Cosmology

Prof. Dr. Matteo Maturi

Winter Semester 2023/24

Contents

00 introduction . . . 2
01a Special relativity concept . . . 20
01b Lorentz geometry . . . 26
01c Lie-algebra and the Lorentz group . . . 32
02 Relativistic mechanics . . . 36
03 Energy-momentum tensor cut . . . 43
04a General relativity in a nutshell . . . 46
04b Curvature-tenosr more details (optional) . . . 56
05 Lambda-CDM cosmological model . . . 65
06 Distances and horizons . . . 75
07 Dynamical dark energy . . . 78
08a Cosmological inflation . . . 83
08b Inflationary perturbation seeds . . . 91
09a Thermal history . . . 97
09a Thermal history key concepts and Boltzmann eq . . . 117
09b Thermal history quantum gasses . . . 121
09c Thermal history baryons aboundance and recombination and photons decoupling . . . 128
09d Thermal history cosmological nucleosynthesis and neutrinos decouplint . . . 132
10 Cosmic structure formation Newtonian . . . 141
11 Linear power spectrum . . . 150
13a Spherical collapse and dark matter haloes . . . 157
14a Non-linear structure formation with Numerical simulations . . . 165
14b Merger-tree and semi-analitic modelling . . . 174
15 Supernovae type-1a . . . 183
16 Gravitational lensing . . . 188
16 Gravitational lensing extra pp . . . 201
17a CMB intro and power spectrum . . . 203
17b CMB anisotropies . . . 215
18 Euclid Space mission . . . 223


UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386
Cosmology
WS 2023/24
apl. Prof. Dr. Matteo Maturi
Center for Astronomy & Institute for Theoretical Physics, Heidelberg University

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Bureaucracy first...

Nice to meet you:
Matteo Maturi, Madonna di Campiglio (Dolomites)
Center for Astronomy & Institute for Theoretical Physics
Cosmology, gravitational lensing, galaxy clusters
Involved in Euclid, KiDS, DESC-LSST, J-PAS

Contact:
maturi@uni-heidelberg.de (Matteo, lecturer)
serena.giardino@aei.mpg.de (Serena, head tutor)

Lectures:
From October 16th to February 10th
Tuesday 09:15-11:00 (INF227/HS 2)
Thursday 09:15-11:00 (INF227/HS 2)

Website, Uebungen:
https://uebungen.physik.uni-heidelberg.de/vorlesung/20232/1762

- Enroll!
- Literature
- Lecture notes
- Additional material (slides, pdf files,…)
- Tutorials / Exercises
- dates of exam and all…
- answers to possible questions

No uni-id?
- Write to Serena to enroll you by hand

Tutorials:
- First tutorial next week!
- The exercises will not be corrected and no mark will be given
- Ask the tutors and comment your solutions with them!
- It is possible to hand in exercises to get a feedback

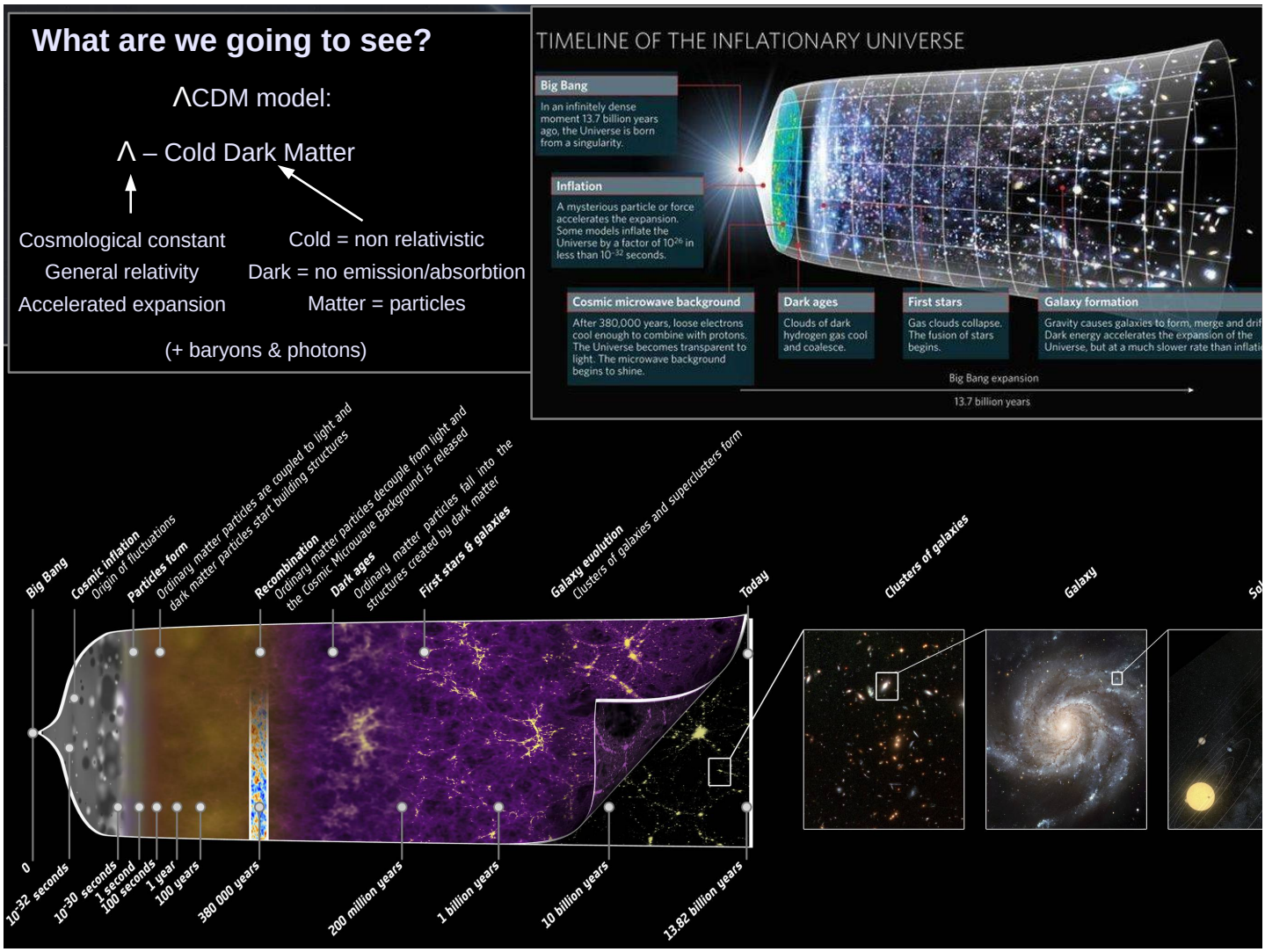
Exam:
Written
Same style of exercises

Admission to the exam: (50% attendance + (3 points)

