)}}_{\bar{v}=\partial_t\bar{x}=0}\Big]$$

$$=-\int_V d^3x\,x^ix^j\,\partial_k(\rho v^k)=-\int_V d^3x\Big[\underbrace{\partial_k(\rho v^kx^ix^j)}_{=0\text{ Gauss}}-\partial_k(x^ix^j)\rho v^k\Big]=+\int_V d^3x\Big(x^j\underbrace{\partial_kx^i}_{\delta^i_k}+x^i\underbrace{\partial_kx^j}_{\delta^j_k}\Big)\rho v^k$$

$$=\int_V d^3x\,(v^ix^j+v^jx^i)\rho$$

$$\frac{d^2I^{ij}}{dt^2}=\int_V d^3x\left[x^j\partial_t(\rho v^i)+x^i\partial_t(\rho v^j)\right]$$

• link $x^j\partial_t(\rho v^i)$ to T^{ij} :

a) $x^j\partial_0T^{0i}=x^j\partial_0(\rho c v^i)=c\,x^j\partial_0(\rho v^i)=x^j\partial_t(\rho v^i)$

b) mom. cons. : $\partial_0T^{0i}+\partial_jT^{ji}=0$ $\partial_0T^{0i}=-\partial_jT^{ji}$ ↘ $x^j\partial_0T^{0i}=-x^j\partial_jT^{ji}=-\underbrace{\partial_j(x^jT^{ij})}_{\text{gradient will drop}}+T^{ij}\underbrace{\partial_jx^j}_{\delta^j_j=1}$

$\Rightarrow T^{ji}=x^j\partial_t(\rho v^i)\ \underbrace{-c^{ij}}_{\text{irrelevant}}$

$$=\int_V d^3x\,(T^{ji}+T^{ij})=2\int_V d^3x\,T^{ij}=2\left[\int_V\rho v^iv^j\,dx^3+\delta^{ij}\int_V P\,dx^3+U^{ij}\right]=0$$

(for perfect fluid : $T^{ij}=\rho v^iv^j+P\delta^{ij}+T^{ij}_{grav}$)   (if $=0\Rightarrow$ system is in equilibrium)

$\Rightarrow$ Virial tensor theorem :

$$\boxed{K^{ij}+\frac{1}{2}\left[\delta^{ij}\int d^3x\,P+U^{ij}\right]=0}$$

(1) Kinetic (translational + rotational) $K^{ij}\equiv\frac{1}{2}\int_V\rho v^iv^j\,dx^3$

(2) work by pressure

(3) grav. potential energy $U^{ij}=\int_V d^3x\,T^{ij}_{grav}$

Grav. force :

$$T_{grav}{}^{\mu}{}_{\nu} = \frac{1}{4\pi G}\left(\partial_\nu\varphi\,\partial^\mu\varphi - \frac{1}{2}\delta^\mu{}_\nu\,\partial_\alpha\varphi\,\partial^\alpha\varphi\right)$$

$$\partial_j T^{ij}_{grav} = \frac{1}{4\pi G}\Big[(\partial_j\partial^i\varphi)\partial^j\varphi + \partial^i\varphi\underbrace{(\partial_j\partial^j\varphi)}_{\nabla^2\varphi = 4\pi G\rho \text{ Poisson eq.}} - \frac{1}{2}\overbrace{\delta^{ij}\partial_j}^{\partial^i}\partial_\alpha\varphi\,\partial^\alpha\varphi - \frac{1}{2}\,\partial_\alpha\varphi\,\overbrace{\delta^{ij}\partial_j}^{\partial^i}\partial^\alpha\varphi\Big]$$

$$= \frac{1}{\cancel{4\pi G}}\left(\cancel{\overset{\leftrightarrow}{\partial_j\partial^i}\varphi\,\partial^j\varphi} + \partial^i\varphi\,\cancel{4\pi G}\rho - \cancel{\partial^i\partial_\alpha\varphi\,\partial^\alpha\varphi}\right) \simeq \rho\,\partial^i\varphi$$

approx : drop $\frac{1}{c^2}\partial_\epsilon\varphi\,\partial^\epsilon\varphi$

Part V

Plasmas, propagation of light in a medium and magneto-hydrodynamics

Main transport mechanisms: summary

- Heat conduction

Energy conservation eq. → as a function of specific entropy $\frac{d\tilde{s}}{dt}$

→ 1° law thermodynamics $\frac{d\hat{s}}{dt} \rightarrow \frac{dT}{dt}$

⇒ $\frac{dT}{dt} = \chi \bar{\nabla} T$ $\quad \chi \equiv \frac{K}{\rho c_p}$ diffusion equation for T

⇒ Transport of thermal energy

- Convection

- Presence of gravity $\downarrow \vec{g}$ + gradient of temperature $\bar{\nabla} T$
- Motion of convective cells due to bojancy
- Criteria for convection instability $\left.\frac{ds}{dz}\right|_z < 0$

⇒ $\boxed{\frac{d\ln T}{dz} > \frac{(1-\gamma)}{\gamma}\frac{d\ln P}{dz}}$ Schwartzschild criterion

- You need a temperature gradient large enough or the cell thermalizes on spot beforore "moving up"

- Tourbolence

- For large R insurgence of tourbolence (inertial forces ≫ viscous forces)
- Creation of cascade of eddies with smaller and smaller size
- Transfer of energy from large (coherent) scales to small (caotic scales)

Heat conduction

Transport of thermal energy

typically (!)

- Systems can reach mechanical equilibrium faster than thermal one

⟹ Systems can be in mechanical equilibrium but not in Thermal one

- Example: Star → mechanical hydrodynamical equilibrium

→ energy radiated away

⟹ Temperature gradient maintained by energy production

— ∽ —

- Entropy equation (from energy conservation)

$$\boxed{\rho T \frac{d\tilde{s}}{dt} = \bar{\nabla}(\kappa \bar{\nabla} T) + \underbrace{v_{ij} T_d^{ij}}}$$

Viscous friction

if $\bar{v}$ gradient $v_{ij} \equiv \frac{\partial v^i}{\partial x_j}$ is too small to drive matter current

⟹ neglect it

thermal capacity at const. pressure

- We have: $dq = c_p dT$

(also for $p = \text{const}$) → : $dq = T d\tilde{s}$ (1° principle) thermodynamic

$$\left.\begin{matrix} dq = c_p dT \\ dq = T d\tilde{s} \end{matrix}\right\} \quad c_p dT = T d\tilde{s} \rightarrow \frac{c_p}{T}\frac{dT}{dt} = \frac{d\tilde{s}}{dt}$$

$$\Rightarrow \quad \rho \cancel{T} \frac{c_p}{\cancel{T}} \frac{dT}{dt} = \kappa \bar{\nabla}^2 T \quad \Rightarrow \quad \boxed{\frac{dT}{dt}} = \frac{\kappa}{\rho c_p} \bar{\nabla}^2 T = \boxed{\chi \bar{\nabla}^2 T}$$

entropy eq. as a diffusion eq.

$\kappa = \text{const.}$

$\chi \equiv \frac{\kappa}{\rho c_p}$ Transport coefficient (thermal diffusivity)

thermal inertia of the material

- It is an energy (bulk) transport mechanism

$\boxed{\text{Convection}}$ Transport of energy and matter

- Presence of a gravitational field $\downarrow \vec{g}$ (it breaks isotropy : priviledge directio
- If temperature gradients $\vec{\nabla} T$ are too large $\Rightarrow$ convection

↳ rising hot bubbles can not cool down sufficiently fast

↳ no time to thermalize

↳ remain warmer than surroundings

↳ they move upward $\Rightarrow$ bulk motion (transport)

- Consider volume $V(P, s)$ $\overset{V}{\textcircled{P\,s}}$ with characteristic pressure and entropy
- Ignore thermal adaptation, mechanical equilibrium will be reached first.

$$V(P', s) = V' \longrightarrow V'' = V(P', s') \quad \text{same volume adapted to the environment}$$

Δz Bojancy $\uparrow$ $\downarrow T$

$$V(P, s) = V$$

instantaneous adaptation (mechanical)

followed by adaptation of s (thermal)

↳ Stable situation if $\boxed{V' = V(P', s) < V(P', s') = V''}$ (Bojancy not enough, gravity wins)

Stability condition

☐ Quantative approach

- Entropy at $z' = z + \Delta z$: $s' = s + \left.\frac{ds}{dz}\right|_z \Delta z$
- Change in volume (const. P): $dV = \left(\frac{\delta V}{\delta s}\right)_P \overset{\downarrow}{ds} = \left(\frac{\delta V}{\delta s}\right)_P c_P \frac{dT}{T}$
- Bubble in new environment at $z + \Delta z$ and s' : $\boxed{V''} = V' + \left(\frac{\delta V}{\delta s}\right)_P \overset{\searrow}{\Delta s} = \boxed{V' + \left(\frac{\delta V}{\delta s}\right)_P \left.\frac{ds}{dz}\right|_z \Delta z} \overset{\text{Stability}}{>} V'$ $\quad$ (>0)

S from energy conservation $\frac{d\hat{s}}{dt}$ ...

$\Rightarrow$ Stability ($V' < V''$) if $\boxed{\left.\frac{ds}{dz}\right|_z > 0}$

i.e. Entropy increases with height $\quad$ look at $\frac{ds}{dz}$!

- $\Rightarrow$ stability for $\left.\frac{ds}{dz}\right|_z \geq 0$, study $\frac{ds}{dz}$ $\quad s = s(T,P)$

$$\frac{ds}{dz} = \left(\frac{\partial s}{\partial T}\right)_P \frac{dT}{dz} + \left(\frac{\partial s}{\partial P}\right)_T \frac{dP}{dz}$$

← for polytropic eq. of state: $\left(\frac{\partial s}{\partial T}\right)_P = \gamma \frac{c_v}{T}$

$\left(\frac{\partial s}{\partial P}\right)_T = -(\gamma-1)\frac{c_v}{P}$

$$= c_v \gamma \frac{dT}{T dz} - c_v(\gamma-1)\frac{dP}{P dz}$$

$$= c_v\left[\gamma \frac{d\ln T}{dz} - (\gamma-1)\frac{d\ln P}{dz}\right] \underset{\substack{\uparrow \\ \text{stable}}}{>} 0$$

Schwarzschild's criterion for stability

$\Rightarrow$ Convection if: $\boxed{\frac{d\ln T}{dz} > \underbrace{\frac{(1-\gamma)}{\gamma}}_{\nabla_{ad}} \frac{d\ln P}{dz}} \rightarrow \frac{d\ln T}{d\ln P} > \nabla_{ad}$

∇_{ad} "adiabatic temperature gradient"

$\nabla_{ad} > 0$ usually

↑ condition for the T gradient

E.g. Star

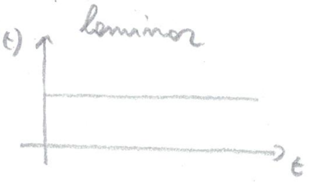
$T_{core} > T_{surface}$

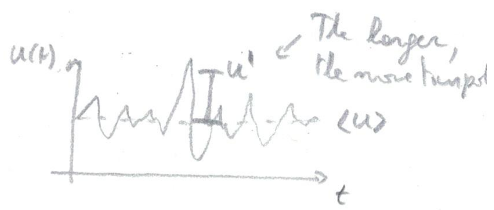
- The condition above tells you which layers are convective / radiative transport dominated

i.e. energy is (mostly) carried by radiation or convection

e.g. Sun: granularity
= convective cells

Tourbolence





- Very complex, here only main concepts
- Flows with large Reynolds number are "unstable" i.e. the laminar flow is broken presence of tourbolence
- This happens when $R \equiv \frac{uL}{\nu} \gtrsim R_{critical}$ ← How to determine it? not really clear....

  $\nu \equiv \frac{\eta}{\rho}$ = kinetic viscosity

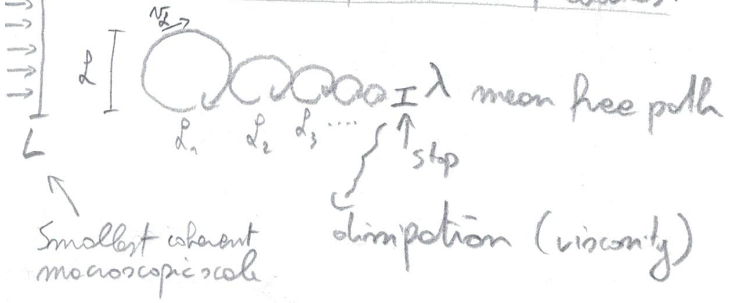
- Tourbolence → transfer of energy from large to small scales until energy dissipated by viscous heat (mean free path scale)

  ↓

  from macroscopic ordered motion to microscopic unordered / thermal motion

  $\underbrace{\qquad\qquad\qquad\qquad}_{\text{Tourbolence}}$

- Formation of cascade of eddies:



- Size of eddie $\underline{\ell}$
- Tangential velocity $\underline{v_\ell}$

$\Rightarrow$ "one turn" in $\underline{\tau_\ell} = \frac{\ell}{v_\ell}$

- $\underline{\tilde{\mathcal{E}}_\ell}$ = energy per unit mass in eddie $= \frac{v_\ell^2}{2}$

  ↑ depends on the material

$\Rightarrow$ flow of specific energy $\tilde{\mathcal{E}}_\ell$

$$\boxed{\dot{\tilde{\mathcal{E}}}_\ell} \approx \frac{d\tilde{\mathcal{E}}_\ell}{d\tau} \approx \left(\frac{v_\ell^2}{2}\right)\left(\frac{1}{\tau_\ell}\right) \approx \boxed{\frac{v_\ell^3}{\ell}}$$

specific energy (kinetic) ↗ inverse time scale ↑

$$L > \ell > \lambda_{viscosity}$$

↑ Hierarchyc cascade

- Energy is not accumulated at any scale but it is transfered from one scale to another

some constant

$$\Rightarrow \quad \dot{\varepsilon} \approx \frac{v_\ell^3}{\ell} \rightarrow v_\ell^3 \approx \dot{\varepsilon}\ell \Rightarrow \boxed{v_\ell \approx \alpha \ell^{1/3}}$$

- Boundary condition at largest scale $v_L = u \approx \alpha L^{1/3} \Rightarrow \alpha = \frac{u}{L^{1/3}}$

$$\Rightarrow \boxed{v_\ell \approx u\left(\frac{\ell}{L}\right)^{1/3}} \quad : \ell \downarrow \Rightarrow v_\ell \downarrow$$

- <u>Vorticity</u>: $\vec{\Omega} \equiv \vec{\nabla} \times \vec{v} \rightarrow \boxed{\Omega_\ell} \approx \frac{v_\ell}{\ell} \approx \frac{u}{\ell}\left(\frac{\ell}{L}\right)^{1/3} \approx \boxed{\frac{u}{(\ell^2 L)^{1/3}}} \quad : \ell \downarrow \Rightarrow \Omega \uparrow$

Smaller eddies have larger vorticity!

- <u>Energy spectrum</u> (distribution of energy on scales → wave number k)

  - <u>Specific energy</u> $\tilde{\varepsilon}_\ell \approx \frac{v_\ell^2}{2} \propto \ell^{2/3}$

Fourier transform: $\hat{\tilde{\varepsilon}}_\ell \equiv \int_{-\infty}^{\infty} \underset{\uparrow \, \propto \ell^{2/3}}{\tilde{\varepsilon}_\ell} \underset{\uparrow \, \text{adimensional}}{e^{i\vec{\omega}\vec{k}}} \underset{\propto \ell^3}{d^3x}$

  - Fourier decomp: $\left(\hat{\tilde{\varepsilon}}_\ell\right) \propto \left(\widehat{\ell^{2/3}_\ell}\right) \propto \ell^3 \ell^{2/3} \propto \ell^{11/3}$

(!)

$$\Rightarrow \text{Energy spectrum } \boxed{E(k)} \propto \underset{\uparrow}{k^2} k^{-11/3} \propto \boxed{k^{-5/3}} \leftarrow \text{Kolmogorov spectrum}$$

number of fourier modes in dk shell

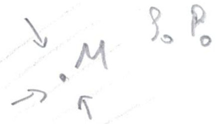
- <u>Dissipation scale λ_{visc}</u>

  - Navier-Stokes eq. $\ldots\ldots = \underbrace{\eta}_{} \vec{\nabla}^2 v^i + \ldots$

viscosity heating rate $h_{visc} \approx \eta$

$$\boxed{\lambda_{visc} = \frac{L}{R^{3/4}}}$$

Bondi accretion



- Extended gas cloud (ρ_0, P_0) inflow because of gravity
- Central point mass (eg. star)
- Stationary flow $(\partial_t f = 0)$ $\Rightarrow$ continuity eq. $\overset{=0}{\cancel{\partial_t \rho}} + \vec\nabla(\rho\vec v) = 0$

  ↳ in spherical symm. $\frac{1}{r^2}\partial_r(r^2\rho v) = 0$

$\Rightarrow r^2 \rho v = \text{const.} \left[\frac{g}{s}\right] \longrightarrow \boxed{4\pi r^2 \rho v = -\dot M}$ accretion rate

$[\dot M]$ = mass per second across a spherical shell (4π)

- Apply Bernoulli's law : $\frac{v^2}{2} + \underset{\uparrow}{\tilde h} + \underset{\uparrow}{\phi} = B$

here : $\phi = -\frac{GM}{r}$ (i.e. assume $M = \text{const.} \Leftrightarrow$ the mass of the accreted material $\ll M$)

$d\tilde h = \frac{dP}{\rho} \Rightarrow$ (a) $\tilde h = \frac{c_s^2 - c_{s0}^2}{\gamma - 1}$ ← for adiabatic gas

(b) $\tilde h = \frac{k_B T}{m}\int\frac{d\rho}{\rho} = c_{s0}^2 \ln\left(\frac{\rho}{\rho_0}\right)$ ← for isothermal ideal gas

Now $\div c_s^2$

B=0 Border conditions $v=0$ $\tilde h=0$ $\phi=0$ away from star

Case b: $\frac{1}{2}\left(\frac{v}{c_{s0}}\right)^2 + \ln\left(\frac{\rho}{\rho_0}\right) + \frac{GM}{r\,c_{s0}^2} = \boxed{\frac{u^2}{2} + \ln\alpha - \frac{1}{x} = 0}$

define $u \equiv \frac{v}{c_{s0}}$ $\alpha \equiv \frac{\rho}{\rho_0}$ $x \equiv \frac{r}{r_B}$

" $r_B \equiv \frac{GM}{c_{s0}^2}$ Bondi

Case a: $\boxed{\frac{u^2}{2} + \frac{\alpha^{\gamma-1} - 1}{\gamma - 1} - \frac{1}{x} = 0}$

Same variables for continuity eq.

accretion rate in units of $\dot M_B$

$4\pi \overset{\downarrow}{r^2}\overset{\downarrow}{\rho}\overset{\downarrow}{v} = 4\pi(r_B^2 x^2)(\rho_0\alpha)(c_{s0}u) = -\dot M \Rightarrow \boxed{x^2\alpha u} = -\frac{\dot M}{4\pi r_B^2 \rho_0 c_{s0}} \equiv -\frac{\dot M}{\dot M_B} \equiv \boxed{\mu}$

with $\dot M_B$ = Bondi accretion rate

- Combine Bernoulli and continuity eq. to get solution

continuity

$$\alpha = \frac{\mu}{x^2 u} \Rightarrow$$

Bernoulli

$$\boxed{\frac{u^2}{2} + \frac{\left(\frac{\mu}{x^2 u}\right)^{\gamma-1} - 1}{\gamma-1} - \frac{1}{x} = 0}$$ Adiabatic (a)

$$\boxed{\frac{u^2}{2} + \ln\left(\frac{\mu}{x^2 u}\right) - \frac{1}{x} = 0}$$ Isothermal (b)

- Investigate the behaviour of the solution

↳ combine total derivatives of continuity eq. and Bernoulli eq.

$$(d) - \alpha x^2 u = \mu \rightarrow (d\alpha) x^2 u + \alpha 2x (dx) u + \alpha x^2 (du) = 0$$ (continuity)

$$(\div x^2 \alpha u) \rightarrow \frac{d\alpha}{\alpha} + 2\frac{dx}{x} + \frac{du}{u} = 0 \rightarrow \left(\frac{d\alpha}{\alpha} = -2\frac{dx}{x} - \frac{du}{u}\right)$$

↙ Plug

$$(d) - \text{Bernoully} \rightarrow u\,du + \frac{\cancel{(\gamma-1)}\alpha^{\gamma-2}}{\cancel{\gamma-1}} d\alpha + \frac{dx}{x^2} = 0$$ adiabatic (a)

$$\rightarrow u\,du + \frac{d\alpha}{\alpha} + \frac{dx}{x^2} = 0$$ Isothermal (b)

$\Rightarrow$ for isothermal: $u\,du - 2\frac{dx}{x} - \frac{du}{u} + \frac{dx}{x^2} = u\,du\left(1 - \frac{1}{u^2}\right) - \frac{dx}{x}\left(2 - \frac{1}{x}\right) = 0$

$$\boxed{u\,du\left(1 - \frac{1}{u^2}\right) = \frac{dx}{x}\left(2 - \frac{1}{x}\right)} \leftarrow \mathcal{M}^2 \equiv u^2 = \left(\frac{v}{c_{s0}}\right)^2$$

$\Rightarrow$ for adiabatic: $\boxed{u\,du\left(1 - \frac{\alpha^{\gamma-1}}{u^2}\right) = \frac{dx}{x}\left(2\alpha^{\gamma-1} - \frac{1}{x}\right)} \leftarrow \mathcal{M}^2 \equiv \frac{u^2}{\alpha^{\gamma-1}}$ ↖ (!)