1) attend at least 50% of the tutorials (your presence will be registered). If attendance < 50%, it is required to hand in 3 full exercise sheets that will be graded.

and

2) gain 3 points by:
- 1 point: present at the black board at least 1/3 of a sheet.
- 1 point: actively participating in the discussion during the tutorials


What are we going to see?
ΛCDM model:
Λ – Cold Dark Matter
Cosmological constant
General relativity
Accelerated expansion
Cold = non relativistic
Dark = no emission/absorbtion
Matter = particles
(+ baryons & photons)
TIMELINE OF THE INFLATIONARY UNIVERSE
Big Bang
In an infinitely dense moment 13.7 billion years ago, the Universe is born from a singularity.
Inflation
A mysterious particle or force accelerates the expansion. Some models inflate the Universe by a factor of 10^{26} in less than 10^{-32} seconds.
Cosmic microwave background
After 380,000 years, loose electrons cool enough to combine with protons. The Universe becomes transparent to light. The microwave background begins to shine.
Dark ages
Clouds of dark hydrogen gas cool and coalesce.
First stars
Gas clouds collapse. The fusion of stars begins.
Galaxy formation
Gravity causes galaxies to form, merge and drif
Dark energy accelerates the expansion of the Universe, but at a much slower rate than inflati
Big Bang expansion
13.7 billion years
Big Bang
Cosmic inflation
Origin of fluctuations
Particles form
Ordinary matter particles are coupled to light and dark matter particles start building structures
Recombination
Ordinary matter particles decouple from light and the Cosmic Microwave Background is released
Dark ages
Ordinary matter particles fall into the structures created by dark matter
First stars & galaxies
Galaxy evolution
Clusters of galaxies and superclusters form
Today
Clusters of galaxies
Galaxy
0
10^{-32} seconds
10^{-30} seconds
1 second
100 seconds
1 year
100 years
380 000 years
200 million years
1 billion years
10 billion years
13.82 billion years

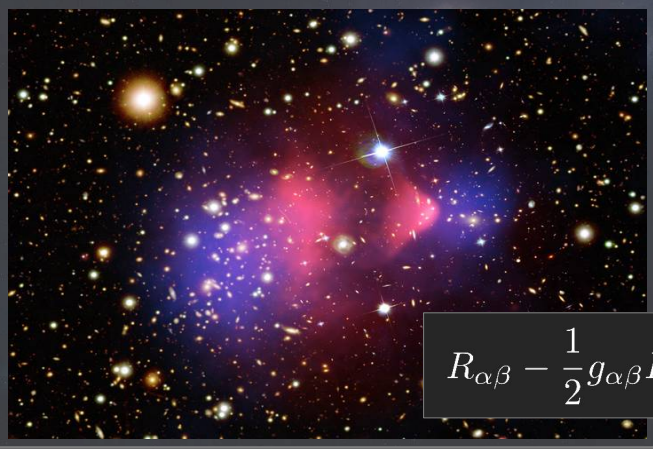
Does the ΛCDM model work?

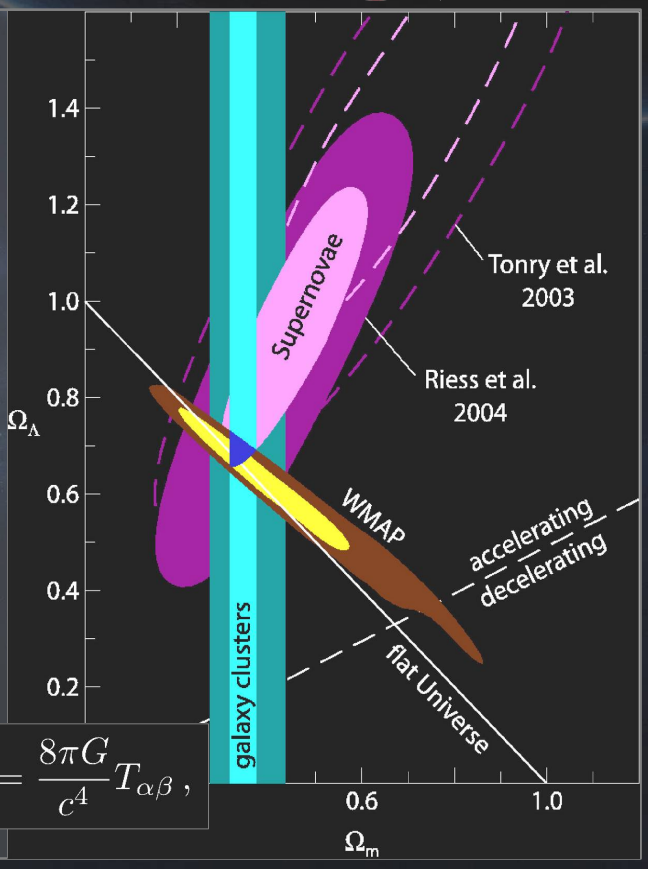
UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Data say so!

- Cosmic Microwave Bckground
- Supernovae luminosity distances
- Gravitational lensing
- Galaxy clusters
- Galaxy clustering
- ....

Dark energy + Dark matter?
or
Is gravity wrong?





$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \,,$$

15/10/23 *Matteo Maturi* 4

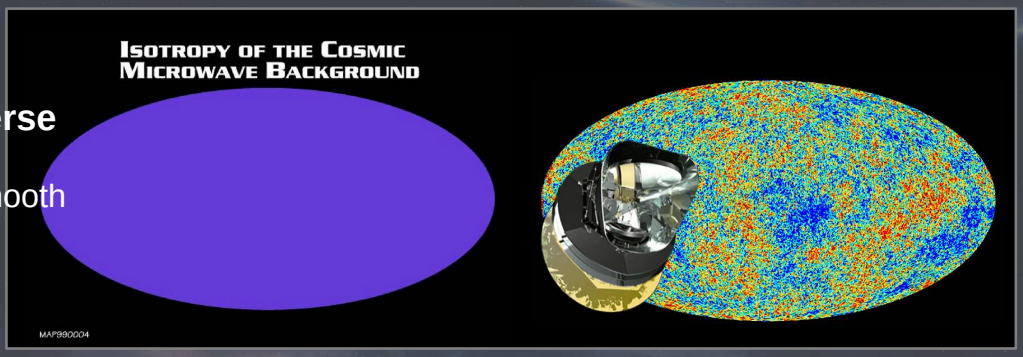
How do we approach it?
Homogeneity and isotropy

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

The young universe

Very uniform and smooth

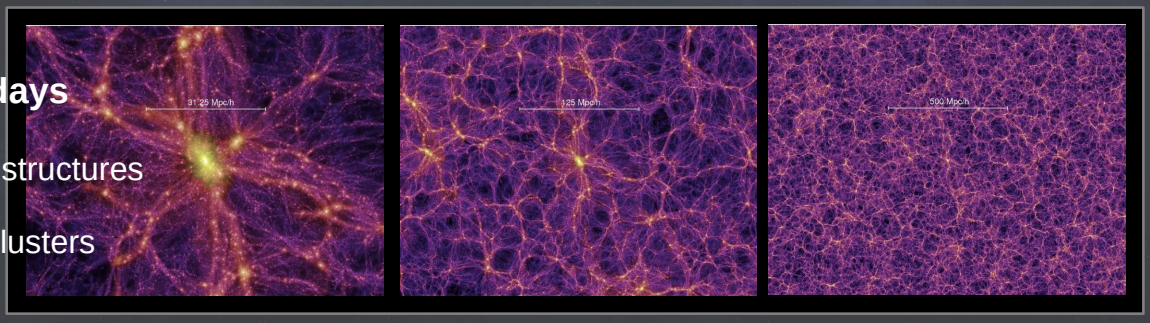
DT/T = 10^{-5}



Now days

Large scale structures

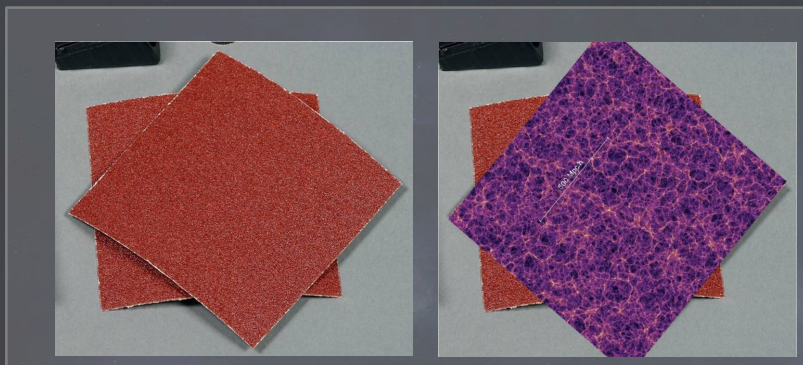
Galaxy clusters



15/10/23 *Matteo Maturi* *5*

Let's build a model of the universe

The universe is like sandpaper: rough but homogeneous and isotropic



To describe sand paper we do not need to describe each single grain!

3 parameters are enough:
*material (**M**), grains dimension (**D**), grains number density (**N**)*

The same for the universe...

15/10/23 *Matteo Maturi* *6*

The ingredients of the cake

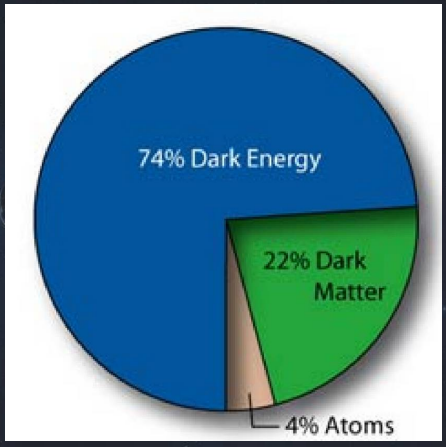
UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Cosmological parameters

Dynamics:
- H_0: Hubble parameter

Ingredients:
- Ω_r : radiation density
- Ω_b : barionic density
- Ω_d : dark matter density
- Ω_Λ: dark energy density

"Raffness":
- σ_8



And the roles of the game?

15/10/23 *Matteo Maturi* *7*

How do the ingredients interact?



UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386



The rules mastering the universe

Theory	Interaction	Behavior	Typical scale
QCD	Strong	$1/r^7$	1.4×10^{-15} m
Electroweak	Weak	$1/r^5$ to $1/r^7$	10^{-18} m
QED	Electromagnetic	$1/r^2$	∞
GR	Gravity	$1/r^2$	∞

We need a long distance force acting on a neutral universe
Gravity!!