↑ Mach number

$\models$ It exists a critical radius

$$\begin{cases} x_c = \frac{1}{2} \\ x_c = \frac{\alpha^{1-\gamma}}{2} \end{cases} \quad u\,du\left(1 - \frac{1}{\mathcal{M}}\right) \overset{(\text{at } x_c)}{=} 0 \models$$

or $u = 0$ (trivial solution)

or $du = 0 \Rightarrow$ max velocity at x_c

or $\mathcal{M} = 1$ (supersonic)

↑ critical radius

↳ these cases depends on the accretion rate $\dot{M}$ ↵

- Studi solution for a specific case

$x = \frac{r}{r_B}$

- If flow is supersonic ($\underline{u = 1}$) at $\underline{x = x_c} = \frac{1}{2}$ (half the Bondi radius)
- E.g. isothermal gas

  - Bernoulli's eq. $\frac{u^2}{2} + \ln\alpha - \frac{1}{x} = 0$

    $\Rightarrow \frac{1}{2} + \ln\alpha - 2 = 0 \Rightarrow \alpha = \frac{\rho}{\rho_0} = e^{3/2}$

    density at x_c is $\rho = \rho_0 e^{3/2} \Rightarrow \alpha = e^{3/2}$

    density of the cloud surrounding the star

  - Continuity $\alpha x^2 u = \mu$ $\left(\mu = -\frac{\dot{M}}{\dot{M}_B}\right)$

    $e^{3/2} \frac{1}{4} 1 = \mu_c \Rightarrow \mu_c = \frac{e^{3/2}}{4}$ critical accretion rate

    $= 1,120$

$\Rightarrow$ for $\mu < \mu_c$: flow speed $< c_{s0}$ at r_c

$\Rightarrow$ for $\mu > \mu_c$: accretion stalls

because the pressure of the gas "reacts" to the in-flow

$\longrightarrow \ \longleftarrow$

Pressure Gravity

Hydrostatic equilibrium

gravitational

gas source of the field (!)

- Static → $\bar{v}=0$ ⇒ Navier-Stokes eq. $\boxed{\underbrace{\bar{\nabla} P}_{\text{gas}} = -\underbrace{\rho}_{}\underbrace{\bar{\nabla}\phi}_{}}$

1) • Take the curl: $\underbrace{\bar{\nabla}\times(\bar{\nabla}P)}_{=0} = \bar{\nabla}\times(-\rho\bar{\nabla}\phi) = -\bar{\nabla}\rho\times\bar{\nabla}\phi + \rho\underbrace{\bar{\nabla}\times\bar{\nabla}\phi}_{=0}$

curl of a gradient vanishes identically

$$\Rightarrow \boxed{\bar{\nabla}\rho\times\bar{\nabla}\phi=0}$$

⇒ Gradient of ρ // Gradient of ϕ

⇒ equi-density surface follows equipotential surfaces 

⇒ The density of a fluid follows the one of ϕ (i.e. the gas profiles traces the one of ϕ ⇒ "direct measure of gravity")

2) • Take the divergence $\boxed{\bar{\nabla}\left(\frac{\bar{\nabla}P}{\rho}\right) = -\overbrace{\bar{\nabla}^2\phi = -4\pi G\rho}^{\text{Poisson eq}}}$

- Plug equation of state $P=P(\rho)$ e.g. $P(\rho)=P_0\left(\frac{\rho}{\rho_0}\right)^\gamma$ Polytropic eq. of state, γ = adiabatic index

↦ Density distribution

— o —

• Example: Self gravitating sphere of gas ⇒ spherical coordinates

$$- \frac{1}{r^2}\partial_r\left(r^2\frac{\partial_r P}{\rho}\right) = \frac{1}{r^2}\partial_r\left(r^2\frac{P_0\gamma}{\rho}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\frac{\partial_r\rho}{\rho_0}\right) = \frac{1}{r^2}\underbrace{\left(\frac{P_0}{\rho_0}\gamma\right)}_{c_{s0}^2}\partial_r\left(\frac{r^2}{\rho}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\partial_r\rho\right) = -4\pi G\rho$$

- It is a Lane-Emden equation

$$\frac{1}{x^2}\partial_x(x^2\partial_x\theta) = -\theta^n \qquad n\equiv(\gamma-1)^{-1} \text{ polytropic index}$$

$$\theta^n \equiv \rho/\rho_0 \qquad x=\frac{r}{r_0} \qquad r_0=\left(\frac{n\,c_{s0}^2}{4\pi G\rho_0}\right)^{1/2}$$

↳ it can be solved for boundary conditions $\partial_x\theta=0$, $\theta=1$ at $x=0$

⇒ solve for ρ: density profile! $\rho(r)$

Instabilities: summary

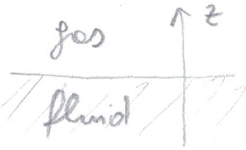
☐ In general

- Perturbative approach: $$f(\bar{x},t) = \overbrace{f_0}^{\text{Solution of background}} + \overbrace{\delta f(\bar{x},t)}^{\text{Perturbation } \delta f \ll f_0} \qquad f = P, \rho, v_i$$
- Decompose in plane waves $$\delta f = a \cdot e^{i(k\bar{x}-\omega t)} = a \underbrace{e^{ik\bar{x}}}_{\text{spatial "shape" } (\bar{x}) \text{ dependency}} \cdot \underbrace{e^{-i\omega t}}_{\text{time evolution}}$$
- Instability for ω purely complex: $\omega = ib$ $b \in \mathbb{R}$ $b > 0$

$e^{-i\omega t} = e^{bt}$ exponential growth

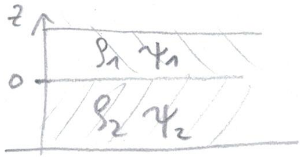
☐ Examples (ideal fluid)

- Gravity waves (counter example)



$\omega \in \mathbb{R} \Rightarrow$ oscillations around equilibrium

- Rayleigh-Taylor instability

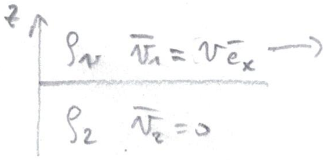


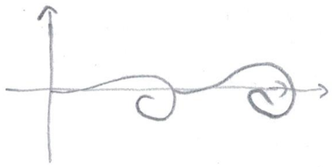
if $\rho_2 < \rho_1 \Rightarrow$ Bojancy

$\omega^2 < 0 \Rightarrow \omega$ has complex part $\Rightarrow$ exponential instability

- Kelvin-Helmotz instability

$\omega^2 < 0$





- Thermal instability

Instabilities

- Instability = perturbation of a system in initial equilibrium
- Perturbative approach $f(\bar{x},t) = f_0 + \delta f(\bar{x},t)$ $f = \rho, P, \bar{v}, \dots$

  ⇓ $\delta f \ll f_0$ small perturbation

  linearize hydro equations

  i.e. keep only 1° order terms ⇒ linear differential equations

- Decompose in plane waves (i.e. Fourier modes)

  ⇒ linear algebraic equation

$$\delta f = a \cdot e^{i(\bar{k}\bar{x} - \omega t)}$$

$$= a \underbrace{e^{i\bar{k}\bar{x}}}_{\text{spatial dependency}} \cdot e^{-i\omega t}$$

$e^{-i\omega t}$ ↳ time evolution, in general $\omega = b + ic \in \mathbb{C}$ $b,c = \mathbb{R}$

if $\omega = b \in \mathbb{R}$ $c = 0$ ⇒ $\cdot\, e^{-ibt}$ oscillations

if $\omega = b + ic$ $c > 0$ ⇒ $\cdot\, e^{ct}$ exponential growth

($c < 0$ ⇒ ↑ damped) ($\tau \equiv \frac{1}{c}$ time scale)

- Goal: get dispersion relation $\omega = \dots$

  if immaginary positive component ⇒ rise of instability

  → amplitudes grow exponentially

- Assumptions
  - No viscosity (ideal) : $\eta = 0 = \zeta$
  - Incompressible fluid : $\bar{\nabla}\rho = 0$ $\bar{\nabla}\bar{v} = 0$

- Shift to frame of unperturbed solution $\bar{v}_0$

$$\bar{v}(x,t) - \bar{v}_0 = \cancel{\bar{v}_0} + \delta\bar{v}(\bar{x},t) - \cancel{\bar{v}_0}$$

to simplify notation $\delta\bar{v} \to \bar{v}$!

- Evolution of vorticity (incompressible)

$$\boxed{\partial_t \Omega} = \bar{\nabla} \times (\bar{v} \times \bar{\Omega}) + \underbrace{\frac{1}{\rho} \bar{\nabla}\rho \times \bar{\nabla} P}_{=0 \text{ incompressible}}$$

remember, here $\bar{v} = \delta \bar{v}$

$$= \bar{v}(\bar{\nabla}\bar{\Omega}) - \bar{\Omega}\underbrace{(\bar{\nabla}\bar{v})}_{=0 \text{ incompressible}}$$

$$= \bar{v} \cdot \bar{\nabla}(\bar{\nabla} \times \bar{v})$$

$$= \boxed{0}$$

⇒ Vorticity can not grow (is constant)

⇒ neglect $\bar{\Omega}$, because on the contrary the other quantities grow exponentially in instabilities

- Bernoulli's law (gravity, incompressible)

$$\partial_t \psi + \frac{v^2}{2} + \tilde{h} + \phi = 0$$

- Neglect $\bar{v}^2 = (\delta \bar{v})^2$ because we keep 1° order terms only
- $\bar{v} = \bar{\nabla}\psi$ ⇒ $\underbrace{\bar{\nabla}\bar{v}}_{=0} = \boxed{\bar{\nabla}^2 \psi = 0}$ ① ψ obeys Laplace eq.
- $d\tilde{h} = \frac{dP}{\rho}$ ⇒ $\tilde{h} = \frac{P}{\rho}$ because $\rho = \text{const.}$ (incompressible)

⇒ $\partial_t \psi + \frac{P}{\rho} + \phi = 0$ ⇒ $\boxed{P = -\rho(\partial_t \psi + \phi)}$

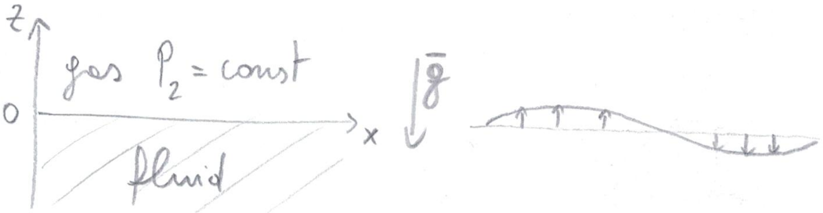
↑

Bernoulli's law allows to link the pressure to velocity (ψ), ρ and grav. potential (ϕ)

↑ assumed fixed

⇒ $P(\rho, \bar{v})$

Gravity waves



Surface "elevation"

$z = \ell(x, y, t)$

$v_z = \frac{d\ell}{dt} = \partial_t \ell + \bar{v}\bar{\nabla}\ell \simeq \partial_t \ell$

very small perturbation

- Bernoulli: $P_2 = -\rho(\partial_t \psi + g\ell)$ P_2 = external pressure on fluid

ϕ = grav. potentional energy

P_2, ρ = const.

$$\boxed{-g\ell} = \frac{P_2}{\rho} + \partial_t \psi = \partial_t\left(\frac{P_2}{\rho} t\right) + \partial_t \psi = \partial_t\left(\frac{P_2 t}{\rho} + \psi\right)$$

$$= \boxed{\partial_t \psi}$$

$\psi \rightarrow \psi - \frac{P_2 t}{\rho}$ $\forall t$

because we care only about $\bar{\nabla}\psi = \bar{v}$ and $\bar{\nabla}\left(\psi - \frac{P_2 t}{\rho}\right) = \bar{\nabla}\psi = \bar{v}$

to get all as a func. of ψ

- $\partial_t^2 \psi = -g\partial_t \ell = -g v_z = -g\partial_z \psi \Rightarrow \boxed{\partial_z \psi = -\frac{1}{g}\partial_t^2 \psi}$ (2)

- Solve (1) $\bar{\nabla}^2\psi = 0$ to get ψ

Laplace eq. ⇒ ansatz $\boxed{\psi = f(z)\cdot e^{i(kx-\omega t)}}$

(1) $\bar{\nabla}^2 \psi = f''(z)\cancel{e^{i(kx-\omega t)}} + f(z)(ik)^2 \cancel{e^{i(kx-\omega t)}} = 0$ $f'' = k^2 f$ $\boxed{f(z) = f_0 e^{(\pm kz)}}$

$$\boxed{\psi = f_0 e^{kz} \cdot e^{i(kx-\omega t)}}$$

us solution "+kz" because we are considering water, i.e $z < 0$

- Combine with (2) to get ω

$\partial_z \psi = k\cancel{\psi} \doteq -\frac{1}{g}\partial_t^2 \psi = -\frac{1}{g}(-i\omega)^2\cancel{\psi}$ $\omega^2 = kg$ $\boxed{\omega = \pm\sqrt{kg} \in \mathbb{R}}$

$k > 0$

⇒ oscillations, no instab..

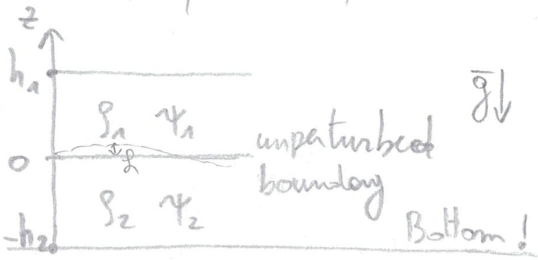
- Group velocity

$v_g = \partial_k \omega = \partial_k \sqrt{kg} = \frac{1}{2}\sqrt{\frac{g}{k}} \Rightarrow \underset{(\text{short }\lambda)}{k\uparrow} \Rightarrow \underset{(\text{slowlier})}{v_g\downarrow}$

Rayleigh-Taylor instability

(convective instability)

2 incompressible fluids



- introduce perturbation $\ell(x,y,t) = z$ ← of boundary
  look for evolution

- Fluids must obey: $\bar{\nabla}^2 \psi = 0$

$$P = -\rho(\partial_t \psi + \phi) \quad \text{at boundary } \phi = g\ell \qquad \text{(Bernoulli's)}$$

- At boundary: $(P_1 - P_2)\big|_{z=\ell} = 0$ (1) $\quad (\partial_z \psi_1 - \partial_z \psi_2)\big|_{z=\ell} = 0$ (2)

$$(1) \Rightarrow \; -\rho_1(\partial_t \psi_1 + g\underset{\sim}{\ell}) + \rho_2(\partial_t \psi_2 + g\underset{\sim}{\ell}) = -\rho_1 \partial_t \psi_1 + \rho_2 \partial_t \psi_2 - g\ell(\rho_1 - \rho_2) = 0$$

$$\ell = \frac{\rho_2 \partial_t \psi_2 - \rho_1 \partial_t \psi_1}{g(\rho_1 - \rho_2)} \rightarrow \underbrace{\partial_t \ell = v_z = \partial_z \psi_1 = \frac{\rho_2 \partial_t^2 \psi_2 - \rho_1 \partial_t^2 \psi_1}{g(\rho_1 - \rho_2)}}_{(3)} \quad \left(\text{solve for } \psi\right)$$

- Ansatz:
  $$\psi_1 = A_1 \cosh[K(z-h_1)] \exp[i(Kx - \omega t)]$$
  $$\psi_2 = A_2 \cosh[K(z+h_2)] \exp[i(Kx - \omega t)]$$
  $\left(\text{obey } \bar{\nabla}^2\psi = 0 \quad \partial_z \psi_1\big|_{z=h_1} = 0 = \partial_z \psi_2\big|_{z=h_2}\right)$

- Plug ansatz in (3)

$$A_1 \sinh[K(z-h_1)] \cdot K \cancel{\exp[-]} = \frac{-\omega^2 [A_2 \rho_2 \cosh[K(z+h_2)] - A_1 \rho_1 \cosh[K(z-h_1)]] \cancel{\exp(-)}}{g(\rho_1 - \rho_2)}$$

- Plug ansatz in (2): link $A_1 \leftrightarrow A_2$

$$A_1 K \sinh[K(z-h_1)] \cancel{\exp[-]} = A_2 K \sinh[K(z+h_2)] \cancel{\exp[-]}$$

- Approximate $z \simeq 0$ and combine the last 2 eq.

$$\mapsto \quad \boxed{\omega^2 = \frac{Kg(\rho_2 - \rho_1)}{\rho_2 \coth(Kh_1) + \rho_1 \coth(Kh_1)}} \quad \text{if } \rho_1 > \rho_2 \Rightarrow \underset{\text{complex}}{\omega^2 < 0}$$

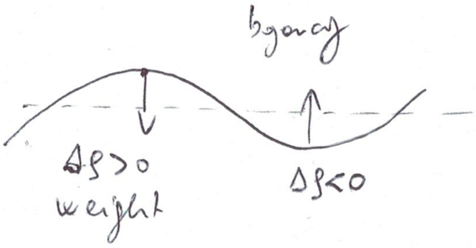
$\}> 0$

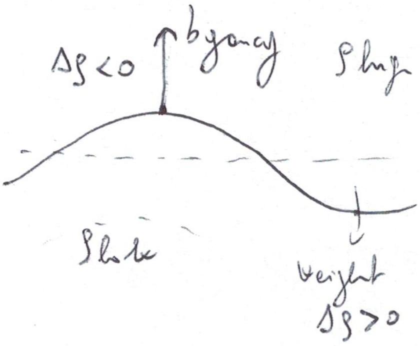
instability!

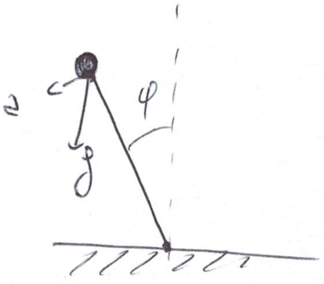
small perturbation lead to exponential evolution

(Analogy with pendulum)

Grav. waves v.s. **Rayleigh-Taylor**







$l\ddot{\varphi} = -g\varphi$

↓

$\varphi = \varphi_0 \exp(\pm i\omega t)$

$\omega = \sqrt{\frac{g}{l}} \in \mathbb{R}$

$(-i\omega)(-i\omega)$

$+ i \cdot i\omega^2 = -$

↓

$l\ddot{\varphi} = +g l$

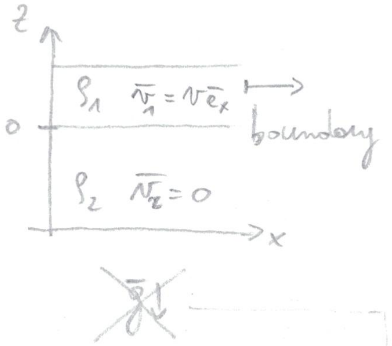
$\varphi = \varphi_0 \exp(\pm i\omega t)$

$\omega = i\sqrt{\frac{g}{l}} \in \mathbb{C}$ pure

Kelvin-Helmoltz instability

(Shearing instability)

- 2 incompressible fluids, no gravity



⇒ shear flow ⇒ excite wave-like perturbations

$$\delta\bar v = \delta\bar v_0 \exp[i(kx-\omega t)] \qquad \bar v = \bar v_0 + \delta\bar v$$

$$\mathcal{L} = \mathcal{L}_0 \exp[i(kx-\omega t)] \qquad z = \overset{=0}{z_0} + \mathcal{L}$$

$$\delta P = \delta P_0 f(z) \exp[i(kx-\omega t)] \qquad P = P_0 + \delta P$$

↳ ($\delta\bar v$, δP stand for small)

const!