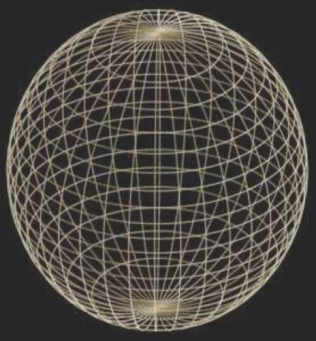
15/10/23 *Matteo Maturi* *8*

Newtonian cosmology

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Consider a spherical portion of the Universe

- radious $l = ar_0$
- density ρ
- mass $M = \rho \frac{4}{3} \pi l^3$
- kinetic energy $A = \frac{kc^2}{2}$ init. condit.



$$acc = -\frac{GM}{l^2}$$

With the previous equations we obtain:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho) \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

15/10/23 *Matteo Maturi* *9*

The standard cosmological model



UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Einstein's field equation

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \,,$$

$$ds^2 = (cdt)^2 - a^2(t) \left[\frac{dr^2}{(1-kr^2)} + r^2 (d\theta^2 + \sin^2\theta d\varphi) \right]$$

Friedmann–Lemaître–Robertson–Walker metric

$$T_{\alpha\beta} = -p g_{\alpha\beta} + \left(p + \rho c^2\right) \mu_\alpha \mu_\beta - \frac{c^4 \Lambda}{8\pi G} g_{\alpha\beta} \,.$$

Universe ingredients

Friedmann's equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

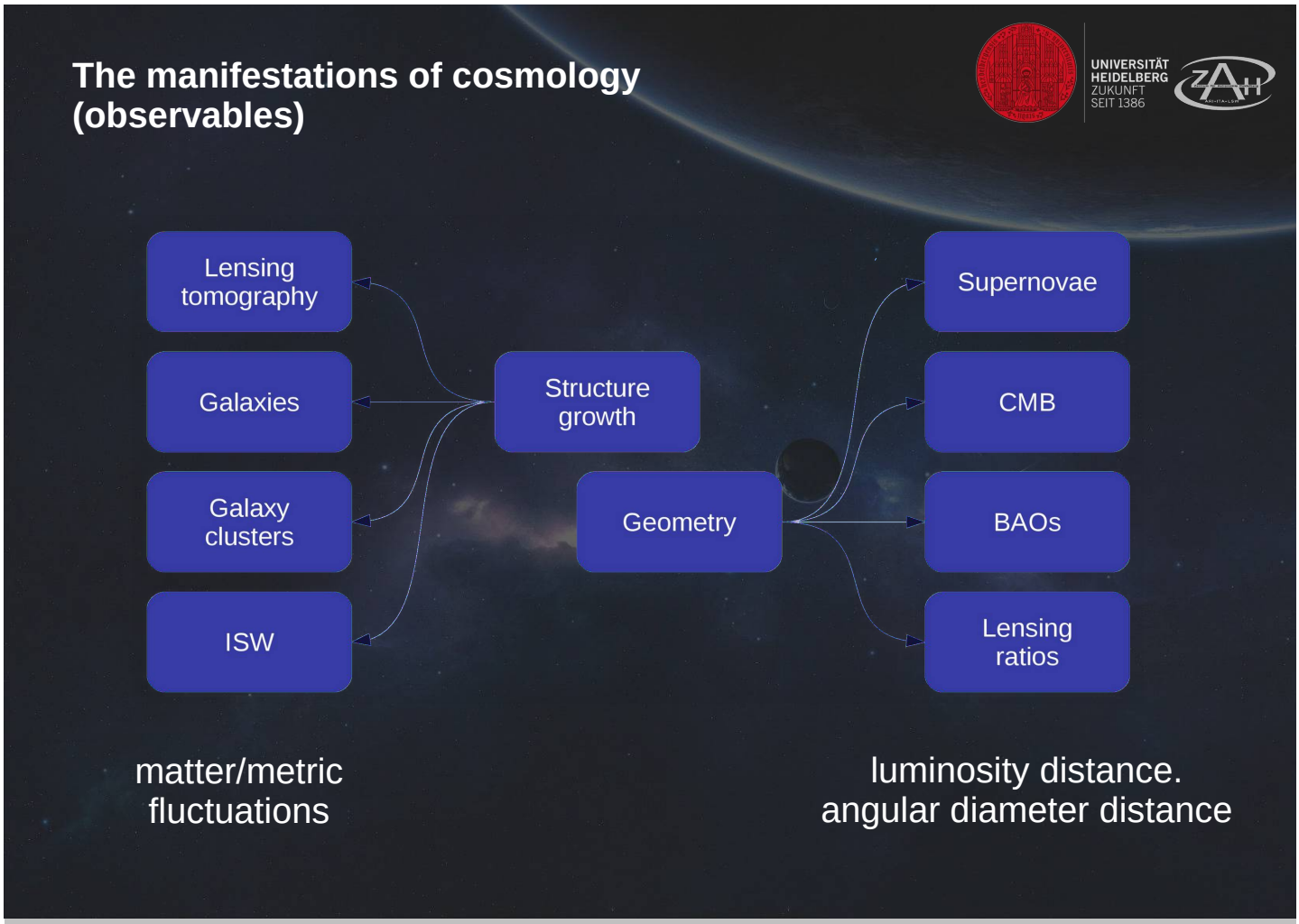
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}$$

Newtonian

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho)$$

15/10/23 *Matteo Maturi* *10*


The manifestations of cosmology
(observables)
UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386
Lensing tomography
Galaxies
Galaxy clusters
ISW
Structure growth
Geometry
Supernovae
CMB
BAOs
Lensing ratios
matter/metric fluctuations
luminosity distance.
angular diameter distance
15/10/23
Matteo Maturi
11

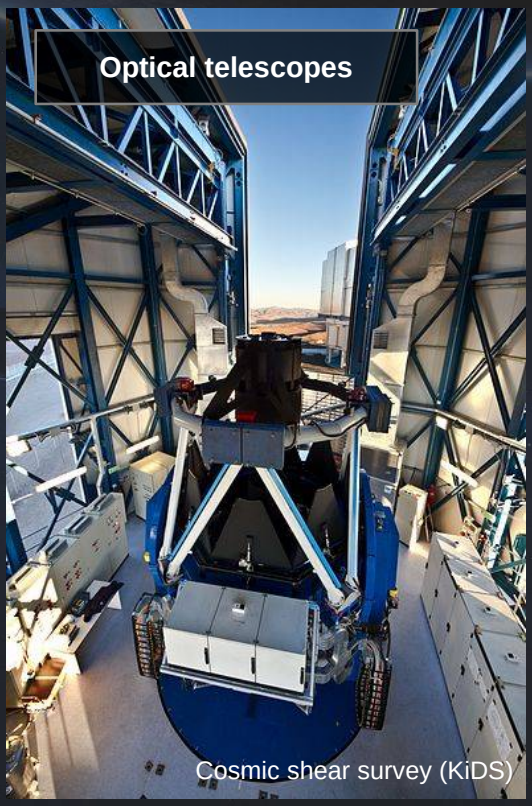
UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