(Perturbative approach)

- $$(\partial_t + \bar v \bar\nabla)\bar v + \frac{\bar\nabla P}{\rho} = 0$$ Euler eq.

const. (unperturb

$$\partial_t(\bar v_0 + \delta\bar v) + (\bar v_0 + \delta\bar v)\bar\nabla(\bar v_0 + \delta\bar v) + \frac{\bar\nabla(P_0 + \delta P)}{\rho} = 0 \leftarrow \boxed{\partial_t \bar v_0 = 0 \quad \bar\nabla \bar v_0 = 0 \quad \bar\nabla P_0 = 0}$$

$$\partial_t\delta\bar v + (\bar v_0 + \delta\bar v)\bar\nabla\delta\bar v + \frac{\bar\nabla\delta P}{\rho} = \partial_t\delta\bar v + \bar v_0\bar\nabla\delta\bar v + \frac{\bar\nabla\delta P}{\rho} = 0 \leftarrow \bar v = (v, 0, 0)^T$$

2° order

const

const

$$\boxed{\partial_t\delta\bar v + v\,\partial_x\delta\bar v + \frac{\bar\nabla\delta P}{\rho} = 0}$$ linearized Euler eq. ①

perturbation

- $\underline{\bar\nabla(\text{eq.}1)}$:

$$\partial_t \underset{\substack{=0\\ \text{incompress}}}{\bar\nabla\delta v} + \partial_x\delta\bar v \underset{=0}{\overset{\text{const}}{\bar\nabla v}} + v\,\partial_x \underset{\substack{=0\\ \text{incompress}}}{\bar\nabla\delta\bar v} + \frac{\bar\nabla^2\delta P}{\rho} = 0$$

$$\boxed{\bar\nabla^2\delta P = 0}$$ δP must satisfy Laplace eq. ⇒ ansatz $\delta P = \delta P_0 f(z)\, e^{i(kx-\omega t)}$

$$\bar\nabla^2(\cancel{\delta P_0} f(z)\exp[i(kx-\omega t)]) = 0 \qquad f''(z)\,\cancel{e[-]} - f(z)k^2\cancel{\exp[-]} = 0$$

$$\Rightarrow \boxed{f(z) = f_0\exp(-kz)} \leftarrow \text{for } z>0 \Rightarrow \boxed{\delta P = \delta P_0 f_0 \exp[i(kx-\omega t) - kz]}$$

$= -f_0 \exp(-kz)$ for $z<0$

②

- Combine, (2) ↷ (1), z component only.... ⟶

$$\Rightarrow \begin{cases} \partial_t \delta\bar{v} + v\,\partial_x \delta\bar{v} + \dfrac{\bar{\nabla}\delta P}{\rho} = 0 & (1) \\ \delta P = \delta P_0\, f_0 \exp[i(kx-\omega t) - kz] & (2) \end{cases}$$

- Combine, (2) ↷ (1) z component

$$(1)\quad \partial_t \delta v_z + v\,\underbrace{\cancel{\partial_x \delta v_z}}_{=0} + \frac{\partial_z \delta P}{\rho} = 0 \qquad \int_0^t \partial_t \delta v_z\, dt' = -\int_0^t \frac{\partial_z \delta P}{\rho}\, dt' = -\int_0^t \overset{(2)}{-\frac{k\delta P}{\rho}}\, dt$$

$$\delta v_z = +\frac{k}{\rho}\,\frac{\delta P}{i(kv-\omega)} \overset{!}{=} \frac{d\ell}{dt} = (\partial_t + v\,\partial_z)\ell = (-i\omega + v\,ik)\ell$$

$$\Rightarrow \begin{aligned} \delta P_1 &= -\frac{\rho_1 \ell}{k}(kv-\omega)^2 && z>0 \text{ (fluid 1)} \\ \delta P_2 &= \frac{\rho_2 \ell}{k}\,\omega^2 && z<0 \text{ (fluid 2)} \quad v=0 \end{aligned}$$

- At the interface layer $\delta P_1 = \delta P_2$

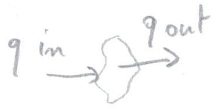
$$-\frac{\rho_1 \cancel{\ell}}{\cancel{k}}(kv-\omega)^2 = \frac{\rho_2 \cancel{\ell}}{\cancel{k}}\,\omega^2 \qquad \rho_2\omega^2 + \rho_1(kv-\omega)^2 = 0$$

$$(\rho_2+\rho_1)\omega^2 - (2\rho_1 kv)\omega + \rho_1 k^2 v^2 = 0 \quad \Rightarrow \quad \boxed{\omega_\pm = \frac{kv}{\rho_2+\rho_1}\left(\rho_1 \pm i\sqrt{\rho_1\rho_2}\right)}$$

if $\rho_2 \neq 0 \Rightarrow \omega \in \mathbb{C}$

$\Rightarrow$ instability

Thermal instability



- So far mechanical instabilities
- Now thermal instability : temperature perturbation

• System can gain heat : compression, radiation (in), gain of hot particles.

• " " loose " : expansion, " (out), loss " " "

• Quantify this with $\mathcal{L}(\rho,T)$ = cooling - heating $\left[\frac{J}{kg\,s}\right]$

↗ Cooling function

source of energy

• In T. equilibrium $\mathcal{L}(\rho,T)=0$

↳ This sets a well defined ρ, T relation
(see example ideal gas below)

▫ Example: mechanical equilibrium

⇒ System adapts to external pressure P_{ext}

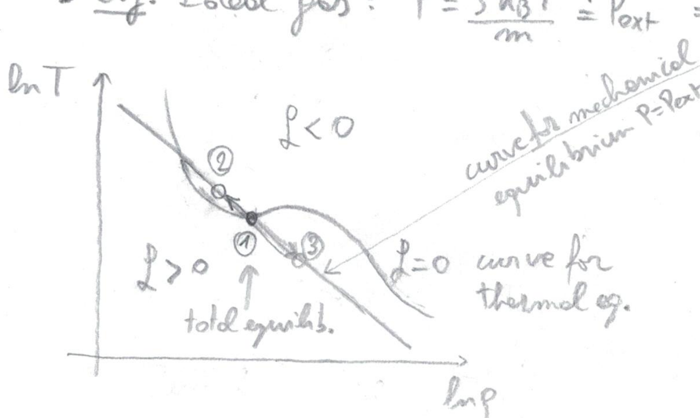
⇒ Eq. of state $\boxed{P=P(\rho,T)\overset{!}{=}P_{ext}}$

sets a ρ-T curve where you have equilibrium.

• If P_{ext} changes ⇒ new equilibrium along a new line

• Usually mechanical equilibrium is reached faster than thermal one

▫ e.g. Ideal gas : $P=\frac{\rho k_B T}{m}\overset{!}{=}P_{ext} \Rightarrow \boxed{T=\frac{P_{ext}\, m}{\rho k_B}}$ (Hyperbole)



• If $\mathcal{L}<0$: heating > cooling $[① \to ②]$

⇒ T↑ ⇒ expansion ⇒ $\rho\downarrow$

⇒ moves from ① to ②
even further away then T. equilibrium

⇒ T↑ even more... Thermal instability

- **If $\mathcal{L} > 0$:** heating < cooling   from ① to ③

  $\Rightarrow$ gain heat but cools more going back to T_0 equilib.

Model for thermal instability

$$\left[\begin{array}{ll} (1) & \partial_t \rho + \bar{\nabla}(\rho \bar{v}) = 0 \\ (2) & \partial_t \bar{v} + \cancel{\frac{\bar{\nabla} v^2}{2}}^{\circledast} - \cancel{\bar{v} \times (\bar{\nabla} \times \bar{v})}^{\circledast} = -\frac{\bar{\nabla} P}{\rho} \\ (3) & T\left[\partial_t s + \cancel{(\bar{v}\bar{\nabla}) s}_{\circledast}\right] = -\mathcal{L}(\rho, T) \end{array}\right.$$

← continuity

← Euler eq.

← Energy conserv. with source term $\mathcal{L}$

$s =$ specific entropy

net energy loss per unit mass

⊛ Move to frame of unperturbed flow ⇒ $\bar{v}_0 = 0$

- At equilib.: T_0, ρ_0, $\mathcal{L}(\rho_0, T_0) = 0$, s_0

- Linearize eq. (1) and (2): perturbation $\delta\rho$, δT, $\delta\bar{v}$, δs

$$(1) \quad \partial_t \delta\rho + \bar{\nabla}(\rho_0 \delta\bar{v}) = 0 \qquad \partial_t^2 \delta\rho + \bar{\nabla}(\rho_0 \partial_t \delta\bar{v}) = 0$$

$$(2) \quad \partial_t \delta\bar{v} = -\frac{\bar{\nabla} P}{\rho_0}$$

$$\Rightarrow \boxed{\partial_t^2 \delta\rho = \bar{\nabla}^2 \delta P} \quad (1\text{-}2)$$

- Linearize eq. (3) $\quad s(P, \rho)$

entropy: $\underline{s} = s_0 + \frac{\partial s}{\partial P}\delta P + \frac{\partial s}{\partial \rho}\delta\rho = s_0 + c_v \frac{\delta P}{P_0} - c_p \frac{\delta\rho}{\rho_0}$

heat capacities

$$\left(\frac{\partial s}{\partial P}\right)_\rho = \frac{c_v}{P} \qquad \left(\frac{\partial s}{\partial \rho}\right)_P = -\frac{c_p}{\rho}$$

(see polytropic eq. of state)

right side: $\underline{\mathcal{L}}(\rho, T) = \mathcal{L}_0 + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial T}\right)_\rho}_{\equiv c_v \mathcal{L}_v} \delta T_\rho + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \rho}\right)_P}_{\equiv c_p \mathcal{L}_p} \delta T_P$

- The contribution s_0 in (3) gives $\mathcal{L}_0$ by def of equilibrium ($s_0 \to \mathcal{L}_0$)

$$\Rightarrow \boxed{\cancel{-\mathcal{L}_0} + T_0 \partial_t\left(c_v \frac{\delta P}{P_0} - c_p \frac{\delta\rho}{\rho_0}\right) = -\left(\mathcal{L}_0 + c_v \mathcal{L}_v \delta T_\rho + c_p \mathcal{L}_p \delta T_P\right)} \quad (3b)$$

- Perturbed eq. of state (ideal gas): $P = \frac{\rho k_B T}{m} \to \delta P = \frac{k_B}{m}(\delta\rho + \delta T)$

$$\delta P = \underbrace{\frac{\rho_0 k_B T_0}{m}}_{P_0}\left(\frac{\delta T}{T_0} + \frac{\delta\rho}{\rho_0}\right) \Rightarrow$$

for $\rho = \text{const}$ $\quad \delta P = P_0 \frac{\delta T_\rho}{T_0} \quad \delta T_\rho = T_0 \frac{\delta P}{P_0}$

for $P = \text{const}$ $\quad 0 = P_0\left(\frac{\delta T}{T_0} + \frac{\delta\rho}{\rho_0}\right) \quad \delta T_P = \frac{-T_0}{P_0}\frac{\delta\rho}{\rho_0}$

- Plug in energy conservation (3b) together with $c_s^2 = \gamma \frac{P_0}{\rho_0}$, $c_p = \gamma c_v$ (Polytrop. eq. of sta

$$\partial_t(\delta P - c_s^2 \delta\rho) = c_s^2 \mathcal{L}_p \delta\rho - \mathcal{L}_v \delta P \quad (3c)$$

- ∇^2 (eq. 3c) $\quad \partial_t(\nabla^2 \delta P - c_s^2 \nabla^2 \delta\rho) = c_s^2 \mathcal{L}_p \nabla^2 \delta\rho - \mathcal{L}_v \nabla^2 \delta P$

eq. (1-2)

Plug $\partial_t^2 \delta\rho = \nabla^2 \delta P$

$$\Rightarrow \boxed{\partial_t(\partial_t^2 \delta\rho - c_s^2 \nabla^2 \delta\rho) = c_s^2 \mathcal{L}_p \nabla^2 \delta\rho - \mathcal{L}_v \partial_t^2 \delta\rho} \leftarrow$$ defines behaviour of ρ

- Decompose in plane waves: $\delta\rho = \hat{\delta\rho} \exp(i(kx - \omega t))$

Fourier mode

$$\Rightarrow \partial_t\left[-\omega^2 \hat{\delta\rho} + c_s^2 k^2 \hat{\delta\rho}\right] = c_s^2 \mathcal{L}_p (-k^2) \hat{\delta\rho} + \mathcal{L}_v \omega^2 \hat{\delta\rho} \qquad \hat{\delta\rho} = \int dx\, \delta\rho\, e^{-i(kx-\omega t)}$$

$$(-\omega^2 + c_s^2 k^2)\,\partial_t \int dx\, \rho\, e^{-i(kx-\omega t)} = (-c_s^2 k^2 \mathcal{L}_p + \mathcal{L}_v \omega^2) \int dx\, \rho\, e^{-i(kx-\omega t)}$$

$$\boxed{-i\omega(-\omega^2 + c_s^2 k^2) = -c_s^2 k^2 \mathcal{L}_p + \mathcal{L}_v \omega^2}$$ (difficult to solve, look for limits)

A — For $c_s^2 k^2 \gg \omega^2$ $\Rightarrow$ $\cancel{-}i\omega \cancel{c_s^2 k^2} \approx \cancel{-c_s^2 k^2} \mathcal{L}_p$ $\boxed{\omega \approx -i \mathcal{L}_p}$ $\quad \mathcal{L}_p \equiv \frac{1}{c_p} \frac{\delta \mathcal{L}}{\delta S}\Big|_T$

$$\delta\rho = \int \frac{d\omega}{2\pi} \hat{\delta\rho}\, e^{i(kx-\omega t)}$$ (back Fourier transform)

$$\approx \int \frac{d\omega}{2\pi} \hat{\delta\rho}\, e^{i(-i\mathcal{L}_p)} \propto \hat{\delta\rho} \exp(-\mathcal{L}_p t)$$

(Small scales)

- For $\mathcal{L}_p < 0$ (heating > cooling) $\Rightarrow$ instability!
- For $\mathcal{L}_p > 0$ $\Rightarrow$ exp. decay of δP $\Rightarrow$ no instability

B — For $c_s^2 k^2 \ll \omega^2$ $\Rightarrow$ $i\omega^3 = \mathcal{L}_v \omega^2$ $\boxed{\omega = -i \mathcal{L}_v}$ $\quad \mathcal{L}_v \equiv \frac{1}{c_v} \frac{\delta \mathcal{L}}{\delta T}\Big|_\rho$

(very large scales)

$\Downarrow$

instability for $\mathcal{L}_v < 0$

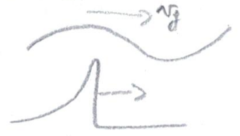
Shock waves

- Hydrodynamic equations: non-linear partial differential eq.

⇒ Can give rise to non-linearities, eg. shock waves (highly non-linear)

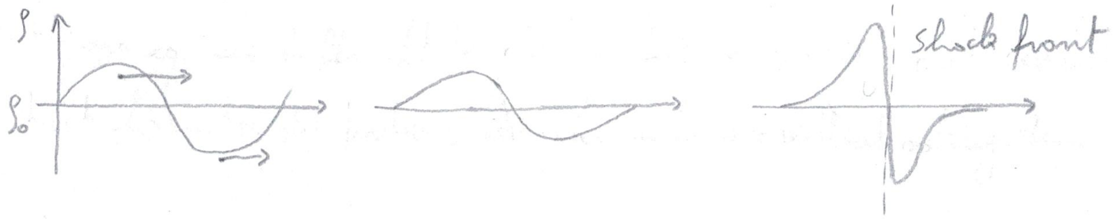
eg. sound waves are linear perturbations → smooth

e.g. sonic "boom" is a non-linear feature → sharp



- If the perturbation moves faster than the sound speed c_s

⇒ the system has no time to react to it ⇒ creation of a shock



- Hydro eq. are difficult to solve in full generality but characteristic features can be found without solving them

— o —

→ Method of characteristics : 2 characteristics $\frac{dx_\pm}{dt} = v \pm c_s$

2 invariants $R_\pm = v \pm \frac{2}{\gamma - 1} c_s$

⇒ in overdense regions ($\rho > \rho_0$) $c_s > c_{s0}$ ⇒ $v_- > 0$

$v_+ < 0$

→ Rankine-Hugoniot conditions :

· Across shocks, matter, energy, momentum flux is preserved

· Find jump in density and pressure between the 2 sides

$\rho_1 \to \rho_2 = r_+ \rho_1$ $r_+ \simeq 4$ for $\mathcal{M}_1 \gg 1$; $T_2 \to \infty$ $P_2 \to \infty$

→ Sedov solution : explosion (e.g. SN)

$R = \left(\frac{3Et^2}{4\pi\rho_1} \right)^{1/5}$

Method of characteristics : short

- Assumptions: no viscosity, polytropic eq. of state, 1D case $\Rightarrow \begin{matrix} \bar{\nabla} \to \partial_x \\ \bar{v} \to v \end{matrix}$

$\Rightarrow$ Hydro eq. $\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0 \\ (\partial_t + v\partial_x) v + \dfrac{\partial_x P}{\rho} = 0 \end{array} \right]$ ①

$\Rightarrow$ Eq. of state: $\left\{ P(\rho) = P_0 \left(\dfrac{\rho}{\rho_0}\right)^{\gamma} \right. \to \partial_x P = c_s^2 \partial_x \rho \Rightarrow \boxed{c_s^2 = \dfrac{P_0}{\rho_0} \gamma \left(\dfrac{\rho}{\rho_0}\right)^{\gamma-1}}$!

$$\underset{2c_s dc_s =}{d c_s^2} = c_s^2 (\gamma-1) \frac{d\rho}{\rho} \Rightarrow d\rho = \frac{2\rho}{\gamma-1} \frac{dc_s}{c_s}$$

$$\partial_x P = c_s^2 \partial_x \rho = c_s^2 \frac{2\rho}{\gamma-1} \frac{dc_s}{c_s}$$ ②

- Plug eq. (2) in (1) $\Rightarrow$ all as func of v, c_s

$$\underbrace{\begin{pmatrix} \frac{2}{\gamma-1} & 0 \\ 0 & 1 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} \partial_t c_s \\ \partial_t v \end{pmatrix}}_{\partial_t \bar{u}} + \underbrace{\begin{pmatrix} \frac{2v}{\gamma-1} & c_s \\ \frac{2c_s}{\gamma-1} & v \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \partial_x c_s \\ \partial_x v \end{pmatrix}}_{\partial_x \bar{u}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{T \partial_t \bar{u} + X \partial_x \bar{u} = \bar{0}} \quad \bar{u} \equiv \begin{pmatrix} c_s \\ v \end{pmatrix}$$ ③

quasi linear diff. eq.

$\Rightarrow$ Study with method of characteristics

i.e. find the directions in the (t,x) plane in which (3) can be written in terms of complete differentials: $\bar{l}^T dt = d\bar{s}^T T$, $\bar{l}^T dx = d\bar{s}^T X$

$$\det(dx\, T - dt\, X) = 0 \Rightarrow \boxed{dx_{\pm} = (v \pm c_s)\, dt}$$ ← these are the so called characteristics

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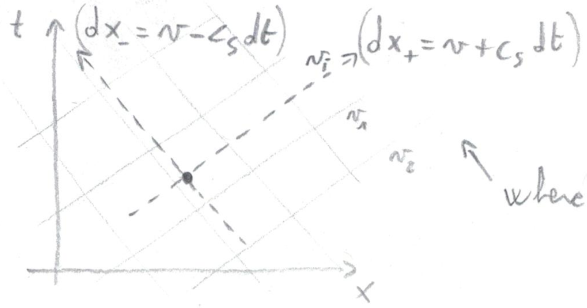
- Characteristics :



← they define a new coordinate system

↖ where they cross $\Rightarrow$ multivalued function

↳ "unphysical" i.e discontinuity (!)

$\boxed{dx_\pm = (v \pm c_s)dt}$ ← characteristics

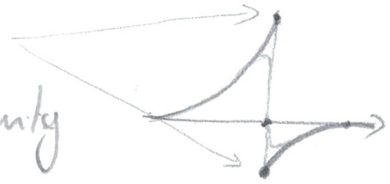
-> curves (lines in this case) along with (eq.2) is an ordinary diff. eq.

-> where they cross => get multi valued function

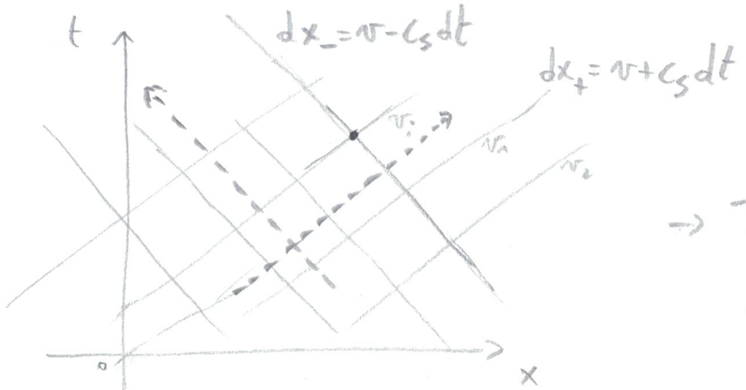
=> "unphysical", i.e. discontinuity

↑

this is the shock wave



-> curves along which the values of the solution run.



-> They defines a "new" coordinate system

- if we have $v(x)$, $c_s(x)$ at $t=0$ (for example)

=> we have a solution for every (x,t)

↳ along each characteristic, we have a Riemann invariant giving (v, c_s)

$$\boxed{v \pm \frac{2}{\gamma-1} c_s = R_\pm} = \text{const}$$

for R_+ if $v\uparrow \Rightarrow c_s\downarrow$

for R_- if $v\downarrow \Rightarrow c_s\uparrow$

Method of characteristics

- Study hydro eq. without solving them
- <u>Assume</u>: inviscous fluid,
  polytropic eq. of state
  1-D case => $\bar{\nabla} \to \delta_x \quad \bar{v} \to v$

$$\Rightarrow \left[\begin{array}{ll} \delta_t \rho + \delta_x(\rho v) = \delta_t \rho + v \delta_x \rho + \rho \delta_x v = 0 & \text{(continuity)} \\ (\delta_t + v\delta_x) v + \dfrac{\delta_x P}{\rho} = \delta_t v + v \delta_x v + \dfrac{\delta_x P}{\rho} = 0 & \text{(Euler eq.)} \end{array}\right] \quad (1)$$

$$\Rightarrow \left[\begin{array}{ll} P = P_0 \left(\dfrac{\rho}{\rho_0}\right)^{\gamma} & \text{(eq. of state)} \end{array}\right] \quad (2)$$

- <u>Combine (1) and (2) to study (1)</u>
  - $\boxed{\delta_x P} = \dfrac{P_0}{\rho_0^\gamma} \gamma \rho^{\gamma-1} \delta_x \rho = \underbrace{\gamma \dfrac{P_0}{\rho_0} \left(\dfrac{\rho}{\rho_0}\right)^{\gamma-1}}_{c_s^2} \cdot \delta_x \rho = \boxed{c_s^2 \delta_x \rho}$ with $c_s = \gamma \dfrac{P_0}{\rho_0}\left(\dfrac{\rho}{\rho_0}\right)^{\gamma-1}$
  - Express P, ρ derivatives in terms of c_s derivatives

$$dc_s^2 = 2 \cancel{c_s} dc_s = \frac{\gamma P_0}{\rho_0 \rho_0^{\gamma-1}} (\gamma-1) \rho^{\gamma-2} d\rho = \underbrace{\frac{\gamma P_0}{\rho_0}}_{} (\gamma-1) \frac{1}{\rho} \underbrace{\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}}_{} d\rho = \cancel{c_s^2} (\gamma-1) \frac{d\rho}{\rho}$$

$$\Rightarrow \boxed{d\rho = \frac{2\rho}{(\gamma-1)} \frac{dc_s}{c_s}}$$

$$\boxed{\delta_x P} = c_s^2 \delta_x \rho = \boxed{c_s \frac{2\rho}{(\gamma-1)} \frac{\delta_x c_s}{c_s}}$$

Plug in (1) and (2)

"reorganize in form of matrix" 3

$$\begin{cases} \frac{2\cancel{\rho}}{\gamma-1}\frac{\partial_t c_s}{c_s} + v\frac{2\cancel{\rho}}{\gamma-1}\frac{\partial_x c_s}{c_s} + \cancel{\rho}\,\partial_x v = 0 \\ \partial_t v + v\,\partial_x v + c_s\frac{2\cancel{\rho}}{\gamma-1}\frac{\partial_x c_s}{\cancel{\rho}} = 0 \end{cases} \qquad \begin{cases} 0\,\partial_t v + \frac{2}{(\gamma-1)\cancel{c_s}}\partial_t c_s + \frac{2v}{(\gamma-1)\cancel{c_s}}\partial_x c_s + c_s\partial_x v = 0 \\ 1\cdot\partial_t v + 0\cdot\partial_t c_s + \frac{2c_s}{\gamma-1}\partial_x c_s + v\,\partial_x v = 0 \end{cases}$$

$$\begin{pmatrix} \frac{2}{\gamma-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \partial_t c_s \\ \partial_t v \end{pmatrix} + \begin{pmatrix} \frac{2v}{(\gamma-1)} & c_s \\ \frac{2c_s}{\gamma-1} & v \end{pmatrix}\begin{pmatrix} \partial_x c_s \\ \partial_x v \end{pmatrix} = \bar{0} \qquad (2)\ \boxed{T\,\partial_t\bar{u} + X\,\partial_x\bar{u} = \bar{0}} \quad \boxed{\bar{u} \equiv \begin{pmatrix} c_s \\ v \end{pmatrix}}$$

$\left(\overset{!}{=} \bar{Z}\right.$ in general$\left.\right)$

- This is a quasi-linear diff. equation

$\Rightarrow$ Study with method of characteristics

i.e. find the directions in (t,x) plane in which (2) can be written in terms of complete differentials :

a) Find the $d\bar{\varsigma}$ such that $\boxed{\begin{matrix}\bar{L}^T dt = d\bar{\varsigma}^T T \\ \bar{L}^T dx = d\bar{\varsigma}^T X\end{matrix}}$ (3) for some $\bar{L} \in \mathbb{R}^2$ (4) $\Rightarrow$ Find $d\bar{\varsigma}$ and $\bar{L}$!

b) $d\bar{\varsigma}\cdot$(eq.2)

$$d\bar{\varsigma}^T\cdot(T\partial_t\bar{u} + X\partial_x\bar{u}) = \bar{L}^T dt\,\partial_t\bar{u} + \bar{L}^T dx\,\partial_x\bar{u} = \bar{L}^T(dt\,\partial_t\bar{u} + dx\,\partial_x\bar{u}) = \boxed{\bar{L}^T d\bar{u} = d\bar{\varsigma}}$$

c) $dx\cdot$(eq.3) : $\bar{L}^T dx\,dt = dx\,d\bar{\varsigma}^T T$

$dt\cdot$(eq.3) : $\bar{L}^T dt\,dx = dt\,d\bar{\varsigma}^T X$

$$dx\,d\bar{\varsigma}^T T = dt\,d\bar{\varsigma}^T X \qquad \boxed{d\bar{\varsigma}(dx\,T - dt\,X) = 0}$$

d) non trivial solution if $\det(dx\,T - dt\,X) = 0$

$$\det\begin{pmatrix} \frac{2(dx - v\,dt)}{\gamma-1} & -c_s dt \\ -\frac{2c_s dt}{\gamma-1} & dx - v\,dt \end{pmatrix} = \frac{2}{\gamma-1}\det\begin{pmatrix} dx - v\,dt & -c_s dt \\ c_s dt & dx - v\,dt \end{pmatrix} = 0$$

$$dx^2 - 2v\,dx\,dt + v^2 dt^2 - c_s^2 dt^2 = dx^2 - 2v\,dx\,dt + (v^2 - c_s^2)dt^2 = 0$$

$$\Rightarrow \boxed{dx_\pm = (v \pm c_s)dt} \leftarrow \text{these are the so called "characteristics"}$$

5

e) Find ds

The $d\bar{s}$ are the eigenvectors with eigenvalue $\bar{0}$

$$\Rightarrow (ds_1, ds_2)\begin{pmatrix} \frac{2(dx-v\,dt)}{\gamma-1} & -c_s dt \\ -\frac{2c_s dt}{\gamma-1} & dx - v\,dt \end{pmatrix} = (0,0) \Rightarrow -c_s dt\, ds_1 + \overset{\text{Plug characteristics}}{\overset{\Downarrow}{(dx - v\,dt)}} ds_2 = 0$$

$$-c_s dt\, ds_1 + \left[(\cancel{v} \pm c_s)dt - \cancel{v\,dt}\right] ds_2 = 0 \quad -\cancel{c_s dt}\, ds_1 \pm \cancel{c_s dt}\, ds_2 = 0 \Rightarrow \boxed{ds_1 = \mp ds_2}$$

f) Find $\bar{L}$ by pluggin $ds_1 = \pm ds_2$ in (eq.3) or (eq.4)

$$\underset{\substack{\uparrow \\ \bar{L} = (L_1, L_2)^T}}{\bar{L}} dt = \overset{\downarrow}{d\bar{s}}^T \overset{\downarrow}{X} = (ds_1, \pm ds_2)\begin{pmatrix} \frac{2v}{\gamma-1} & c_s \\ \frac{2c_s}{\gamma-1} & v \end{pmatrix} \Rightarrow \begin{aligned} L_1 dx &= \frac{2(v \pm c_s)}{\gamma-1} ds_1 \\ L_2 dx &= (c_s \pm v) ds_2 \\ &\overset{!}{=} (c_s \mp v) ds_1 \end{aligned}$$

→ $\bar{L}$ and $\bar{s}$ scales with the same const.

⇒ Only the ratio L_1/L_2 matters

$$\frac{L_1}{L_2} = \pm \frac{2}{\gamma-1} \quad , \text{Set } L_2 = 1 \text{ for convenience} \Rightarrow \boxed{\bar{L} = \left(\frac{\pm 2}{\gamma-1}, 1\right)^T}$$

g) Finally, express initial eq. explicitely

$$\overset{\searrow}{\bar{L}}\, d\bar{u} = \overset{\searrow}{d\bar{s}}\, \bar{Z} \quad \text{where } \bar{Z} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in our case} \quad , \quad \bar{u} \equiv \begin{pmatrix} c_s \\ v \end{pmatrix}$$

$$\left(\frac{\pm 2}{\gamma-1}, 1\right)\begin{pmatrix} dc_s \\ dv \end{pmatrix} = \frac{\pm 2\, dc_s}{\gamma-1} + dv = 0 \qquad dv \pm \frac{2}{\gamma-1} dc_s = d\left(v \pm \frac{2}{\gamma-1} c_s\right) = 0$$

$$\Rightarrow \boxed{v \pm \frac{2}{\gamma-1} c_s = R_\pm} = \text{const} \qquad \underline{\text{Rieman invariants}} \quad \left[\frac{m}{s}\right]$$

↑

- these are constants along the characteristics curves
- For R_+ if $v\uparrow \Rightarrow c_s\downarrow$
- For R_- if $v\downarrow \Rightarrow c_s\downarrow$

What does it mean?

2 characteristics : $\frac{dx_\pm}{dt} = v \pm c_s$ along these lines....

2 invariants : $R_\pm = v \pm \frac{2}{\gamma - 1} c_s$ ....these values are constant.

- For $x<0$

  Points within the perturbations are connectet to this non perturbed one

  by $\frac{dx_-}{dt} = v - c_s = -c_{s0}$ $\quad R_- = v - \frac{2c_s}{\gamma - 1}$

  it is an invariant => at v=0 $\quad R_- = 0 - \frac{2c_{s0}}{\gamma - 1}$ $\Rightarrow \boxed{v_- = \frac{2(c_s - c_{s0})}{\gamma - 1}}$

- For $x>0$

  Points at boundary of perturb.: fluid is unperturbed

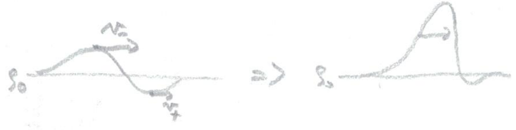
  by(t) $\frac{dx_+}{dt} = v + c_s = c_{s0}$ $\quad R_+ = v + \frac{2c_s}{\gamma - 1}$

  $R_+ = 0 + \frac{2c_{s0}}{\gamma - 1}$ $\Rightarrow v_+ = \frac{2(c_{s0} - c_s)}{\gamma - 1}$

- In overdense regions $(\rho > \rho_0)$ $c_s > c_{s0}$ $\Rightarrow v_- > 0$

  $\Rightarrow v_+ < v_-$ $\Rightarrow$ Discontinuity

this is the shock :



— With this we do not have a solution but...

— .... we have seen that there can be shocks in (any) fluid (!)

— .... we have seen why shocks form!

7

Rankine-Hugoniot conditions

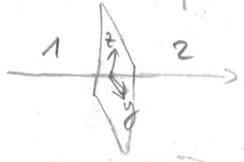
- In shocks, flow variables are discontinuous
- But across the two sides of the shocks must be conserved:

matter current $\rho \bar{v}$ (continuity)

energy current $\bar{q} \equiv \left(\frac{v^2}{2} + \tilde{h}\right)\rho \bar{v}$

momentum current $T^{ij} = \rho v^i v^j + P\delta^{ij}$

shocksurface (where the discontinuity occours)



flux (1), (2) and (3) across the shock

$$\partial_t \rho_1 \overset{!}{=} \partial_t \rho_2 \Rightarrow \rho_1 v_1 = \rho_2 v_2$$

$$\partial_t \rho_1 = -\partial_x(\rho_1 v_1) \qquad \partial_t \rho_2 = -\partial_x(\rho_2 v_2)$$

$$\Rightarrow \begin{cases} (1) & \rho_1 v_1 = \rho_2 v_2 \\ (2) & \left(\frac{v_1^2}{2} + \tilde{h}_1\right)\rho_1 v_1 = \left(\frac{v_2^2}{2} + \tilde{h}_2\right)\rho_2 v_2 \\ (3) & \rho_1 v_1^2 + P_1 = \rho_2 v_2^2 + P_2 \end{cases}$$

- Assume polytropic eq. of state $\Rightarrow$ sound speed: $c_s^2 = \gamma \frac{P}{\rho}$

$\Rightarrow$ specific enthalpy: $\tilde{h} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} = \frac{c_s^2}{\gamma - 1}$

$\Rightarrow$ Mach number $\mathcal{M}^2 = \frac{v^2}{c_s^2}$

- Express quantities on side (2) with ratios of side (1)

$$\rho_2 \equiv r\rho_1 \qquad P_2 = q P_1$$

$$\Rightarrow \boxed{\tilde{h}_2} = \frac{\gamma}{\gamma - 1}\frac{P_2}{\rho_2}\cdot\frac{P_1}{P_1}\cdot\frac{\rho_1}{\rho_1} = \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1}\left(\frac{P_2}{P_1}\right)\left(\frac{\rho_1}{\rho_2}\right) = \boxed{\tilde{h}_1 \cdot \frac{q}{r}}$$

$$(1) \quad v_2 = \frac{\rho_1}{\rho_2} v_1 = \frac{v_1}{r}$$

$$(2) \quad \left(\frac{\mathcal{M}_1^2 \cancel{c_{s1}^2}}{2} + \frac{\cancel{c_{s1}^2}}{\gamma - 1}\right)\cancel{\rho_1 v_1} = \left(\frac{\mathcal{M}_1^2 \cancel{c_{s1}^2}}{2r^2} + \frac{\cancel{c_{s1}^2}}{\gamma - 1}\frac{q}{r}\right)(\cancel{\rho_1 r})\left(\frac{\cancel{v_1}}{\cancel{r}}\right) \Rightarrow$$

$$\boxed{\frac{\mathcal{M}_1^2}{2} + \frac{1}{\gamma - 1} = \frac{\mathcal{M}_1^2}{2r^2} + \frac{1}{\gamma - 1}\frac{q}{r}} \quad (4)$$

$$(3) \quad \cancel{\rho_1} \mathcal{M}_1^2 \cancel{c_{s1}^2} + \frac{\cancel{\rho_1 c_s^2}}{\gamma} = (\cancel{\rho_1} r)\frac{\mathcal{M}_1^2 \cancel{c_{s1}^2}}{r^{\cancel{2}}} + \frac{(\cancel{\rho_1 r})}{\gamma}\cancel{c_{s1}^2}\frac{q}{\cancel{r}} \Rightarrow \boxed{\mathcal{M}_1^2 + \frac{1}{\gamma} = \frac{\mathcal{M}_1^2}{r} + \frac{q}{\gamma}}$$

- Combine equations (2) and (3)

$r(\gamma-1)\times(\text{eq.2})$

$\gamma\times(\text{eq.3})$

eliminate q by subtracting them ⊕

$$\Rightarrow\ r^2\underbrace{\left[\mathcal{M}_1^2(\gamma-1)+2\right]}_{\text{const}_1}-2r\underbrace{\left[\mathcal{M}_1^2\gamma+1\right]}_{\text{const}_2}+\mathcal{M}_1^2\underbrace{\left[\gamma+1\right]}_{\text{const}_3}=0$$

Rankine-Hugoniot conditions

2 solutions: $r_-=1 \Rightarrow (\rho_1=\rho_2)$ i.e. trivial (no jump)

$$\boxed{r_+=\frac{\mathcal{M}_1^2(\gamma+1)}{\mathcal{M}_1^2(\gamma-1)+2}} \Rightarrow (\rho_2=r_+\rho_1)$$ ← density jump

Plug r_+ in (eq.4) $\Rightarrow$ $$\boxed{q=\frac{2\gamma\mathcal{M}_1^2-\gamma+1}{\gamma+1}} \Rightarrow (P_2=q\,P_1)$$ ← pressure jump

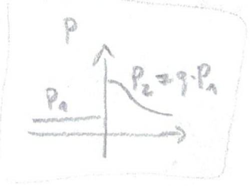
- Assume ideal gas $(PV=NK_BT) \Rightarrow P=\frac{\rho K_BT}{m}$

$$\frac{P_1}{P_2}=\frac{\rho_1K_BT_1}{\rho_2K_BT_2} \Rightarrow \boxed{\frac{T_2}{T_1}}=\frac{P_2}{\rho_2}\frac{\rho_1}{P_1}=\boxed{\frac{q}{r}}$$ ← temperature jump

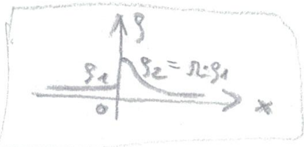
$\Rightarrow$ All quantities $r_+, q, \frac{T_2}{T_1}$ as a function of the $\mathcal{M}$ number

□ In practice: $$r_+=\frac{\gamma+1}{(\gamma-1)+2\mathcal{M}_1^{-2}}=\frac{\gamma+1}{\gamma+1\left(-2+2\mathcal{M}_1^{-2}\right)}=\frac{\gamma+1}{\gamma+1+2(\mathcal{M}_1^{-2}-1)}$$

Pressure increases



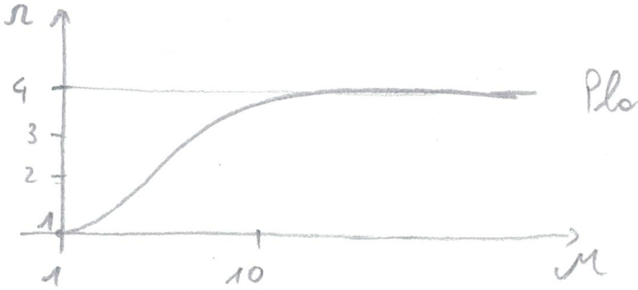
- if $\mathcal{M}_1>1 \Rightarrow r_+>1 \Rightarrow$



ρ increases

$$q=\frac{2\gamma\mathcal{M}_1^2-\gamma+1}{\gamma+1}>\frac{2\gamma1-\gamma+1}{\gamma+1}=\frac{\gamma+1}{\gamma+1}=1 \Rightarrow P_2>P_1$$

- if $\mathcal{M}_1\gg1 \Rightarrow \boxed{r_+=\frac{\gamma+1}{\gamma-1}=4}$ for ideal mono atomic non relat gas ($\gamma=5/3$) ; $T_2\to\infty$ $P_2\to\infty$

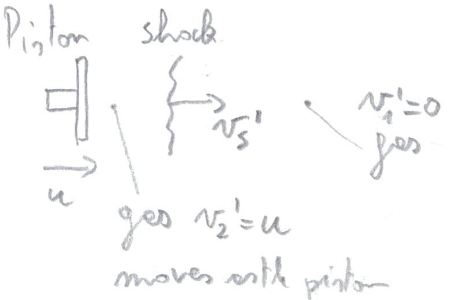


Plateau (this is the max ρ increase that you can have from a shock)

- $\boxed{\text{Velocity propagation of a shock}}$ (i.e. propagation of a non-linear wave)

- Piston moves in pipe with velocity u = const
- istantaneous acceleration ⇒ formation of shock in front of it

→ the shock moves in front of the pyston



in laboratory frame

- Transform to shock frame ⇒ $v_1 = v_1' - v_s$ $v_1' = 0 \Rightarrow v_1 = -v_s$

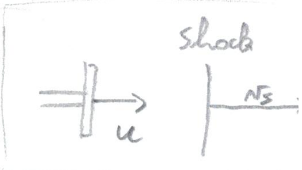
$v_2 = v_2' - v_s = u - v_s$

$$\Rightarrow v_2 - v_1 = \boxed{u} = v_1\left(\frac{v_2}{v_1} - 1\right) = v_1\left(\frac{1}{r} - 1\right) = -v_s\left(\frac{1}{r} - 1\right) \Rightarrow \boxed{v_s = \frac{r u}{r-1}}$$

- Strong shock ($M \gg 1$) : $r = \frac{\gamma+1}{\gamma-1}$ ⇒ $\boxed{v_s = \frac{\gamma+1}{2} u}$

e.g. ideal, mono-atomic → $= 4$ ⇒ $v_s = \frac{4}{3} u$

$\gamma = \frac{5}{3}$



- We do not always have a piston...

⇒ Express v_s as a function of r

⇒ Take one of the Rankine-Hougonot conditions

$$r_+ = \frac{\mathcal{M}_1^2(\gamma+1)}{\mathcal{M}_1^2(\gamma-1)+2} \rightarrow \mathcal{M}_1^2 = \frac{2r}{(\gamma+1)-(\gamma-1)r}$$

↙

$$\boxed{v_s} = -v_1 = |\mathcal{M}_1| c_{s1}$$

↖

OR

$$q = \frac{2\gamma \mathcal{M}_1^2 - \gamma + 1}{\gamma+1} \rightarrow \mathcal{M}_1^2 = \frac{(\gamma+1)q+(\gamma-1)}{2\gamma}$$

Plasma : summary

- Charged particles ⇒ e^-, Ze treated as a single fluid

↓

ideal p.: tight interactions: collisions + E.M.

- Charge shielding: - displacement of charges due to test charge $q(+)$

- thermal motion counteract this displacement
- Use Poisson eq. to find interaction potential

$$\phi(r) = \frac{q}{r} \exp(-K_D \sqrt{Z+1} \cdot r)$$ (Yukawa potential type

(plasma parameter : λ_D)

$$\lambda_D \equiv K_D^{-1} = \sqrt{\frac{K_B T}{4\pi \tilde{n} e}}$$ Debye length

exponential cut-off = screaming

- Plasma frequency: (time scale)$^{-1}$ necessary for the plasma to readjust itself after it has been perturbed

(Plasma parameter ω_p) $$t_D = \frac{\lambda_D}{\sqrt{\langle v^2 \rangle}} \qquad \omega_p \equiv \frac{1}{t_D} = \sqrt{\frac{4\pi \tilde{n}_e e^2}{m_e}}$$

average thermal velocity of particles in plasma

Plasma physics

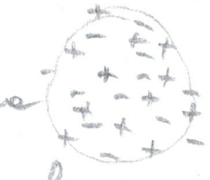
scale of system ↘ average part ↓ separat

$L \gg \lambda$

on average ensembles

$\# \oplus = \# \ominus$

- Gas: collection of neutral particles
- Plasma: collection of charged particles
- E.g. totally/partially ionized ⇒ macroscopically neutral



- It is not microscopically neutral only when there are mechanisms to keep ⊕ and ⊖ separated

e.g. Electric field

⇒ Plasma = single fluid (not 1⊕ fluid + 1⊖ fluid)

if: [short range collisions (like gas) $\lambda \ll L$] + [short range (coulomb) interaction random, fast (!)]

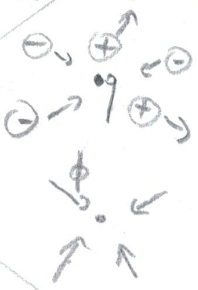
↓ possible even if there is charge shielding.