The tools

Space telescopes

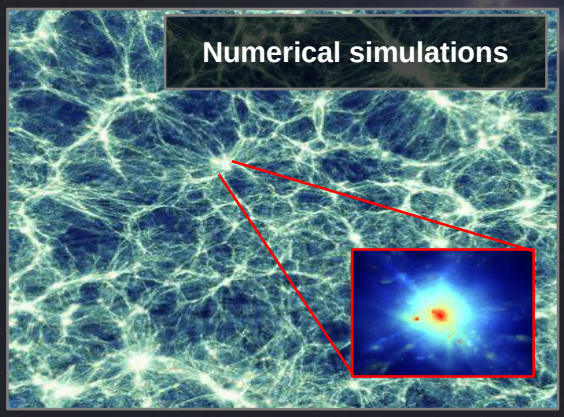


Optical telescopes

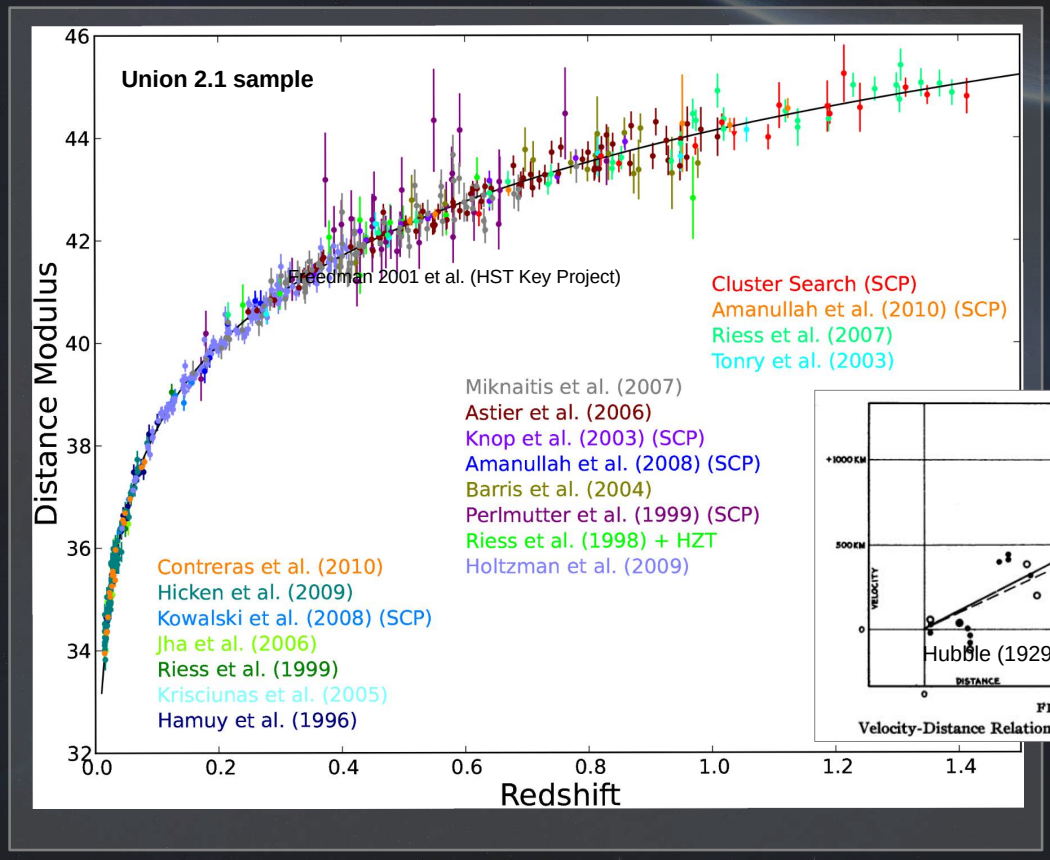


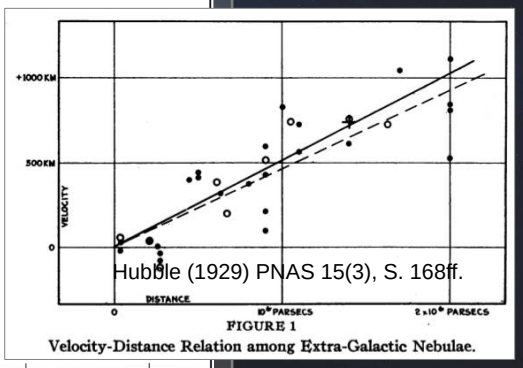
Analytical: 8+1 ~ 10

Numerical simulations



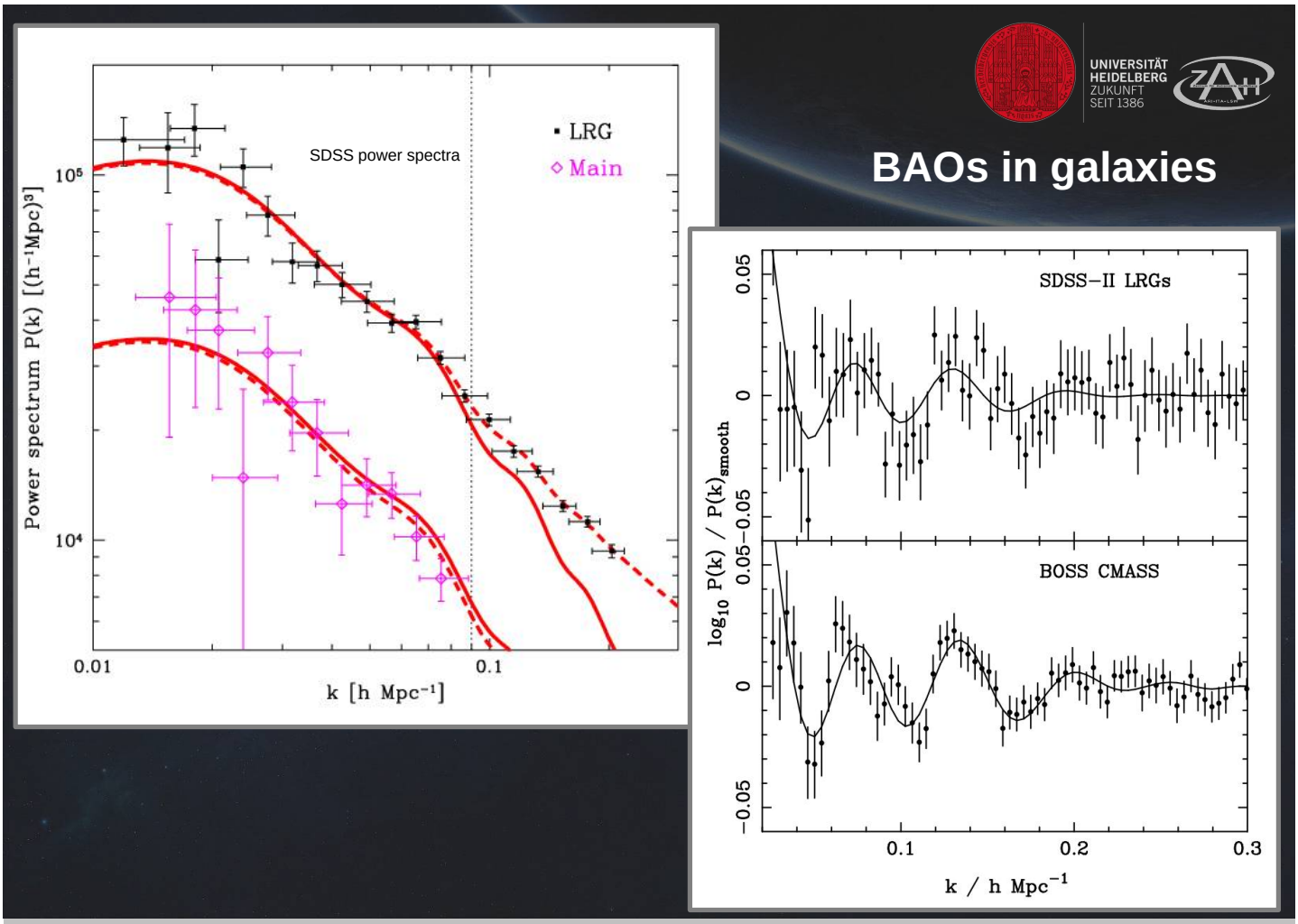
Super Novae

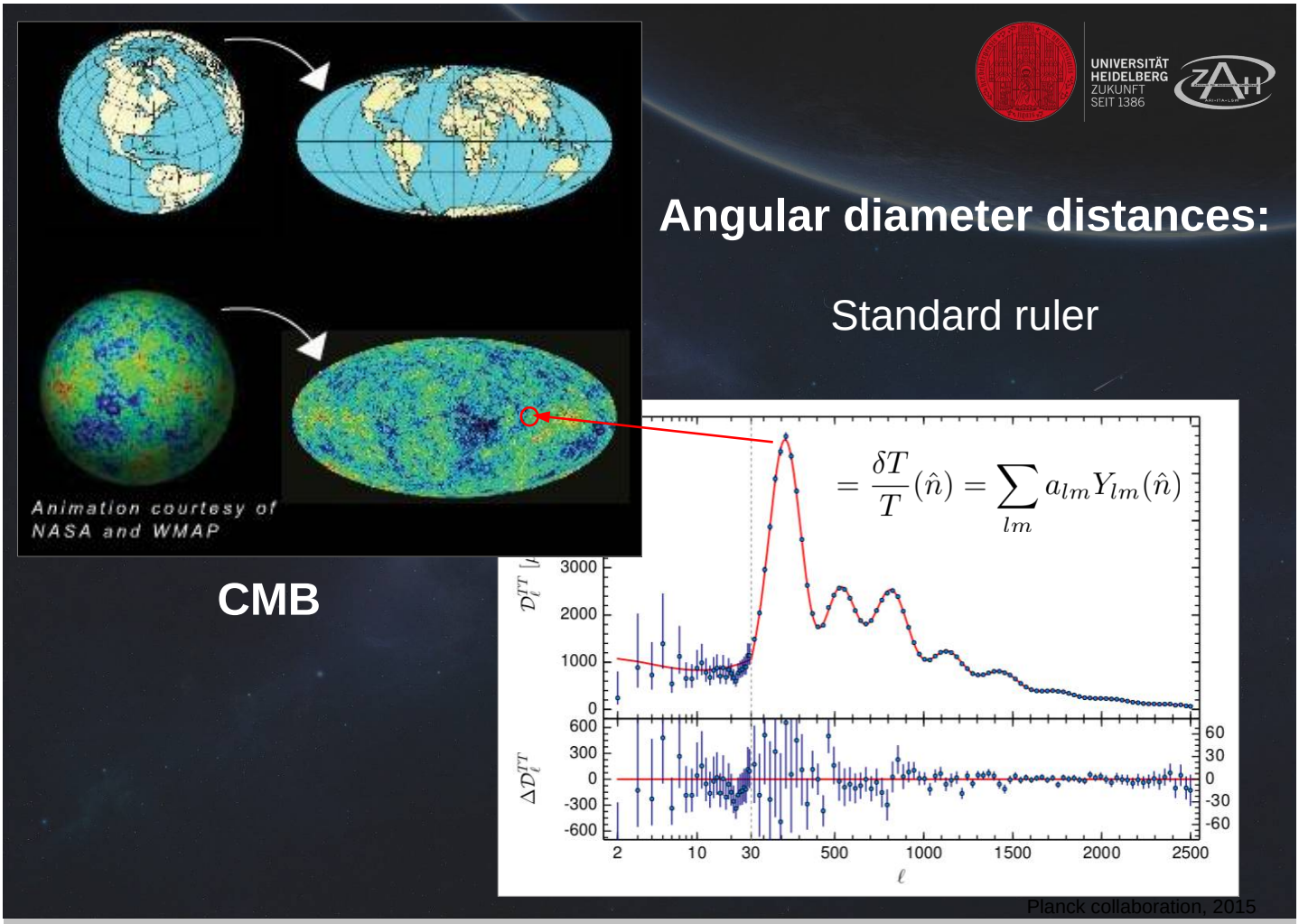




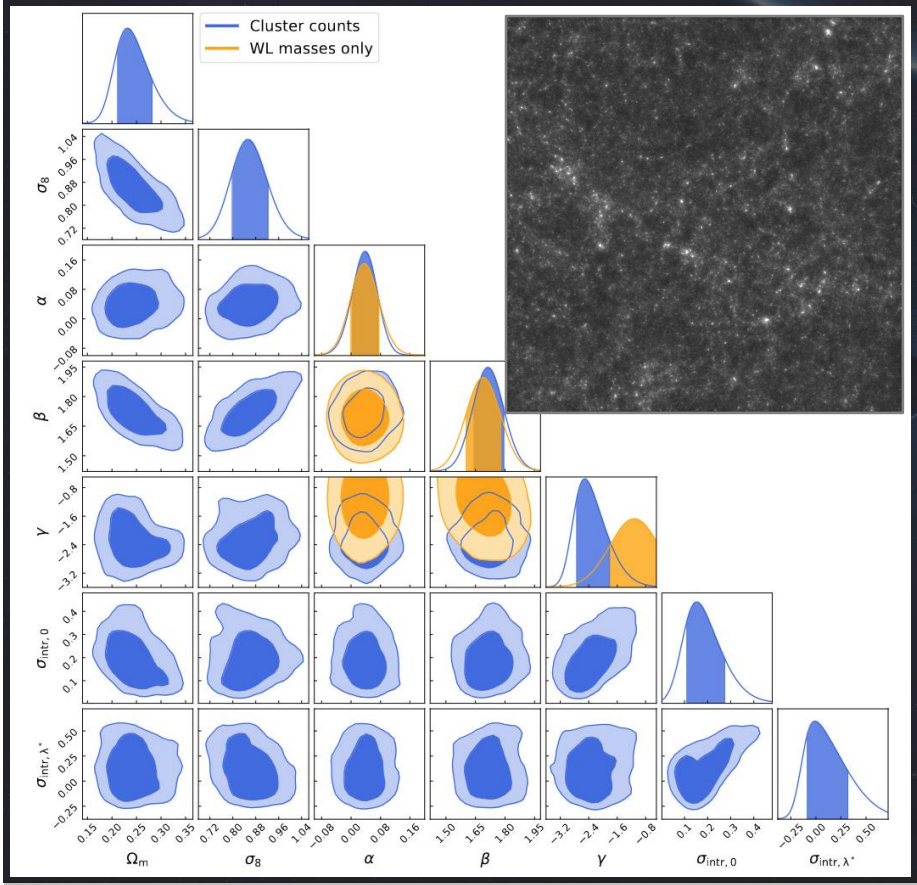
Hubble (1929) PNAS 15(3), S. 168ff.

15/10/23 *Matteo Maturi* *13*


UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386
BAOs in galaxies
SDSS power spectra
LRG
Main
Power spectrum P(k) [(h⁻¹Mpc)³]
10⁵
10⁴
0.01
0.1
k [h Mpc⁻¹]
SDSS−II LRGs
BOSS CMASS
log₁₀ P(k) / P(k)smooth
0.05
0
−0.05
0.1
0.2
0.3
k / h Mpc⁻¹
15/10/23
Matteo Maturi
14


UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386
Angular diameter distances:
Standard ruler
Animation courtesy of
NASA and WMAP
CMB