— ∘ —

~~Charge shielding~~

equilibrium ↓

- ~~Plasma of e^- + Z ions, neutral ⇒ $e\bar{n}_e = Ze\bar{n}_i$~~
- ~~Place charge q (e.g. ⊕) somewhere ⇒~~



~~local change of charge~~

~~⇒ perturbe charge density (unbalance)~~

~~⇒ electric potential $\phi \neq 0$~~

- ~~Thermal motion counteract this~~

~~⇒ Particles minimizes their potential energy $-e\phi$ $Ze\phi$~~

~~⇒ equilibrium density $\bar{n}_e$ $\bar{n}_i$ → $n_i = \frac{\bar{n}_e}{Z} \exp$~~

Charge shielding – Debye length

- $-e$ Ze $\rightarrow$ $-e n_e$ $Ze n_i$ neutral $\Rightarrow$ $\boxed{Z\cancel{e} n_i = \cancel{e} n_e}$ $\boxed{n_i = \frac{n_e}{Z}}$

  assumption: 1 specie only!

- Test charge in plasma q (e.g. positive) $\Rightarrow$

  $n_e \uparrow$ $n_i \downarrow$ $\Downarrow$ $\phi \neq 0$ electric potential

  $\Rightarrow$ change in local densities

  $\Rightarrow$ net electric potential

- Thermal motions counteract this displacement (randomize)

  $\Rightarrow$ particles rearrange such to minimize potential energies: $Ze\phi$, $-e\phi$

  $\Rightarrow$ density corrected by Boltzmann factor (at equilibrium)

$$n_i = \left(\frac{\tilde{n}_e}{Z}\right) \exp\left(\frac{Ze\phi}{k_BT}\right) \qquad n_e = \tilde{n}_e \exp\left(-\frac{e\phi}{k_BT}\right)$$

$\tilde{n}_i, \tilde{n}_e$ unperturbed density

- ϕ from Poisson eq. and $\phi \to 0$ at $|\vec{x}| = \infty$

$$\vec{\nabla}^2 \phi = -4\pi\left(Ze\, n_i(\vec{x}) - e\, n_e(\vec{x}) + q\,\delta(\vec{x})\right)$$

origin of particle $\vec{x}_1 = \vec{0}$

$$= -4\pi\, \tilde{n}_e e \left[\frac{\cancel{Z}}{\cancel{Z}} \exp\left(\frac{Ze\phi}{k_BT}\right) - \exp\left(-\frac{e\phi}{k_BT}\right)\right] - 4\pi q\,\delta(\vec{x})$$

- Assume $Ze\phi \ll k_BT$, $e\phi \ll k_BT$ i.e. plasma not too cold, q not too large

$$\simeq -4\pi\, \tilde{n}_e e \left(\frac{Ze\phi}{k_BT} + \frac{e\phi}{k_BT}\right) - 4\pi q\,\delta(\vec{x})$$

$$= -4\pi \frac{\tilde{n}_e e^2}{k_BT} \phi\, (Z+1) - 4\pi q\,\delta(\vec{x})$$

$$k_D^2 \equiv 4\pi \frac{\tilde{n}_e e^2}{k_BT}$$ Debye wave number

- Solve in Fourier space: $-k^2 \underset{\sim}{\hat{\phi}} = -k_D^2 (Z+1) \underset{\sim}{\hat{\phi}} - 4\pi q$ $\left(\hat{\delta} = 1\right)$

  $(ik)\cdot(ik)$

$$\hat{\phi}(k) = \frac{+4\pi q}{(Z+1)k_D^2 + k^2}$$

$\int dx^2 \delta(x) e^{i\vec{k}\vec{x}} = 1$

$=1$ $x=0$

- Back to real space (radial dependency only ⟷ not $\vec{k}$ but k)
  ↳ isotropy ⇒ polar coordinates

$$\phi(r) = 2\pi \int_0^\infty \frac{k^2 dk}{(2\pi)^2} \int_{-1}^{1} d(\cos\theta)\, \hat{\phi}(k)\, e^{ikr\cos\theta} = 4\pi \int_0^\infty \frac{k^2 dk}{(2\pi)^2}\, \hat{\phi}\, \frac{\sin(kr)}{kr}$$

$$= +\frac{q}{r} \exp\left(-k_D \sqrt{z+1} \cdot r\right)$$

$$\boxed{\lambda_D \equiv k_D^{-1} = \sqrt{\frac{k_B T}{4\pi \tilde{n} e^2}}}$$ Debye length (if T↑ ⇒ λ_D↑)

⇒ from Coulomb $\xrightarrow{\text{to}}$ Yukawa potential

screening given by thermal motion

Note: - no dependency on test charge q
⇒ intrinsic property of the plasma

Ideal plasma → if many particles within $V = \lambda_D^3$

→ i.e. case in which only weak electric interactions have small effects on thermal motions

↓

- To see it, compare Electric potential with average Kinetic energy of one ion

- $U = \frac{Ze^2}{\tilde{r}}$ $\quad K = \frac{3}{2} K_B T$ (in thermal equilibrium) (average k!)

and

- $\frac{1}{\frac{4\pi}{3}\tilde{r}^3} \approx \tilde{n}_e \Rightarrow \tilde{r} \simeq \left(\frac{3}{4\pi \tilde{n}_e}\right)^{1/3}$ (Typical scale for U)

average separation between particles

$$\Rightarrow \frac{U}{K} = \frac{Ze^2}{\tilde{r}} \cdot \frac{2}{3K_BT} = \frac{Ze^2 \tilde{r}^2}{\tilde{r}^3} \frac{2}{3K_BT} = \frac{Ze^2 \tilde{r}^2 4\pi \hat{n}_e}{3} \cdot \frac{2}{3K_BT} = \frac{2Z}{9} \frac{\tilde{r}^2}{\lambda_D^2}$$

$$\frac{U}{K} << 1 \quad \text{if} \quad \boxed{\tilde{r}^2 \ll \lambda_D^2}$$

! as already mentioned

$\boxed{\text{Plasma frequency}}$

- Average thermal velocity (equilibrium T)

$\langle v^2 \rangle = \frac{k_B T}{m_e}$ ⇒ Typical time scale on Debye scale (ideal)

$$t_D \equiv \frac{\lambda_D}{\sqrt{\langle v^2 \rangle}} = \sqrt{\frac{\cancel{k_B T}}{4\pi \tilde{n}_e e^2} \frac{m_e}{\cancel{k_B T}}} = \sqrt{\frac{m_e}{4\pi \tilde{n}_e e^2}}$$

$\downarrow$

<u>Plasma frequency</u> > $\omega_p \equiv \frac{1}{t_D} = \sqrt{\frac{4\pi \tilde{n}_e e^2}{m_e}} = 5{,}6 \cdot 10^4 \, Hz \left(\frac{\tilde{n}_e}{cm^{-3}} \right)^{1/2}$ (if $\tilde{n}_e \uparrow$ $\omega_p \uparrow$)

- t_D: Time scale under which the plasma has no time to readjust itself after an external electric field has applyed to it

⇒ use ω_p because usually the perturbing E is from an E.M. wave

$$|\!\Rightarrow \underline{\text{parameters of a plasma}} \begin{array}{l} \nearrow \lambda_D \\ \searrow \omega_p \end{array}$$

Electromagnetism in media: summary

- E.M. waves in plasma act on its particles (wave → particles)
- Backreaction of particles to the E.M. wave (wave ← particles)

⇒ Include this in Maxwell's equations

↳ they are linear ⇒ $\bar{D} = \epsilon \bar{E}$ $\quad$ (dielectric/magnetic permeability $\epsilon, \mu\,(\bar{x}, t)$ tensors

(dielectric/magnetic displacements) → (auxiliary fields) $\quad \bar{H} = \mu \bar{B}$

- Here we assume $\mu = 1 \Rightarrow \bar{H} = \bar{B}$ for simplicity (particles have $\bar{m} = 0$) magnetic dipole $\tau = \bar{m} \times \bar{B}$
- Incoming E.M. ⇒ polarization charge $\rho_{pol} = -\bar{\nabla}\bar{P}$ superposition

  " current $\bar{j}_{pol} = \frac{\delta \bar{P}}{\delta t}$ from continuity eq.

⇒ Modify Gauss law: $\bar{\nabla}\bar{E} = 4\pi(\rho + \rho_{pol}) \Rightarrow \bar{\nabla}\bar{D} = 4\pi\rho \quad \bar{D} \equiv \bar{E} + 4\pi\bar{P}$

Maxwell's eq.:

" Ampere law: $\bar{\nabla} \times \bar{B} = \frac{1}{c}\frac{\delta \bar{E}}{\delta t} + \frac{4\pi}{c}(\bar{j} + \bar{j}_{pol}) = \frac{1}{c}\frac{\delta \bar{D}}{\delta t} + \frac{4\pi}{c}\bar{j}$

Induction eq.: $\frac{\delta \bar{B}}{\delta t} = -c\bar{\nabla} \times \bar{E}$, $\quad \bar{\nabla}\bar{D} = 0$, $\quad \bar{\nabla}\bar{B} = 0$

- Other importan relation $J = \sigma \bar{E}$ $\quad \sigma$ = conductivity
- Dielectric tensor $\underline{\epsilon}$: $\hat{D}^i(\omega, k) = \underbrace{\hat{\epsilon}^i_j(\omega, k)\,\hat{E}^j(\omega, k)}$ in Fourier space

  convolution ⇒ superposition of E

  ⇒ $\hat{\epsilon}$ "weights the superposition
- Decompose $\hat{\epsilon}$: $\quad \hat{\epsilon}_{ij} = \hat{\epsilon}_{\parallel}\Pi^{\parallel}_{ij} + \hat{\epsilon}_{\perp}\Pi^{\perp}_{ij}$

  $\Pi^{\parallel}_{ij} = \bar{k}_i \bar{k}_j$, $\quad \Pi^{\perp}_{ij} = (\delta_{ij} - \Pi^{\parallel}_{ij})$ projectors parallel/orthogonal to $\bar{k}$ incoming wave
- From Maxwell eq: dispersion relation: $\det\left[\underset{①}{\left(1 - \frac{\omega^2}{c^2k^2}\hat{\epsilon}\right)\Pi^{\perp}_{ij}} - \underset{②}{\frac{\omega^2}{c^2k^2}\hat{\epsilon}_{\parallel}\Pi^{\parallel}_{ij}}\right] = 0$

(1): transversal (2): longitudinal waves

$\boxed{\text{Electromagnetism in media}}$ (Maxwell's equations in medium)

- Medium in general, not only plasmas (e.g. crystal)
- Maxwell's eq. in vacuum not valid (Homogeneous eq. are not changed)
- Plasma reacts to E.M. wave altering them
  (↳ charges/currents)

⇒ Inhomogeneous eq. affected by charges and currents in medium

- In vacuum, fields are defined by forces on test charges ($\bar{E}$) & test current loops ($\bar{B}$)
  (negligible on system (idealization))

- In medium, the force on charged particles is given by $\bar{D}$ dielectric displacement not $\bar{E}$

  test current loop experience magnetic force by $\bar{H}$ magnetic field not $\bar{B}$ " "

$\boxed{\text{Construct } \bar{D} \text{ and } \bar{H}}$

- Maxwell eq. are linear ⇒ natural to assume $\bar{D}, \bar{H}$ linearly related to $\bar{E}, \bar{B}$

$$\begin{bmatrix} \bar{D} = \epsilon \bar{E} & \text{dielectricity } \epsilon \\ \bar{H} = \mu \bar{B} & \text{magnetic permeability } \mu \end{bmatrix} \quad (\epsilon = 1 = \mu \text{ Gauss system})$$

- In general: $\epsilon(\bar{x}, t) \in \mathbb{R}$
  $\mu(x, t) \in \mathbb{R}$
  (if medium isotropic)

  or Tensors if medium is non isotropic
  (e.g. principal axis of crystal
  mag. field lines in plasma)
  ⇓
  direction of $\bar{D}, \bar{H}$ could be $\neq \bar{E}, \bar{B}$

Particles in a plasma might have a magnetic momentum

- Magnetic moment : defined as a vector giving the torque (τ) excerted by an external $\bar{B}$ on particles

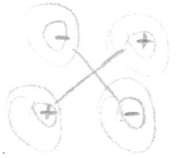
$$\bar{\tau} \equiv \underline{\bar{m}} \times \bar{B}$$

(decomposition in)

- Magnetic dipole = magnetic north and south very close one to each other

"a small magnet"

- " quadrupole = higher order
  ⋮
  " exapole
  ⋮



- " monopole ⊕ or ⊖ not observed in nature (see cosmic inflation)

• If particles in medium have $\bar{m} \neq 0$ $\Rightarrow$ $\mu \neq 1$ $(\bar{B} = \mu \bar{H})$

$$\downarrow$$

$$\bar{B} \neq \bar{H}$$

• Here we assume $\bar{m} = \bar{0}$ $\Rightarrow$ $\mu = 1$ $\Rightarrow$ $\bar{B} = \bar{H}$

↳ i.e. the particles of the medium have no relevent magnetic mom.

↳ $\bar{m} = 0$ particles do not "react" to a $\bar{B}$
$\Rightarrow$ all parts in equations involving $\bar{B}$ are not affectet
$\Rightarrow$ $\bar{B} = \bar{H}$

- Response of medium to external $\bar{E}$ only

- External $\bar{E}$ displaces charges in medium
- Charge displacement treated in a multipole expansion
  ↳ (dipole moments)
- Local charge density as superposition of microscopic dipoles
  ↳ ("sum" over a volume)

⇒ Define macroscopic polarization $\bar{P}$ as linear superposition of dipoles

$$\boxed{\rho_{pol} \equiv -\bar{\nabla}\,\bar{P}}$$

Polarization charge density ρ_{pol} $\left(q = \frac{d}{dx}\int dx\, q \right)$ Scatch!

- Charge conservation must apply to ρ_{pol}

$$\partial_t \rho_{pol} + \bar{\nabla}\,\bar{j}_{pol} = 0 \qquad -\partial_t \bar{\nabla}\bar{P} + \bar{\nabla}\,\bar{j}_{pol} = \bar{\nabla}(-\partial_t \bar{P} + \bar{j}_{pol}) = 0 \quad \Longmapsto \quad \boxed{\bar{j}_{pol} = \partial_t \bar{P}}$$

- Plug $\rho_{pol}, \bar{j}_{pol}$ in Maxwell's eq.

response of medium free particles

$$\bar{\nabla}\bar{E} = 4\pi(\rho + \rho_{pol}) = 4\pi\rho - 4\pi\bar{\nabla}\bar{P} \quad \Rightarrow \quad \boxed{\bar{\nabla}\underbrace{(\bar{E} - 4\pi\bar{P})}_{\equiv \bar{D}} = 4\pi\rho}$$

(Gauss' law)

auxiliary response fiel

$$\boxed{\bar{\nabla}\times\bar{B}} = \frac{1}{c}\partial_t \bar{E} + \frac{4\pi}{c}(\bar{j} + \bar{j}_{pol}) = \frac{1}{c}\partial_t(\bar{E} + 4\pi\bar{P}) + \frac{4\pi}{c}\bar{j}$$

$$= \boxed{\frac{1}{c}\partial_t \bar{D} + \frac{4\pi}{c}\bar{j}}$$

(Ampere's law)

⇒ Inhomogeneous Maxwell's eq:

$$\bar{\nabla}\,\bar{D} = 4\pi\rho$$
$$\bar{\nabla}\times\bar{B} = \frac{1}{c}\partial_t \bar{D} + \frac{4\pi}{c}\bar{j}$$

- Assume macroscopically neutral medium $\rho = 0$

($\bar{J} \neq 0$ may be the case anyway (remember that))

- Assume no-net magnetic momentum for microscopic constituents of media

$\Rightarrow$ $\bar{B} = \bar{H}$ ($\bar{m} = 0 \Rightarrow \mu = 1$)

$$\Rightarrow \left[\begin{array}{ll} \bar{\nabla} \bar{D} = 0 & \dot{\bar{D}} + 4\pi \bar{J} = c \bar{\nabla} \times \overset{\overset{\bar{B}}{\downarrow}}{\bar{H}} \\ \bar{\nabla} \bar{B} = 0 & \dot{\bar{B}} = - c \bar{\nabla} \times \bar{E} \end{array} \right]$$ Maxwell's eq.

(needed to describe E.M. waves)

with $\underset{\substack{\uparrow \\ \text{dielectricity}}}{\bar{D} = \varepsilon E}$ $\underset{\substack{\uparrow \\ \text{conductivity}}}{\bar{J} = \sigma E}$ (may be tensors!)

- ~~For now set $\bar{J} = 0$ (no free-currents)~~

The dielectric Tensor

$$\bar{D} = \overset{\downarrow}{\epsilon} \bar{E}$$

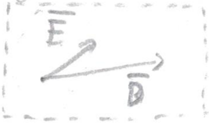
1• <u>$\bar{D}$ may be delayed with respect to the external $\bar{E}$</u>

$\Rightarrow$ $\bar{D}(\bar{x}, t)$ may depend on $E(\underbrace{\bar{x}'}_{\text{other place}}, \underbrace{t'}_{\text{earlier time}})$ with $t' < t$ (the medium must have the time to react to ext $\bar{E}$)

$\Rightarrow$ <u>Express $\bar{D}$ as $\underbrace{\text{convolution}}_{\text{in Fourier space it is a multiplication}}$ of $\bar{E}$ with some kernel</u>

2• <u>Medium might have a priviledged direction</u> (crystal or local mag. field)

$\Rightarrow$ $\bar{D}$ may point in a different direction of $\bar{E}$



$\Rightarrow$ ϵ is a <u>tensor</u> (in general), a scalar can only affect amplitudes not direction.

$$\Rightarrow \boxed{\hat{D}^i(\omega, \bar{k}) = \hat{\epsilon}^i_j(\omega, \bar{k}) \cdot \hat{E}^j(\omega, \bar{k})} \leftarrow \text{Dielectric Tensor } \epsilon_{ij} \ (\bar{D} = \epsilon \bar{E})$$

$\in \mathbb{R}$ $\Rightarrow$ $\epsilon_{ij}(-\omega, -k) = \epsilon^*_{ij}(\omega, k)$ $\Leftarrow$ $\in \mathbb{R}$

- If medium has no priviledged direction $\Rightarrow$ $\hat{\epsilon}_{ij}$ can be spanned by the $\bar{k}$ of the incoming E.M. wave only (no other direction is present

- <u>Ansatz</u> $\boxed{\hat{\epsilon}_{ij}(\omega, \bar{k}) = \underbrace{\hat{A}}_{\substack{\text{isotropic}\\\text{(amplitude)}}} \delta_{ij} + \hat{k}_i \hat{k}_j \underbrace{\hat{B}}_{\substack{\text{anisotropic}\\\text{(direction}}}}$ with $\hat{k} \equiv \frac{\bar{k}}{k}$ just the direction

$\hookrightarrow$ This is symmetric $(\hat{\epsilon}_{ij} = \hat{\epsilon}_{ji})$. <u>If you need anti-symmetric part, add:</u>
$[+ \alpha \epsilon_{ijk} \hat{k}^k]$ $\alpha \in \mathbb{R}$ ϵ_{ijk} Levi civita (purely antisymmetric)

- Investigate δ_{ij}, $\hat{k}_i\hat{k}_j$ and their relation

– To do it, look at the elements of $\hat{\epsilon}_{ij}$: δ_{ij}, $\hat{k}_i\hat{k}_j$ by....

1) Noticing that $\hat{k}_i k^i = \frac{k_i}{k} k^i = \frac{k^2}{k} = k$

2) Look at direction of $\hat{k}_i\hat{k}_j$: $\underbrace{\hat{k}_i\hat{k}_j}_{\equiv \Pi''_{ij}} \cdot k^j = \hat{k}_i \cdot k = k_i \Rightarrow$ parallel to $\bar{k}$

$\equiv \Pi''_{ij}$ projector to direction // to $\bar{k}$ $(\mathrm{Tr}\,\Pi''=1)$

3) Look at off diagonals : $(\underbrace{\delta_{ij}-\hat{k}_i\hat{k}_j}) k^j = k_i - k_i = 0 \Rightarrow \perp$ to k^j

$\equiv \Pi^{\perp}_{ij}$ projector to direction $\perp$ to $\bar{k}$

$= \delta_{ij} - \Pi''_{ij}$ $(\mathrm{Tr}\,\Pi^{\perp} = 2)$

Clearly : $\Pi''\Pi'' = \Pi''$, $\Pi^{\perp}\Pi^{\perp} = \Pi^{\perp}$, $\Pi^{\perp}\Pi'' = 0 = \Pi'' T^{\perp}$

$\Rightarrow$ You can use Π'' and $\Pi^{\perp}$ to decompose $\hat{\epsilon}$ in components // and $\perp$ to incoming E.M. wave (characterized by $\bar{k}$)

$$\boxed{\hat{\epsilon}_{ij} = \hat{\epsilon}_{\parallel}\Pi''_{ij} + \hat{\epsilon}_{\perp}\Pi^{\perp}_{ij}}$$

← Combine with ansatz to find $\hat{A}$ and $\hat{B}$

$$\hat{\epsilon}_{ij} = \hat{A}\underbrace{\delta_{ij}}_{\Pi^{\perp}_{ij}+\Pi''_{ij}} + \underbrace{\hat{k}_i\hat{k}_j}_{\Pi''_{ij}}\hat{B} = \hat{A}\Pi^{\perp}_{ij} + \hat{A}\Pi''_{ij} + \Pi''_{ij}\hat{B} = \hat{\epsilon}_{\perp}\Pi^{\perp}_{ij} + \underbrace{(\hat{\epsilon}_{\perp} + \hat{B})}_{=\hat{\epsilon}_{\parallel}}\Pi''_{ij}$$

$$\Downarrow \qquad \boxed{\hat{A} = \hat{\epsilon}_{\perp}}$$

(affects amplitude only)

(affects direction) $\rightarrow$ $\boxed{\hat{B} = \hat{\epsilon}_{\parallel} - \hat{\epsilon}_{\perp}}$

8

E.M. waves in plasma: dispersion relation

- It tells you how waves propagate
- In vacuum: $c^2\omega^2=\bar K^2$
- Whith a medium: we need the Maxwell eq. with the dielectric tensor:

$$\boxed{\begin{array}{llll}(1) & \bar\nabla\bar D=0 & \dot{\bar D}+4\pi\bar j=c\bar\nabla\times\bar B & (3)\\ (2) & \bar\nabla\bar B=0 & \dot{\bar B}=-c\bar\nabla\times\bar E & (4)\end{array}}$$

(assuming particles of plasma with magnetic moment $\bar m=0$ $\Rightarrow \mu=0 \Rightarrow \bar H=\bar B$)

- Decompose incoming wave into plane waves: $\exp(i(\bar K\bar x-\omega t))$

$$(1)\ \boxed{\cancel{i}\bar K\hat{\bar D}=0} \qquad -i\omega\hat{\bar D}+4\pi\hat{\bar j}=c\,i\bar K\times\hat{\bar B} \quad \underline{\text{set } \bar j=0} \Rightarrow \boxed{\bar K\times\hat{\bar B}=-\frac{\omega\hat{\bar D}}{c}}\ (3)$$

$$(2)\ \boxed{\cancel{i}\bar K\hat{\bar B}=0} \qquad \cancel{-}i\omega\hat{\bar B}=\cancel{-}c\cancel{i}\bar K\times\hat{\bar E} \qquad \Rightarrow \boxed{\bar K\times\hat{\bar E}=\frac{\omega\hat{\bar B}}{c}}\ (4)$$

(for simplicity)

- $\overset{(-iK)}{\bar\nabla}\times(4)$: $$\bar K\times(\overbrace{\bar K\times\hat{\bar E}}^{(4)})=\frac{\omega}{c}\overbrace{\bar K\times\hat{\bar B}}^{(3)}=\frac{\omega}{c}\cdot\frac{-\omega\hat{\bar D}}{c}=-\frac{\omega^2\hat{\bar D}}{c^2}$$

we also have: $\bar K\times(\bar K\times\hat{\bar E})=\bar K(\bar K\hat{\bar E})-\hat{\bar E}(\bar K\bar K)$ $\quad(\bar a\times(\bar b\times\bar c)=\bar b(\bar a\bar c)-\bar c(\bar a\bar b))$

$$\Rightarrow \boxed{\frac{\omega^2\hat{\bar D}}{c^2}=\hat{\bar E}K^2-\bar K(\bar K\hat{\bar E})}$$

- Plug $\hat D^i=\hat\epsilon^i_j\hat E^j$

$$\boxed{\frac{\omega^2}{c^2}\hat\epsilon^i_j\hat E^j}=\hat E^iK^2-K^iK_j\hat E^j=K^2(\hat E^i-\hat K^i\hat K_j\hat E^j) \qquad \hat K=\frac{\bar K}{K}$$

$$=\boxed{K^2\underbrace{(\delta^i_j-\hat K^i\hat K_j)}_{\Pi^\perp_{ij}}\hat E^j} \qquad \hat E^i=\delta^i_j\hat E^j$$

$$\Rightarrow \left(\Pi^\perp_{ij}-\frac{\omega^2}{c^2K^2}\hat\epsilon_{ij}\right)\hat E^j=0$$ non-trivial solution for (i.e. $\bar E\neq0$) $$\boxed{\det\left(\Pi^\perp-\frac{\omega^2}{c^2K^2}\hat\epsilon\right)=0}$$

Dispersion relation!!

- Decompose ϵ in $\perp$ and $\parallel$ components

$$\det\left[\Pi^\perp_{ij}-\frac{\omega^2}{c^2K^2}\left(\hat\epsilon_\parallel\Pi^\parallel+\hat\epsilon_\perp\Pi^\perp\right)\right]=\boxed{\det\left[\left(1-\frac{\omega^2}{c^2K^2}\hat\epsilon_\perp\right)\Pi^\perp-\frac{\omega^2}{c^2K^2}\hat\epsilon_\parallel\Pi^\parallel\right]=0}$$

- It tells you how transversal and longitudinal waves propagate

Transversal / longitudinal waves

- In vacuum only transversal waves exist
- In media you also have longitudinal waves (see dispersion relat.)

- **Transversal waves** (dispersion relation)

$\bar{E}$ orthogonal to $\bar{k}$ $\Rightarrow$ $\Pi^{\parallel}_{ij}\hat{E}^j = 0 \qquad \Pi^{\perp}_{ij}\hat{E}^j = \hat{E}_i$

$$\Rightarrow \left(1 - \frac{\omega^2}{c^2k^2}\hat{\epsilon}_\perp\right)\hat{E}_i = 0 \quad \rightarrow \quad \det\left(1 - \frac{\omega^2}{c^2k^2}\hat{\epsilon}_\perp\right) = 0 \qquad \boxed{\omega^2 = \frac{c^2k^2}{\hat{\epsilon}_\perp}}$$

- Like in vacuum ($\omega^2 = c^2k^2$) but with a reduced velocity $\left(\frac{c^2}{\hat{\epsilon}_\perp}\right)$

- **Longitudinal waves** (dispersion relation)

in this case : $\Pi^{\parallel}_{ij}\hat{E}^j = \hat{E}^i \qquad \Pi^{\perp}_{ij}\hat{E}^j = 0$

$$\Rightarrow \left(\frac{\omega^2}{c^2k^2}\hat{\epsilon}_\parallel \underbrace{\Pi^{\parallel}_{ij}\right)\hat{E}^j}_{\hat{E}^i} = 0 \qquad \boxed{\frac{\omega^2}{c^2k^2}\hat{\epsilon}_\parallel \hat{E}^i = 0}$$

in general, it means that $\hat{\epsilon}_\parallel = 0$

to understand this condition we need $\hat{\epsilon}_\parallel$ explicit (see later on)

-

$$\left[\left(1 - \frac{\omega^2}{c^2k^2}\hat{\epsilon}_\perp\right)\Pi^{\perp}_{ij} - \frac{\omega^2}{k^2k^2}\hat{\epsilon}_\parallel \Pi^{\parallel}_{ij}\right]\hat{E}^j = 0$$

- This is not the complete story....
- To have a full understanding of the dispersion relation we need to understand $\hat{\epsilon}_\perp$ and $\hat{\epsilon}_\parallel$ explicitely!

Dielectric tensor ⟷ Phase space : summary

- $\hat{\epsilon}_{ij}$ depends on the Phase-Space distribution of the plasma particles (!)

  ↓ probability density $f(\bar{q},\bar{p},t)$ $\bar{q}\equiv\bar{x}$ here

(collisionless)
- Boltzmann eq. : evolution of $f \doteq f_0 + \delta f$ (Perturbative approach)

  ↑ small perturbation given by incoming E.M. wave

$$\frac{\partial \delta f}{\partial t} + \bar{v}\bar{\nabla}\delta f - e\underbrace{\left(\bar{E}+\frac{\bar{v}}{c}\times\bar{B}\right)}\frac{\partial f_0}{\partial \bar{p}} = 0$$

⇓ F = Lorentz force from incoming E.M. wave on plasma particles

Solution : $\delta f = \dfrac{e\bar{E}}{i(\bar{k}\bar{v}-\omega)}\dfrac{\partial f_0}{\partial \bar{p}}$ (Phase-space distribution of perturbation)

- Assume Homogeneous, Isotropic, Stationary plasma

$$\Rightarrow \quad \rho_{pol} = -e\int d^3p\,\delta f \qquad \bar{J}_{pol} = -e\int d^3p\,\bar{v}\,\delta f$$

- Together with $\rho_{pol} = -\bar{\nabla}P$ and $\bar{J}_{pol} = \dot{\bar{P}}$ you get

$$\hat{P}_i = -\frac{eE^j}{\omega}\int d^3p\,\frac{v_i}{(\bar{k}\bar{v}-\omega)}\frac{\partial f_0}{\partial p^j} \quad \longrightarrow \quad 4\pi\hat{P}_i = \hat{D}_i - \hat{E}_i = \hat{\epsilon}_{ij}\hat{E}^j - \hat{E}_i$$

$$\hat{\epsilon}_{ij} = \delta_{ij} - \frac{4\pi e^2}{\omega}\int d^3p\,\underbrace{\frac{v_i}{(\bar{k}\bar{v}-\omega)}}_{\text{Pole}}\frac{\partial f_0}{\partial p^j} \quad \rightarrow \text{Decompose } \hat{\epsilon}^{\parallel}_{ij},\ \hat{\epsilon}^{\perp}_{ij}$$

- Pole for $\bar{k}\bar{v}=\omega$ ⇒ damping $\doteq$ dissipated energy

$$Q = \frac{\partial}{\partial t}\left(\frac{\bar{E}^2}{8\pi}\right) + \bar{E}\bar{J}_{pol} = \begin{pmatrix}\text{damping of}\\ \text{E.M. wave}\end{pmatrix} + \begin{pmatrix}\text{Ohmic}\\ \text{heating}\end{pmatrix} = \frac{\bar{E}\dot{\bar{D}}}{4\pi} \longleftarrow \hat{D} = \underline{\underline{\epsilon}}\,\bar{E}$$

$$\langle Q\rangle = \frac{\omega}{8\pi}|E|^2\,\mathrm{Im}\left(\hat{\epsilon}_{\parallel}\right) = -|E|^2\frac{\pi e^2 m_e\omega}{2k^2}\frac{d\bar{f}_0}{dp_x}\bigg|_{p_x=\frac{\omega m_e}{k}} \longleftarrow \text{Landau damping}$$

↑ time average

Boltzmann equation

- One particle in phase space $(\bar{q},\bar{p})$, PDF: $f(q,p,t)$

$$\int d^3\bar{q}\, d^3\bar{p}\, f(p,q,t) = 1$$, total derivative conserved $\Rightarrow$ $$\boxed{\frac{\delta f}{\delta t} + \bar{\nabla}_q(\underline{f\dot{\bar{q}}}) + \bar{\nabla}_p(\underline{f\dot{\bar{p}}}) = 0}$$

flux of f

- Analogy with matter conservation

$$\int d^3\bar{q}\, \rho(\bar{q},t) = M \quad \Rightarrow \text{ continuity eq. } \quad \frac{\delta\rho}{\delta t} + \bar{\nabla}(\rho\dot{\bar{q}}) = 0$$

- Together with Hamilton's equations $\dot{q} = \frac{\delta H}{\delta p}$ $\quad\dot{p} = -\frac{\delta H}{\delta q}$

$$\frac{\delta}{\delta\bar{q}}(f\dot{\bar{q}}) + \frac{\delta}{\delta\bar{p}}(f\dot{\bar{p}}) = \frac{\delta}{\delta\bar{q}}\left(f\frac{\delta H}{\delta\bar{p}}\right) - \frac{\delta}{\delta\bar{p}}\left(f\frac{\delta H}{\delta\bar{q}}\right)$$
$$= \frac{\delta H}{\delta\bar{p}}\frac{\delta f}{\delta\bar{q}} + \cancel{f\frac{\delta^2 H}{\delta\bar{p}\,\delta\bar{q}}} - \frac{\delta H}{\delta\bar{q}}\frac{\delta f}{\delta\bar{p}} - \cancel{f\frac{\delta^2 H}{\delta\bar{q}\,\delta\bar{p}}} \qquad \circledast$$
$$= \dot{\bar{q}}_i\frac{\delta f}{\delta\bar{q}} + \dot{\bar{p}}_i\frac{\delta f}{\delta\bar{p}}$$

$$\Rightarrow \boxed{\underbrace{\frac{\delta f}{\delta t} + \dot{\bar{q}}\frac{\delta f}{\delta\bar{q}} + \dot{\bar{p}}\frac{\delta f}{\delta\bar{p}}}_{[f,H]} = 0} \longleftarrow \text{Vlasov equation} \qquad \left(\frac{\delta f}{\delta t} = [H,f]\right) \text{ Poisson Brackets}$$

(Hamilton's equations)

- Using $H = \frac{\bar{p}^2}{2m} + \phi(\bar{x},t)$: $\dot{\bar{q}} = \frac{\delta H}{\delta\bar{p}} = \frac{2\bar{p}}{2m} = \frac{\bar{p}}{m}$ $\quad -\dot{\bar{p}} = \frac{\delta H}{\delta\bar{x}} = \frac{\delta\phi}{\delta\bar{x}}$ $\quad (\bar{q}\equiv\bar{x})$ just in cartesian coordinates in $\circledast$

$$\Rightarrow \boxed{\frac{\delta f}{\delta t} + \frac{\bar{p}}{m}\frac{\delta f}{\delta\bar{x}} - \frac{\delta\phi}{\delta\bar{x}}\frac{\delta f}{\delta\bar{p}} = 0} \longleftarrow \text{Collisionless Boltzmann eq.}$$

only one particle $\Rightarrow$ can't be collisions

$\bar{\nabla}\phi = \bar{F}$
external forces

particle diffusion

add source term for collisions: $\left(\frac{\delta f}{\delta t}\right)_{\text{collision}}$

Dielectricity: longitudinal / transverse components

- Describe response of particles in plasma due to incoming E.M. wave

$\Rightarrow$ Use collisionless Boltzmann equation: $f(\bar q, \bar p, t)$ distribution in phase-space $(\bar q, \bar p)$

$$\frac{\partial f}{\partial t} + \underbrace{\dot{\bar q}\frac{\partial f}{\partial \bar q} + \dot{\bar p}\frac{\partial f}{\partial p}}_{[f,H]} = 0$$

no collisions

$\bar q = \bar x$ cartesian coordinates $\bar q$ = generalized coord.

$\dot{\bar q} = \bar v$

$\dot{\bar p} = \bar F$ external forces

- Neglect contribution of ions "small accelerations with respect to e^-" $(m_i \gg m_{e^-})$

$\Rightarrow$ negligible emission (Larmor)

- Perturbative approach (E.M wave is a small perturbation)

$f = f_0 + \delta f$, f_0 = unperturbed distrib. $\downarrow$ δf

$$\Rightarrow \quad \frac{\partial \delta f}{\partial t} + \bar v \bar\nabla \delta f - e\Big(\bar E + \underbrace{\frac{\bar v}{c}\times \bar B}_{\text{neglect}}\Big)\cdot\frac{\partial f_0}{\partial \bar p} = 0$$

$\dot{\bar p} = \bar F$ = Lorentz force

1° order approximation

$\bar v, \bar p$ equilibrium plasma vel. and mom.

- Locally isotropic distribution $\frac{\partial f_0}{\partial \bar p} /\!/ \bar v$

$\Rightarrow (\bar v \times \bar B)\cdot\frac{\partial f_0}{\partial \bar p} = 0$ $/\!/ \bar v$

$$\Rightarrow \quad \frac{\partial \delta f}{\partial t} + \bar v \bar\nabla \delta f = e\bar E\frac{\partial f_0}{\partial \bar p} \longrightarrow -i\omega\hat{\delta f} + i\bar v \bar K \hat{\delta f} = e\bar E\frac{\partial f_0}{\partial \bar p}$$

plane wave approx $\exp[i(\bar k \bar x - \omega t)]$

$$\boxed{\delta f = \frac{e\bar E}{i(\bar K\bar v - \omega)}\frac{\partial f_0}{\partial \bar p}}$$

Phase-space distribution

derivative of f_0

- Now that you have $\delta f \rightarrow$ get $\bar P \rightarrow$ get $\hat{\varepsilon}$

- get polarization P
  - For simplicity assume f_0 locally: $\left\{\begin{array}{l}\text{homgeneous}\\ \text{isotropic}\\ \text{stationary}\end{array}\right\}$ $\Rightarrow$ $\rho_{pol}\overset{(\bar{x},t)}{=} -e\int d^3p\,\delta f(\bar{x},\bar{p},t)$

$$\bar{j}_{pol} = -e\int d^3p\,\delta f\cdot\bar{v}$$

$(\bar{j} = e\bar{v}$ point like$)$

$\left(\rho_{pol},\ \bar{j}_{pol}\text{ are }\propto \exp[i(\bar{k}\bar{x}-\omega t)]\right)$

- We have seen that $\rho_{pol} \equiv -\bar{\nabla}\bar{P}$ $\Rightarrow$ $\hat{\rho}_{pol} = -i\bar{k}\hat{\bar{P}}$

$$\bar{j}_{pol} = \frac{\delta\bar{P}}{\delta t} \qquad \hat{j}_{pol} = -i\omega\hat{\bar{P}}$$

$$\Rightarrow\ -i\omega\hat{\bar{P}} = -e\int d^3p\,\delta f\cdot\bar{v} = -e\int d^3p\,\frac{e\hat{\bar{E}}}{i(\bar{k}\bar{v}-\omega)}\frac{\delta f_0}{\delta\bar{p}}\cdot\bar{v}\ \left(\cdot\frac{i}{i}\right) = i e^2\hat{\bar{E}}\int d^3p\,\frac{\bar{v}}{(\bar{k}\bar{v}-\omega)}\frac{\delta f_0}{\delta\bar{p}}$$

$$\boxed{\hat{P}_i = -\frac{e^2\hat{E}^j}{\omega}\int d^3p\,\frac{v_i}{(\bar{k}\bar{v}-\omega)}\frac{\delta f_0}{\delta p^j}}$$ polarization

- Get $\hat{\epsilon}$
  - And we have $4\pi\hat{P}_i \equiv \hat{D}_i - \hat{E}_i = (\hat{\epsilon}_{ij} - \delta_{ij})\hat{E}^j$ $\quad(\hat{D}_i \equiv \hat{\epsilon}_{ij}\hat{E}^j)$

(definition of $\hat{D}$)

$$\boxed{\hat{\epsilon}_{ij} = \delta_{ij} - \frac{4\pi e^2}{\omega}\int d^3p\,\frac{v_i}{(\bar{k}\bar{v}-\omega)}\frac{\delta f_0}{\delta p^j}}$$

- Decompose in $\parallel$ and $\perp$ components

$$\hat{\epsilon}^{ij} = \epsilon_\perp \Pi_\perp^{ij} + \epsilon_\parallel \Pi_\parallel^{ij} \qquad (\text{apply } \Pi_\perp^{ij} \text{ and } \Pi_\parallel^{ij} \text{ to } \uparrow)$$

$$\boxed{\begin{aligned}\hat{\epsilon}_\perp &= 1 - \frac{2\pi e^2}{\omega}\int d^3p\,\bar{v}_\perp\frac{\delta f_0}{\delta\bar{p}_\perp}\frac{1}{(\bar{k}\bar{v}-\omega)}\\ \hat{\epsilon}_\parallel &= 1 - \frac{4\pi e^2}{k^2}\int d^3p\,\bar{k}\frac{\delta f_0}{\delta\bar{p}}\frac{1}{(\bar{k}\bar{v}-\omega)}\end{aligned}}$$

$$\bar{v}_\perp \equiv \bar{v} - \frac{(\bar{k}\bar{v})\bar{k}}{k^2} \qquad \bar{p}_\perp = m\bar{v}_\perp$$

pole at $\bar{k}\bar{v} = \omega$ $\Rightarrow$ $\hat{\epsilon}_\parallel$ is complex

$\Rightarrow$ ω is immaginary

$\Rightarrow$ damping of the wave

(Landau damping)

[The unperturber phase-space distribution of particles in plasma sets $\hat{\epsilon}$!]