$= \frac{\delta T}{T}(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$
$\mathcal{D}_\ell^{TT}$
$\Delta\mathcal{D}_\ell^{TT}$
3000
2000
1000
0
600
300
0
-300
-600
60
30
0
-30
-60
2
10
30
500
1000
1500
2000
2500
ℓ
Planck collaboration, 2015
15/10/23 Matteo Maturi 15

Galaxy clusters



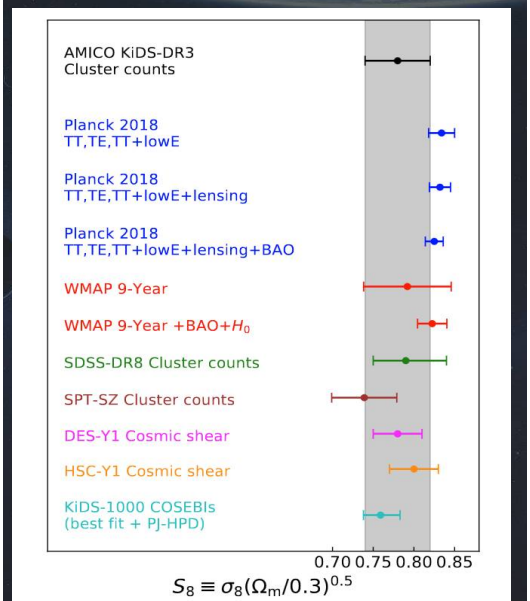


Fig. 7. Comparison of the constraints on $S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$ obtained, *from top to bottom*, from the joint analysis of cluster counts and weak lensing in the AMICO KiDS-DR3 catalogue (black dot), from the results obtained by Planck Collaboration VI (2020) (blue dots), Hinshaw et al. (2013) (red dots), Costanzi et al. (2019) (green dot), Bocquet et al. (2019) (brown dot), Troxel et al. (2018) (magenta dot), Hikage et al. (2019) (orange dot), and Asgari et al. (2021) (cyan dot). The median, as well as the 16th and 84th percentiles are shown.