Landau damping

- Consider longitudinal waves
- $\hat{\epsilon}_{\parallel}$ has an immaginary part ⇒ ω is immaginary
  ⇒ damping = dissipation of its energy

- **Energy dissipation rate** $Q = \underbrace{\frac{\delta}{\delta t}\left(\frac{\bar{E}^2}{8\pi}\right)} + \underbrace{\bar{E}\,\bar{j}_{pol}}$ ← (only $\bar{j}_{pol}$ because we assumed $\bar{j}=0$ for the free charges)

  ↳ Ohmic heating

  → Damping of E.M. wave ($\frac{\bar{E}^2}{8\pi}$ = energy of the field)

  ($\bar{j}_{pol} = \dot{\bar{P}}$) → from continuity

$$= \frac{\bar{E}\dot{\bar{E}}}{4\pi} + \bar{E}\dot{\bar{P}} = \frac{\bar{E}}{4\pi}\left(\dot{\bar{E}} + 4\pi\dot{\bar{P}}\right) = \frac{\bar{E}\dot{\bar{D}}}{4\pi} \qquad (\bar{E}+4\pi\bar{P}=\bar{D})$$

- Consider 1 monocromatic wave: $\bar{E} = \hat{\bar{E}}\exp[i(\bar{k}\bar{x}-\omega t)]$
  $\dot{\bar{D}} = -i\omega\,\hat{\bar{D}}\exp[i(\bar{k}\bar{x}-\omega t)]$ $\qquad \hat{\bar{D}} = \hat{\epsilon}_{\parallel}\hat{\bar{E}}$

- $Q \in \mathbb{R}$ ⇒ Take only real part of $\mathrm{Re}(\bar{E}) = \frac{\hat{\bar{E}}+\hat{\bar{E}}^*}{2}$ $\qquad \mathrm{Re}(\bar{D}) = \frac{\hat{\bar{D}}+\hat{\bar{D}}^*}{2}$

$$Q = -\frac{i\omega}{16\pi}(\bar{E}+\bar{E}^*)(\hat{\epsilon}\hat{\bar{E}} - \hat{\epsilon}^*\bar{E}^*)$$ – from exp[–] of $\hat{\epsilon}_{\parallel}$

- Average over time ⇒ $\langle\bar{E}\bar{E}\rangle = 0 = \langle\bar{E}^*\bar{E}^*\rangle$ because $\propto [\exp(-2i\omega t)]$ (oscillating around 0)

$$\langle Q\rangle = -\frac{i\omega}{16\pi}\left(\hat{\epsilon}_{\parallel}\hat{\bar{E}}\hat{\bar{E}}^* - \hat{\epsilon}_{\parallel}^*\hat{\bar{E}}^*\hat{\bar{E}}\right) = -\frac{i\omega}{16\pi}|E|^2\underbrace{\left(\hat{\epsilon}_{\parallel} - \hat{\epsilon}_{\parallel}^*\right)}_{=2i\,\mathrm{Im}(\hat{\epsilon}_{\parallel})} = \frac{\omega}{8\pi}|E|^2\,\mathrm{Im}(\hat{\epsilon}_{\parallel})$$

- Take expression of $\hat{\epsilon}_{\parallel}$ and remove pole

$$\frac{1}{\bar{k}\bar{v}-\omega} \rightarrow \frac{1}{\bar{k}\bar{v}-\omega-i\delta}$$ small deviation. we only need Im(–)

$$\mathrm{Im}\left(\frac{1}{\bar{k}\bar{v}-\omega-i\delta}\right) = \mathrm{Im}\left(\frac{1}{\bar{k}\bar{v}-\omega-i\delta}\,\frac{\bar{k}\bar{v}-\omega+i\delta}{\bar{k}\bar{v}-\omega-i\delta}\right) = \mathrm{Im}\left(\frac{\bar{k}\bar{v}-\omega-i\delta}{(\bar{k}\bar{v}-\omega)^2+\delta^2}\right) = \frac{i\delta}{(\bar{k}\bar{v}-\omega)^2+\delta^2}$$

$$\lim \frac{i\delta}{(\bar{k}\bar{v}-\omega)^2+\delta^2} = \pi\,\delta_D(\bar{k}\bar{v}-\omega)$$

- Esplicitly:

$$\mathrm{Im}\left(\hat{\mathcal{E}}_{\parallel}\right) = \mathrm{Im}\left[\underset{\uparrow \mathbb{R}}{1} - \frac{4\pi e^2}{K^2}\int d^3p\, \bar{K}\, \frac{\delta f_0}{\delta \bar{P}}\, \frac{1}{(\bar{K}\bar{v}-\omega)}\right]$$

$$= -\frac{4\pi e^2}{K^2}\int dP_x \int dP_y \int dP_z\, \bar{K}\, \frac{\delta f_0}{\delta \bar{P}}\, \pi\, \delta_D(\bar{K}\bar{v}-\omega)$$

- rotate x axes along $\bar{K}$ $\Rightarrow \bar{K}=(K,0,0)$

$$= -\frac{4\pi^2 e^2}{K^2}\int dP_x\, K\, \frac{\delta \bar{f}_0}{\delta P_x}\, \delta_D(K v_x-\omega)$$

$$\bar{f}_0(P_x) \equiv \int dP_y \int dP_z\, f^0(\bar{P})$$

$$= -\frac{4\pi^2 e^2}{K^2}\int dP_x\, K\, \frac{\delta \bar{f}_0}{\delta P_x}\, \frac{m}{K}\, \delta_D\left(P_x - \frac{\omega m}{K}\right)$$

$$\delta_D(K v_x-\omega) = \delta_D\left(\frac{K P_x}{m}-\omega\right) = \delta_D\left[\frac{K}{m}\left(P_x-\frac{\omega m}{K}\right)\right] = \frac{m}{K}\,\delta_D\left(P_x-\frac{\omega m}{K}\right)$$

$$= -\frac{4\pi^2 e^2 m_e}{K^2}\left.\frac{d\bar{f}_0}{dP_x}\right|_{P_x=\frac{m_e\omega}{K}}$$

$$\Rightarrow \langle Q\rangle = \frac{\omega}{8\pi}|E|^2\,\mathrm{Im}\left(\hat{\mathcal{E}}_{\parallel}\right)$$

$$= -|E|^2\,\frac{\pi e^2 m_e \omega}{2K^2}\left.\frac{d\bar{f}_0}{dP_x}\right|_{P_x=\omega m/K} \longleftarrow \text{Landau damping}$$

?

details depend on $\bar{f}_0$!

- **Meaning:** the incoming E.M. wave gives energy to the plasma
  - e^- faster than wave get slow down
  - e^- slowlier " " " speed up , more than those

$\Rightarrow$ more electrons get speed (in general you have less low energy e^- than high energy once)

$\Rightarrow$ gain kinetic energy by "sucking" from the incoming E.M. wave

Magnetohydrodynamics: summary

- Hydrodynamics of plasma ⇒ 2 fluids
- Hypotesys:
  1) Non relativistic motions $\beta \ll 1, \gamma \approx 1$
  2) Plasma with high conductivity σ ⇒ [$c\rho' \ll |\bar{j}'|$ small drifts can create long $\bar{j}$]
  3) Presence of small drifts

     $\bar{v}_{drift} = \bar{v}_e - \bar{v}_i$ ⇒ non perfect coupling between e^- fluid and Ze fluid

$\models$ $c\rho' \ll |\bar{j}'|$, $c\rho \ll |\bar{j}|$, $|\bar{E}| \ll |\bar{B}|$, $\frac{4\pi}{c}\bar{j} = \bar{\nabla}\times\bar{B}$

1• $$\boxed{\frac{\delta \bar{B}}{\delta t} = \underbrace{\frac{c^2}{4\pi\sigma}\bar{\nabla}^2\bar{B}}_{\text{diffusion}} + \underbrace{\bar{\nabla}\times(\bar{v}\times\bar{B})}_{\text{advection}}}$$

▫ Induction eq. for ideal perfect coupling (no drift)

- No $\bar{B}$ can be generated by plasma
- $R_M \equiv \frac{\text{advection}}{\text{diffusion}} = \frac{4\pi\sigma v L}{c^2}$ Magnetic Reynolds number

• (Induction eq.) + $\boxed{\bar{\nabla}P_e \times \frac{\bar{\nabla}n_e}{n_e^2}}$ ▫ Additional term when breaking ideal coupling ⇒ presence of small drift

⇒ Is a source term, now plasma can generate $\bar{B}$ on its own if $\bar{\nabla}P_e$ and $\bar{\nabla}n_e$ are not //

2• $$\boxed{\rho\frac{d\bar{v}}{dt} = -\bar{\nabla}\left(P + \frac{\bar{B}^2}{8\pi}\right) - \frac{1}{4\pi}(\bar{B}\bar{\nabla})\bar{B}}$$

▫ Euler eq., now energy density of $\bar{B}$ contributes to pressure

3• $$\boxed{\delta_t\rho + \bar{\nabla}(\rho\bar{v}) = 0}$$

▫ Continuity eq. is not changed

4• $$\boxed{P = P(\rho)}$$

▫ Eq. of state

$\models$ (1), (2), (3), (4) Closed system of eq. ⇒ fix $\rho, \bar{v}, P, \bar{B}$

↳ $\bar{E}$ (from Ampere Ohm laws)

• $$\boxed{\rho T\frac{d\hat{s}}{dt} = \bar{\nabla}(K\bar{\nabla}T) + v_{ij}T_d^{ij} + \underbrace{\frac{c^2}{16\pi^2\sigma}(\bar{\nabla}\times\bar{B})^2}_{\text{new}}}$$

Entropy

⇒ even if $K = 0 = \eta = \zeta$ entropy density evolves

Magnetohydrodynamic

(simplest case: $e^- + Ze$)

- Hydrodynamic : gas, neutral particles, 1 fluid
- Magnitohydrodynamic : plasma ⇒ 2 fluids coupled by E.M. interaction
- If perfect coupling ⇒ you can treat them as one fluid

but... ⇒ | no separation of charges | ⇒ can not generate $\bar{E}$ and $\bar{B}$
| no net electric current |

- To allow generation of $\bar{E}, \bar{B}$ ⇒ slightly loosen assumption of perfect coupling

⇒ allow small drift between e^- and Ze : $\bar{v}_{drift} = \bar{v}_e - \bar{v}_i$

(so that you have region ⊕ and other ⊖, and current of net charge)

- Even small drifts

- Assume non-relativistic motions : $v \ll c, \gamma \approx 1$

- Move to plasma rest-frame : (') = rest frame
- Lorentz transforms $c\rho' = c\rho - \bar{\beta}\bar{j}$

$\bar{j}' = \bar{j} - \bar{\beta} c\rho$

plasma rest frame

- Even small drifts (⇒ small ρ') can create strong currents $\bar{j}'$: $\boxed{c\rho' \ll |\bar{j}'|}$

$j^0 \ll \sqrt{j_i j^i}$

$$\Rightarrow \quad c\rho' = c\rho - \bar{\beta}\bar{j} \ll |\bar{j}'| \qquad c\rho - \bar{\beta}\bar{j} \ll \bar{j} - \bar{\beta}c\rho$$

$$(\beta \ll 1) \longrightarrow \quad c\rho + \underbrace{\bar{\beta}\rho}_{\approx 0} \ll \bar{j} + \underbrace{\bar{\beta}\bar{j}}_{\approx 0} \quad \Rightarrow \quad \boxed{c\rho \ll |\bar{j}|}$$

(also for the observer)

- This is the case for large conductivity σ

Ohm's law : $\bar{j}' = \sigma \bar{E}'$ (even if $\bar{E}'$ is small, σ is large ⇒ $\bar{j}'$ is large)

Note: $[\sigma] = \text{time}^{-1}$, $\left[\frac{\delta}{\delta t}\right] = \text{time}^{-1}$ ⇒ $[\sigma \bar{E}] = \left[\frac{\delta E}{\delta t}\right]$ ⇒ $|\bar{j}'| \gg \left|\frac{\delta E}{\delta t}\right|$ i.e. σ large is equivalent to say: charges respond very fast to changes of $\bar{E}$

$\Rightarrow$ Maxwell eq: $\underbrace{\dot{\bar{E}}' + 4\pi \bar{j}'}_{} = c\,\bar{\nabla}\times\bar{B}' \longrightarrow \boxed{\frac{4\pi}{c}\bar{j}' = \bar{\nabla}\times\bar{B}'}$

σ is large $\Rightarrow |\dot{\bar{E}}'| \ll |\bar{j}'|$

$\Rightarrow$ Maxwell eq: $\bar{\nabla}\bar{E}' = 4\pi\rho'$

$\bar{\nabla}\times\bar{B} = 4\pi\frac{\bar{j}}{c}$

$c\rho' \ll |\bar{j}'| \Rightarrow 4\pi\rho' \ll 4\pi\left|\frac{\bar{j}}{c}\right| \Rightarrow \bar{\nabla}E' \ll |\bar{\nabla}\times\bar{B}|$

$\Rightarrow \boxed{|\bar{E}| \ll |\bar{B}|}$ (they are smooth functions)

$\boxed{\text{Induction equation}}$ (For now, assume perfect coupling)

- $\bar{j} \approx \bar{j}' = \sigma E'$

$\bar{E}' = \bar{E} + \bar{\beta} \times \bar{B} \quad \Rightarrow \quad \bar{E} = \bar{E}' - \bar{\beta} \times \bar{B} = \frac{\bar{j}}{\sigma} - \bar{\beta} \times \bar{B}$ (Lorentz transf.)

$$\mapsto \boxed{\frac{\delta \bar{B}}{\delta t}} = -c \bar{\nabla} \times \bar{E} = -c \bar{\nabla} \times \left(\frac{\bar{j}}{\sigma} - \bar{\beta} \times \bar{B} \right)$$

plug Ampere's law $\bar{j} = \frac{c}{4\pi} \bar{\nabla} \times \bar{B}$

$$= -c \bar{\nabla} \times \left(\underbrace{\frac{c}{4\pi\sigma} \bar{\nabla} \times \bar{B}}_{} - \bar{\beta} \times \bar{B} \right)$$

$$\frac{c^2}{4\pi\sigma} \left(\bar{\nabla} \times (\bar{\nabla} \times \bar{B}) \right) = \frac{c^2}{4\pi\sigma} \left(\nabla (\underbrace{\bar{\nabla} \bar{B}}_{=0}) - \bar{\nabla}^2 \bar{B} \right) = -\frac{c \bar{\nabla}^2 \bar{B}}{4\pi\sigma}$$

$=0$ Maxwell's eq. $\bar{\nabla} \bar{B} = 0$

$$= \boxed{\underset{(1)}{\frac{c^2}{4\pi\sigma} \bar{\nabla}^2 \bar{B}} + \underset{(2)}{\cancel{c} \bar{\nabla} \times \left(\frac{\bar{v}}{\cancel{c}} \times \bar{B} \right)}}$$

for observer

$\hookrightarrow$ Evolution of $\bar{B}$ in plasma (Pre existing $\bar{B}$)

(1) Diffusion of $\bar{B}$ due to conductivity σ of plasma (diffusion)

- if $\sigma \to \infty \Rightarrow (1) = 0 \mapsto \bar{B}$ is "locked" to the plasma
  - $\hookrightarrow$ (each change in $\bar{B}$ creates strong currents that counteract its change)

(2) Transport of $\bar{B}$ with the fluid flow (advection)

- Ratio of their order of magnitudes: (1): $\frac{c^2 B}{4\pi\sigma L^2}$ (2): $\frac{vB}{L}$

$$R_M \equiv \frac{\text{advection}}{\text{diffusion}} = \frac{vB}{\cancel{L}} \frac{4\pi\sigma L^{\cancel{2}}}{c^2 \cancel{B}} = \frac{4\pi\sigma v L}{c^2}$$ (Magnetic Reynolds number)

- if $R_M \gg 1 \Rightarrow$ diffusion is negligible (Astrophysics) $\frac{\delta \bar{B}}{\delta t} = \bar{\nabla} \times (\bar{v} \times \bar{B})$
- if $R_M \ll 1 \Rightarrow$ " dominates (Labs)

$$\frac{\delta \bar{B}}{dt} = \frac{c^2}{4\pi\sigma} \bar{\nabla}^2 \bar{B} \rightarrow i\omega \cancel{\hat{\bar{B}}} = \frac{c^2}{4\pi\sigma} (-) K^2 \cancel{\hat{\bar{B}}} \rightarrow \omega_{diff} = \frac{i c^2 K^2}{4\pi\sigma} \in \mathbb{C} \mapsto \text{damping}$$

$\Rightarrow \bar{B}$ decays on time scale $\tau_{diff} \equiv \frac{2\pi}{\mathrm{Im}(\omega_{diff})}$

Euler's equation

- How $\bar{B}$ changes because of plasma flow (induction eq.)
- How the plasma is affected by $\bar{B}$: Euler equation ← Lorentz force

Lorentz. force ($|E| \ll |B|$)

$$\rho \frac{d\bar{v}}{dt} = -\bar{\nabla} P + \bar{F}_B$$

$$\uparrow$$

$$\left(\partial_t \bar{v} + (\bar{v}\bar{\nabla})\bar{v}\right)$$

$$\bar{F}_B = n \cdot \frac{e}{c}(\bar{v} \times \bar{B}) = \frac{1}{c}\Big[\underbrace{(ne\bar{v})}_{\bar{j}} \times \bar{B}\Big]$$

$\bar{j}$ ← Ampere's law $\bar{j} = \frac{c}{4\pi}\bar{\nabla}\times\bar{B}$

$$= \frac{1}{4\pi}(\bar{\nabla}\times\bar{B})\times\bar{B}$$

$$= \frac{1}{4\pi}\left[(\bar{B}\bar{\nabla})\bar{B} - \frac{1}{2}\bar{\nabla}(\bar{B}^2)\right]$$

↳ gradient of $\bar{B}$ energy density

→ how $\bar{B}$ changes along $\bar{B}$ lines

$$\boxed{\rho\frac{d\bar{v}}{dt} = -\bar{\nabla}P + \frac{1}{4\pi}(\bar{\nabla}\times\bar{B})\times\bar{B}} = -\bar{\nabla}P + \frac{1}{4\pi}\left[(\bar{B}\bar{\nabla})\bar{B} - \frac{1}{2}\bar{\nabla}(\bar{B}^2)\right] = -\bar{\nabla}\left(P + \frac{\bar{B}^2}{8\pi}\right) - \frac{1}{4\pi}(\bar{B}\bar{\nabla})\bar{B}$$

- energy density of $\bar{B}$ contribute to pressure (Magnetic pressure)

- **Alternatively** : as conservation law, add T^{ij}_{magnetic}

$$\partial_\mu T^{\mu i} = \partial_t(\rho v^i) + \partial_j T^{ij} = 0 \quad \text{with} \quad T^{ij} = \rho v^i v^j + P\delta^{ij} + T_d^{ij} + T_m^{ij}$$

Magnetic contribution

$$T_m^{ij} = -\frac{1}{4\pi}\left(B^i B^j - \frac{|\bar{B}|^2}{2}\delta^{ij}\right)$$

Diffusion contribution

$$T_d^{ij} = -\eta\left(\partial^j v^i + \partial^i v^j - \frac{2}{3}\delta^{ij}\bar{\nabla}\bar{v}\right) - \zeta\,\delta^{ij}\bar{\nabla}\bar{v}$$

Generation of Magnetic fields

- If e^- and Ze are ideally tightly related $\Rightarrow$ "neutral" flow

$\Rightarrow$ No generation of $\bar{B}$

$\Rightarrow$ To generate $\bar{B}$ you need a net current density:
$$\bar{J} = Ze\overset{\overset{n_i = \frac{n_e}{Z}}{\downarrow}}{n_i}\bar{v}_i - en_e\bar{v}_e = en_e(\bar{v}_i - \bar{v}_e) \Rightarrow \bar{v}_i \neq \bar{v}_e$$

- Induction equation with ideal coupling

$$\frac{\partial \bar{B}}{\partial t} = \underbrace{\frac{c^2}{4\pi\sigma}\bar{\nabla}^2\bar{B} + \bar{\nabla}\times(\bar{v}\times\bar{B})}_{\propto B}$$

$\Rightarrow$ if $B=0$ from the start, $\frac{\partial \bar{B}}{\partial t} = 0$ $\forall$ time

i.e. $\bar{B}$ are not generated

to have $\frac{\partial \bar{B}}{\partial t} \neq 0$ you need a $\bar{B}$ from the start

$\rightarrow$ this is because we assumed perfect coupling between e^- and Ze

- To loose this assumption, consider v_{e^-}, v_i separately (2 fluids)

$\Rightarrow$ 2 Euler's equations $\Rightarrow$

(ϕ = grav. potential)

$$\left[\begin{array}{l} n_e m_e \frac{d\bar{v}_e}{dt} = -\bar{\nabla}P_e - n_e e\left(\bar{E} + \frac{\bar{v}_e}{c}\times\bar{B}\right) - n_e m_e\bar{\nabla}\phi \\ n_i m_i \frac{d\bar{v}_i}{dt} = -\bar{\nabla}P_i + n_i Ze\left(\bar{E} + \frac{\bar{v}_i}{c}\times\bar{B}\right) - n_i m_i\bar{\nabla}\phi \end{array}\right.$$

e^-, Ze coupled by $\bar{E}, \bar{B}, \phi$

- Combine these 2 equations: $(m_i \gg m_e)$

$$\frac{d(\bar{v}_e - \bar{v}_i)}{dt} = -\frac{\bar{\nabla}P_e}{n_e m_e} - \frac{e}{m_e}\left(\bar{E} + \frac{\bar{v}_e}{c}\times\bar{B}\right) \cancel{-\bar{\nabla}\phi} + \frac{\bar{\nabla}P_i}{n_i m_i} + \frac{Ze}{m_i}\left(\bar{E} + \frac{\bar{v}_i}{c}\times\bar{B}\right) \cancel{+\bar{\nabla}\phi}$$

$$\approx -\frac{\bar{\nabla}P_e}{n_e m_e} - \frac{e}{m_e}\left(\boxed{\bar{E}} + \frac{\bar{v}_e}{c}\times\bar{B}\right) + \leftarrow$$

(gravity vanishes!)

- Include collision term (phenomenological): $-\left(\frac{\bar{v}_e - \bar{v}_i}{\tau}\right)$

collision time

- Assume stationary flow, i.e. $\frac{d(\bar{v}_e-\bar{v}_i)}{dt}=0$

$$\Rightarrow \quad \bar{j} = e n_e \underbrace{(\bar{v}_i-\bar{v}_e)}\ (= \text{const})$$

$$\Rightarrow \quad \bar{E} = -\frac{\bar{\nabla} P_e}{e n_e} - \frac{\bar{v}_e}{c}\times\bar{B} + \frac{m_e}{e}\frac{(\bar{v}_e-\bar{v}_i)}{\tau}$$

$$+ \frac{m_e \bar{j}}{e^2 n_e \tau}$$

Note:

- Lorentz force: $\bar{F}_{\text{Lorentz}} = e\left(\bar{E}+\frac{\bar{v}_e}{c}\times\bar{B}\right) = e\left(\underset{(1)}{-\frac{\bar{\nabla}P_e}{e n_e}} \cancel{-\frac{\bar{v}_e}{c}\times\bar{B}} + \underset{(2)}{\frac{m_e\bar{j}}{e^2 n_e\tau}} \cancel{+\frac{\bar{v}_e}{c}\times\bar{B}}\right)$

- Without (2) you have equilibrium when F_{Lortz} balance the pressure gradient force

$\Rightarrow$ it means that (2) enhances $\bar{E}$

- $\bar{E}$ effects evolution of $\bar{B}$ (Faraday's law)

$$\frac{\delta\bar{B}}{\delta t} = -c\bar{\nabla}\times\bar{E} = \frac{c}{e}\underset{①}{\nabla\times\frac{\bar{\nabla}P_e}{n_e}} + \bar{\nabla}\times(\bar{v}_e\times\bar{B}) - \frac{m_e c}{e^2\tau}\underset{②}{\bar{\nabla}\times\frac{\bar{j}}{n_e}}$$

Ampere's law $\bar{j} = \frac{c}{4\pi}\bar{\nabla}\times\bar{B}$

$\left(\begin{aligned}&\bar{\nabla}\times(\varphi\bar{F}) =\\ &=\bar{\nabla}\varphi\times\bar{F}+\varphi\bar{\nabla}\times\bar{F}\end{aligned}\right)$

$$① \quad \bar{\nabla}\times\frac{\bar{\nabla}P_e}{n_e} = \bar{\nabla}n_e^{-1}\times\bar{\nabla}P_e + n_e^{-1}\underset{=0}{\bar{\nabla}\times\bar{\nabla}P_e} = -\frac{\bar{\nabla}n_e}{n_e^2}\times\bar{\nabla}P_e = \bar{\nabla}P_e\times\frac{\bar{\nabla}n_e}{n_e^2}$$

$$② \quad \bar{\nabla}\times\frac{\bar{j}}{n} = \underset{\approx 0}{\bar{j}\times\frac{\bar{\nabla}n_e}{n_e^2}} + n_e^{-1}\bar{\nabla}\times\bar{\nabla}\bar{j}$$

$\bar{j} = \frac{c}{4\pi}\bar{\nabla}\times\bar{B}$ Ampere's law

$\bar{j}\,/\!/\,\bar{\nabla}n_e$ good approx

$$\approx \frac{c}{4\pi n_e}\bar{\nabla}\times(\bar{\nabla}\times\bar{B})$$

$$= \frac{c}{4\pi n_e}\left[\bar{\nabla}(\bar{\nabla}\bar{B})-\bar{\nabla}^2\bar{B}\right] = -\frac{c\bar{\nabla}^2\bar{B}}{4\pi n_e}$$

$\bar{\nabla}\bar{B}=0$ Maxwell

(Battery mechanism)

$$= \underbrace{\bar{\nabla}P_e\times\frac{\bar{\nabla}n_e}{n_e^2}} + \bar{\nabla}\times(\bar{v}_e\times\bar{B}) + \frac{m_e c^2}{e^2\tau n_e 4\pi}\bar{\nabla}B^2$$

$\frac{m_e c^2}{e^2\tau n_e} \to \sigma^{-1}$

$\sigma \equiv \frac{e^2\tau n_e}{m_e}$ conductivity

- new source term!

$\Rightarrow$ generation of $\bar{B}$ if $\bar{\nabla}P$ and $\bar{\nabla}n_e$ are misaligned (even if $\bar{B}(t=0)=0$)

Continuity equation

$$\boxed{\partial_t \rho + \bar{\nabla}(\rho \bar{v}) = 0}$$

it is just matter conservation

⇒ it is the same for neutral or charged particles

⇒ it does not effected by any $\bar{B}$

— o —

⊨ The full set of equations is:

- Induction equation $\bar{B} \longleftrightarrow \bar{v}$	3 equations
- Euler / Navier-Stokes	3 "
- Continuity	1 "
- Eq. of state $P = P(\rho)$	1 "
	8 equations

$\Updownarrow$ (closed system)

- Variables are: ρ, $\bar{v}$, P, $\bar{B}$

(1) (3) (1) (3) = 8 variables

$\Downarrow$

$$\bar{E} = \frac{\bar{J}}{\sigma} - \bar{\beta} \times \bar{B} \qquad \bar{J} = \frac{c}{4\pi} \bar{\nabla} \times \bar{B}$$

$\wr$ derived from $\bar{B}$ Ampere's law

Ohm's law Lorentz tranf

$$\bar{J} \approx \bar{j}' = \sigma \bar{E}' \qquad \bar{E}' = \bar{E} + \bar{\beta} \times \bar{B}$$

Entropy

- For gas we have:

$$\rho T \frac{d\tilde{s}}{dt} = \underbrace{\overset{\text{heat conduction}}{\bar{\nabla}(\kappa \bar{\nabla} T)} + \overset{\text{viscous friction}}{\nu_{ij} T_d^{ij}}}_{} \quad \text{(Entropy evolution)}$$

[$= 0$ if $\kappa = 0 = \eta = \zeta$ (no transport)
$\Downarrow$
entropy density is conserved]

- For plasma: need to include Ohmic heat

$$\frac{dP}{dV} = \bar{J}' \cdot \bar{E}' = \frac{\bar{j}'^2}{\sigma} \approx \overset{\text{maxwell (Ampere)}}{\frac{\bar{J}^2}{\sigma}} = \frac{c^2}{16\pi^2 \sigma} (\bar{\nabla} \times \bar{B})^2$$

$\bar{J}'\bar{E}'$ (it is in plasma rest frame because it is a plasma property)

$\bar{E} = \frac{\bar{J}}{\sigma}$ (conductivity)

$$\Rightarrow \quad \boxed{\rho T \frac{d\tilde{s}}{dt} = \bar{\nabla}(\kappa \bar{\nabla} T) + \nu_{ij} T_d^{ij} + \underbrace{\frac{c^2}{16\pi^2 \sigma} (\bar{\nabla} \times \bar{B})^2}_{\text{New term}}}$$

$\frac{1}{\sigma}$ = resistivity

$(J^2 R)$

$\Downarrow$

Even if there are no heat conduction and viscous friction (i.e. diffusion processes)
specific entropy evolves because of $\bar{\nabla} \times \bar{B}$

$\Big(= \frac{4\pi}{c} \bar{J}$ currents in plasma $\Big)$

Energy

- Instead of $\tilde{s}$ expres in terms of internal energy ε $\quad q = \overset{\text{internal energy}}{\bar{q}} + \bar{q}_\varepsilon + \bar{q}_{\text{friction}}$

- For gas we have: $\partial_t \varepsilon + \underset{\text{energy flux}}{\bar{\nabla}(\varepsilon \bar{v})} + \underset{\text{work}}{P \bar{\nabla} \bar{v}} = \underset{\text{(diffusive heat transfer)}}{\bar{\nabla}(K \bar{\nabla} T)} + \underset{\text{(heat by friction)}}{v_{ij} T_d^{ij}}$

- Add energy density of the $\bar{B}$ field:

$$\frac{|\bar{B}|^2}{8\pi}$$

, Poynting vector of the E.M field

$$\left(\frac{\rho}{2}\bar{v}^2 + \varepsilon \rightarrow \frac{\rho}{2}\bar{v}^2 + \varepsilon + \frac{|\bar{B}|^2}{8\pi} \right)$$

$$\bar{S} = \frac{c}{4\pi} \bar{E} \times \bar{B} \qquad \bar{E} = \frac{\bar{j}}{\underset{\text{conductivity}}{\sigma}} - \bar{\beta} \times \bar{B} \qquad \bar{j} = \frac{c}{4\pi} \bar{\nabla} \times \bar{B} \ \text{Ampere}$$

$$= \frac{c^2}{16\pi^2\sigma} (\bar{\nabla} \times \bar{B}) \times \bar{B} - \frac{1}{4\pi} (\bar{\beta} \times \bar{B}) \times \bar{B}$$

$$= \frac{c^2}{16\pi^2\sigma} \left[(\bar{B}\bar{\nabla})\bar{B} - \frac{1}{2}\bar{\nabla}(\bar{B}^2) \right] - \frac{1}{4\pi} \left[(\bar{B}\bar{v})\bar{B} - (\bar{B}^2)\bar{v} \right]$$

- Add contribution to current densities in energy conservation

$$q^i = \underbrace{\underbrace{\rho\left(\frac{v^2}{2} + \hat{h}\right) v^i}_{1} - \underbrace{K \partial^i T}_{2} - \underbrace{v_j T_d^{ij}}_{3}}_{\text{gas}} - \underbrace{\boxed{\underset{\uparrow (!)}{\frac{c^2}{16\pi^2\sigma}} \left[\underbrace{\text{——}}_{4} - \underbrace{\text{——}}_{5} \right] - \frac{1}{4\pi} \left[\underbrace{\text{——}}_{6} - \underbrace{\text{——}}_{7} \right]}}_{\text{additional term!}}$$

1) Transport of kinetic energy and specific enthalpy
2) Entropy loss by heat (work)
3) " " " viscous friction

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4) Magnetic tension
5) Gradient of the internal energy of $\bar{B}$

} ≈ for $\sigma \rightarrow \infty$ ideal conductive

6) Change of $\bar{B}$ along the flow lines of the fluid
7) Transport of $\bar{B}$ energy with the fluid

$\boxed{\text{For incompressible flow}}$: $\bar{\nabla}\bar{v} = 0$

⇒ induction equation simplify:

$$\frac{\delta B}{\delta t} = \frac{c^2}{4\pi\sigma}\bar{\nabla}^2\bar{B} + \bar{\nabla}\times(\bar{v}\times\bar{B}) = \frac{c^2}{4\pi\sigma}\bar{\nabla}^2\bar{B} + \bar{v}(\bar{\nabla}\bar{B}) - \underbrace{\bar{B}(\bar{\nabla}\bar{v})}_{=0}$$

⇒ Diffusive stress energy tensor simplify:

$$T_d^{ij} = -\eta\Big(\underbrace{\partial^i v^j + \partial^j v^i}_{\equiv 2v^{ij}} - \underbrace{\frac{2}{3}\delta^{ij}\bar{\nabla}\bar{v}}_{=0}\Big) - \underbrace{\xi\,\delta^{ij}\bar{\nabla}\bar{v}}_{=0} = -2\eta v^{ij}$$

↑

velocity gradient tensor

⇒ ξ disappears : only the viscosity η remains in the Navier-Stokes eq.

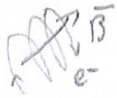
Other facts about plasmas

- In case you have a gas + plasma

- gas does not feel $\bar{B}$ ⇒ moves free from it
- Coupling between the 2 given by collisions
  ⇒ Friction between the two
  ↳ affects induction equation : diffusion of $\bar{B}$

- Faraday rotation

☐ Left and circular polarization states in one waves propagates differently
- they have different dispersion relations
- expected : given the $\bar{B}$ in the magnetized plasma $\bar{e}$ orbit in the same way
  this brakes the symmetry between the 2 polariz. states



— o —

▭ Linearly polarized light = superposition of 2 circular polarization with same intensity and phase
- The 2 propagates differently ⇒ change of phase (shift) ψ

⇓

(Faraday) rotation of the polarization angle

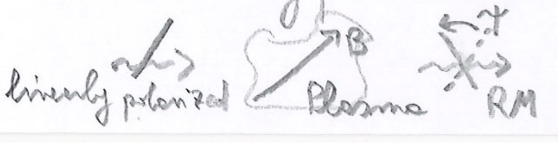
$$\boxed{\psi = \frac{2\pi e^3}{m^2 c^2 \omega^2} \underbrace{\int dz\, n_e B}_{\equiv RM}}$$

component along the line of sight

Rotation measure

- RM has a frequency dependency (!)

⇒ if linearly polarized emission (e.g. synchrotron)



$\cancel{B}$ ⟼ observing in at least 2 ≠ ω and knows n_e ⇒ measure B!

Part VI

Stellar dynamics

Stellar dynamics: summary

- No direct collisions
- Interaction via gravity ⇒ no screening as in plasma (for General Relativity)
  ⇒ long range force
  ⇒ one particle feels all other particles
- Evolution of the system ⟶ Boltzman equation

$$\partial_t f + \bar{v}\bar{\nabla} f - \bar{\nabla}\phi\,\bar{\nabla}_{\bar{v}} f = 0$$

$f(\bar{x}, \bar{v}, t)$ probability distribution of particles in the phase-space

- 1° momentum of Boltzmann eq., i.e. $\int d^3v$ (Boltzmann eq.) $= 0$

(1) $\Rightarrow \partial_t n(\bar{x}, t) + \bar{\nabla}(n\langle\bar{v}\rangle) = 0$ continuity eq.

n is a probability not the actual # of stars

- 2° momentum of Boltzman eq., i.e. $\int d^3v\, x^M$ (Boltzmann eq.) $= 0$

(2) $\Rightarrow \partial_t v^j + v^i\partial_i v^j = -\partial_i\phi - \underbrace{\partial_i[n(\sigma^2)^{ij}]}$ Euler eq.

↳ Equivalent to pressure term in hydrodynamics

- (1) and (2) goes under the name of Jeans equations
- Spherical systems, static, caotic orbits: $\partial_r(n\sigma_r^2) + (n\sigma_r^2)\frac{2\beta(r)}{r} = n\partial_r\phi$

$$\partial_r(n\sigma_r^2) + (n\sigma_r^2)\frac{2\beta(r)}{r} = -n\partial_r\phi \quad \text{(Euler eq.)}$$

$$n\sigma_r^2 = G\int_r^\infty \frac{M(y)n(y)}{y^2}\exp\left(2\int_r^y \frac{\beta(x)}{x}dx\right)$$

Solution of Euler eq.
⇒ measure M via observations

Stellar dynamic

star = any masspoint

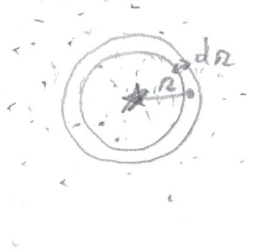
- Hydrodynamics: direct collisions $\lambda \ll L$ system
- Plasma physics: screened coulomb interaction $l \ll \lambda_D$

} short range interactions

- Stellar systems: no direct collisions
  gravitational interaction → long range!

⇒ Each star feels the force of all other stars

example:



2D infinite, homogeneous distributed stars

- [n] surface density
- All stars with same mass

$$dN = n \cdot 2\pi R\, dR \qquad \text{\# of stars in ring}$$

$$dF = n\, 2\pi R\, dR \frac{Gm^2}{R^2}$$

$$F = \int dF = n 2\pi G m^2 \int \frac{dR'}{R'} = 2\pi n G m^2 \underline{\underline{\ln(R)}}$$

!

Force on the star ★ comes from all distances: ln(R)

macroscopic force as in hydrodynamics
(from collection of many particles)
⇓
here: neglect microscopic interactions on large scales

⇒ Use Collisionless Boltzmann eq. (Phase space distribution f)

$$d_t f(\bar{x},\bar{v},t) = \partial_t f + \overset{\swarrow}{\dot{\bar{x}}}\bar{\nabla} f + \overset{\swarrow}{\dot{\bar{v}}}\bar{\nabla}_{\bar{v}} f = 0 \qquad \dot{\bar{v}} = \frac{\bar{F}}{m} = -\bar{\nabla}\phi(\bar{x},t)$$

$$\boxed{\partial_t f + \bar{v}\bar{\nabla} f - \bar{\nabla}\phi\bar{\nabla}_{\bar{v}} f = 0} \qquad \dot{\bar{x}} = \bar{v}$$

- $\int d^3v\, f(\bar{v},\bar{x},t) = n(\bar{x},t)$ ← this is a probability, not an actual number density ≠ from smooth matter density of a fluid
- $\frac{1}{n}\int d^3v\, v f(\bar{v},\bar{x},t) = \langle\bar{v}(\bar{x},t)\rangle$
  - discrete collection of points ⇒ considerable fluctuations (Poissonian realization) $\sqrt{n}$
  - $n \to \rho$ difficult, stars have large range of masses

- Integrate over velocities to get spatial distribution $f(\bar{x}, t)$ (1° moment)

$$\underbrace{\partial_t \int_{-\infty}^{\infty} d^3v\, f}_{} + \underbrace{\int_{-\infty}^{\infty} d^3v\, \bar{v}\bar{\nabla} f}_{} - \bar{\nabla}\phi \underbrace{\int_{-\infty}^{\infty} d^3v\, \bar{\nabla}_{\bar{v}} f}_{=0} = 0$$

($\phi(\bar{x})$, no dependency on $\bar{v}$)

(particles have finite velocities)

$f(\bar{x}, \bar{v}, t) \to 0$ for $|\bar{v}| \to \infty$

$$\langle \bar{v} \rangle = \frac{1}{n}\int d^3v\, \bar{v} f(\bar{v}, \bar{x}, t)$$

$$(1) \quad \boxed{\partial_t n(\bar{x}, t) + \bar{\nabla}(n\langle \bar{v} \rangle) = 0}$$

"Continuity equation"

- Go for velocity dispersions : (2° moments) $\int d^3v\, (\;)^i \cdot v^j$

$$\partial_t \int d^3v\, f v^j + \int dv^3\, \partial_i f\, v^i v^j - \partial_i \phi \int d^3v\, \frac{\partial f}{\partial v^i} v^j = 0$$

integrate by part

$$(2) \quad \partial_t(n\langle v^j \rangle) + \partial_i(n\underbrace{\langle v^i v^j \rangle}_{}) - \partial_i\phi \underbrace{\Big[\underbrace{f v \Big|_{-\infty}^{\infty}}_{=0} - \underbrace{\int d^3v\, f \frac{\partial v^j}{\partial v^i}}_{\circledast} \Big]}_{-\partial_j \phi \cdot n} = 0$$

$$\circledast = \int d^3v\, f \delta^{ij} = \delta^{ij} \cdot n$$

$$\langle v^i v^j \rangle = \langle (v^i - \langle v^i \rangle)(v^j - \langle v^j \rangle) \rangle + \langle v^i \rangle \langle v^j \rangle = (\sigma^2)^{ij} + \langle v^i \rangle \langle v^j \rangle$$

Velocity dispersion tensor

- Combine (1) and (2) : (eq 2) $-\langle v^j \rangle \cdot$ (eq. 1)

$$\underbrace{\partial_t(n\langle v^j \rangle)}_{} + \partial_i(n\langle v^i v^j \rangle) + n\partial_j\phi - \cancel{\langle v^j \rangle \partial_t n} - \langle v^j \rangle \partial_i(n\langle v^i \rangle) = 0$$

$$(n\partial_t\langle v^j \rangle + \cancel{\langle v^j \rangle \partial_t n})$$

$$n\partial_t\langle v^j \rangle + \partial_i(n\langle v^i v^j \rangle) - \langle v^j \rangle \partial_i(n\langle v^i \rangle) = -n\partial_j\phi$$

$$n\partial_t\langle v^j \rangle + \partial_i\Big(n(\sigma^2)^{ij} + \underbrace{n\langle v^i \rangle \langle v^j \rangle}_{\langle v^j \rangle \partial_i(n\langle v^i \rangle) + n\langle v^i \rangle \partial_i \langle v^j \rangle}\Big) - \cancel{\langle v^j \rangle \partial_i(n\langle v^i \rangle)} = -n\partial_j\phi$$

$$\cancel{\langle v^j \rangle \partial_i(n\langle v^i \rangle)} + n\langle v^i \rangle \partial_i \langle v^j \rangle$$

(velocity dispersion plays the role of a pressure)

$$\Rightarrow \quad \boxed{\partial_t\langle v^j \rangle + \langle v^i \rangle \partial_i \langle v^j \rangle = -\partial_j\phi - \partial_i\left[n(\sigma^2)^{ij}\right]\frac{1}{n}}$$

"Euler" eq.

- These are the Jeans equations:

$$\boxed{\begin{aligned} &\partial_t n + \overline{\nabla}(n\overline{v}) = 0 \\ &\partial_t v^j + v^i \partial_i v^j = -\partial_i \phi - \partial_i \left[n (\sigma^2)^{ij} \right] \frac{1}{n} \end{aligned}}$$

(here $\langle v^i \rangle \equiv v^i$)

$$\downarrow$$

$$\frac{\nabla P}{\rho} = \rho^{-1} \delta^{ij} \partial_j P$$

Equivalent of pressure in Hydro. eq. (Euler eq.)

$\boxed{\text{Spherical systems}}$ (e.g. elliptical galaxies)

– Assumptions:

- $\langle v_\varphi \rangle = 0 = \langle v_\theta \rangle$ φ: polar
- $\langle v_r v_\theta \rangle = 0 = \langle v_r v_\varphi \rangle$ velocity components are statistically uncorrelated

$\Rightarrow$ Isotropy in velocity space (caotic motions) ⊛

- Static system $\Rightarrow$ all time derivatives $= 0$

$$\Rightarrow \partial_r (n \sigma_r^2) + \frac{n}{r} \left[2\sigma_r^2 - (\sigma_\theta^2 + \sigma_\varphi^2) \right] = -n \partial_r \phi$$

$$\sigma^2_{r,\theta,\varphi} = \int d^3v \, v^2_{r,\theta,\varphi} \cdot f$$

Velocity dispersion

(derived straight from Boltzmann eq.)

- Because of ⊛ $\sigma_\theta^2 = \sigma_\varphi^2$
- Define anisotropic parameter β: $\sigma_\theta^2 = \sigma_\varphi^2 = \sigma_r^2 (1-\beta)$

$\beta > 0 \Rightarrow \sigma_\theta^2 = \sigma_\varphi^2 < \sigma_r^2 \Rightarrow$ radial motions dominates

$\beta < 0 \Rightarrow \sigma_\theta^2 = \sigma_\beta^2 > \sigma_r^2 \Rightarrow$ tangential " "

- In general $\beta(r)$

1° order, linear, ordinary, diff. eq.

$$\Rightarrow \partial_r (n \sigma_r^2) + \frac{n}{r} \left[\cancel{2\sigma_r^2} - 2(\cancel{1}-\beta)\sigma_r^2 \right] = \boxed{\partial_r (n\sigma_r^2) + (n\sigma_r^2) \frac{2\beta(r)}{r} = -n \partial_r \phi}$$

- Solve with variation of constants

$$n\sigma_r^2 = G\int_r^\infty dy\left[\frac{M(y)\,n(y)}{y^2}\exp\left(2\int_r^y\frac{\beta(x)}{x}dx\right)\right]$$

DM $\Uparrow$

- Here we used $\partial_r\phi = \frac{GM(r)}{r^2}$, ϕ is the total potential (not stars only)

- Observations:
  - $n(r)$ obtain by luminosity of the galaxy (stellar density)
  - Assume a $\beta(r)$
  - $\sigma(r)$ obtaind with spectroscopy (Projected along the line of sight)

$\Rightarrow$ you can measure the mass M