Content of the lectures



UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386



1) Introduction
- Special and General Relativity

2) Uniform universe
- FLRW metric
- Friedmann equations
- Dark energy
- Cosmic inflation
- Cosmological Nucleosynthesis

3) Thermal History
- Summary of equilibrium thermodynamics
- When reactions starts/end
- Neutronos decouplings (CνB)
- Electrons decoupling (CMB)
- Nucleosynthesys

4) Structure formation
- Hydrodinamics (Continuity, Euler, Poisson)
- Linear theory
- Dark matter halos
- N-Body numerical simulations

5) Current Observations
- Galaxy clustering
- Cosmic Microwave Background (CMB)
- Gravitational lensing
- Galaxy clusters
- Other probes...

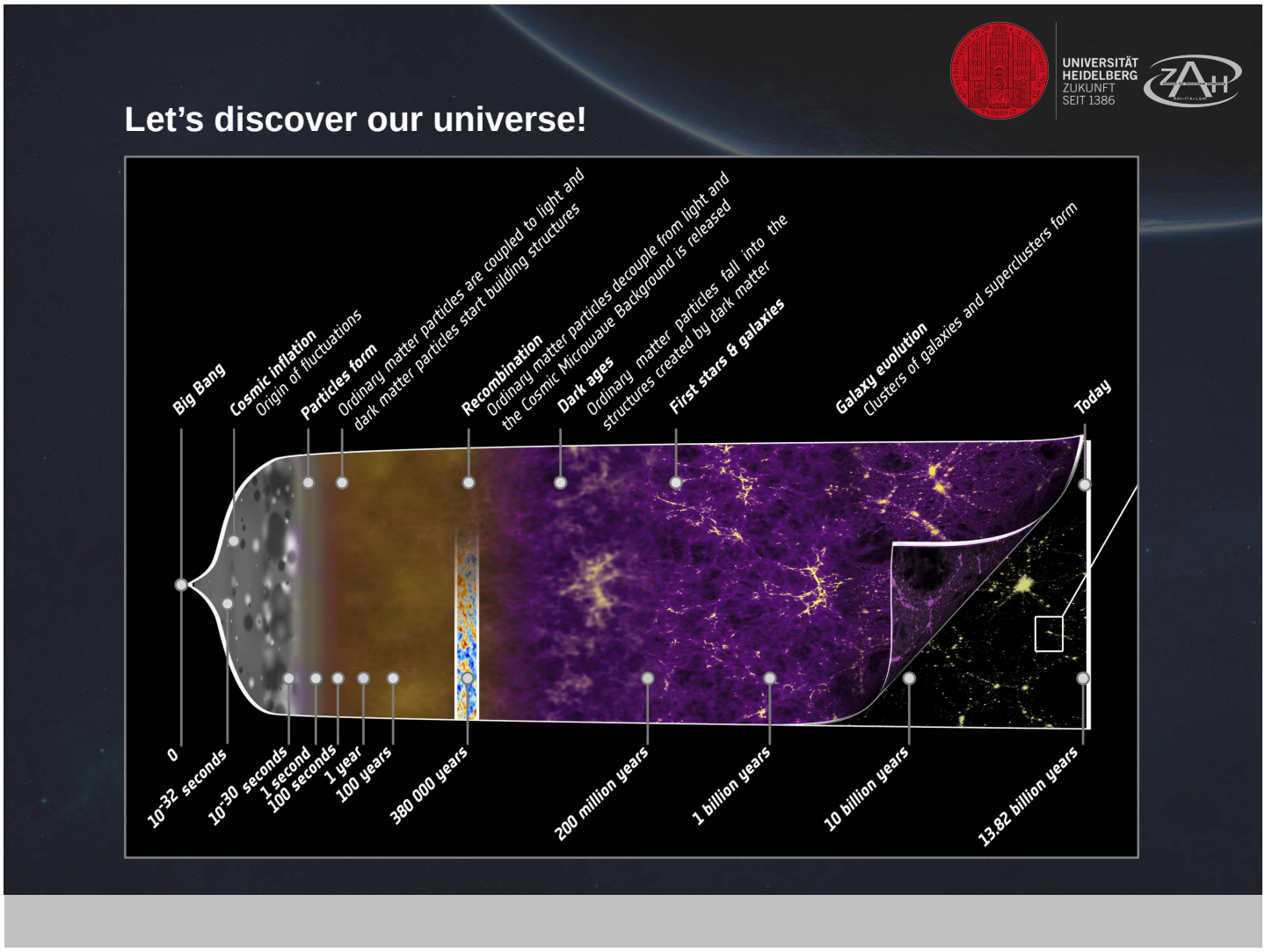
Lectures style:

1- Black Board as much as I can
2- I will skip trivial steps… yes
3- All (all!) steps in lecture notes
4- Script from other local lecturers
5- Books, e.g.
Cosmological Physics (John A. Peacock)
Modern Cosmology (S. Dodelson, F. Schmidt)

6- I will start super slow… I will accelerate
7- Recap at beginning/end of each lectures
8- A mega recap along the way

Recommendations:

- Come in presence!
- Use videos as an auxiliary instrument
- Discuss with other people!
- Be active at the tutorials!


UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386
Let's discover our universe!
Big Bang
Cosmic inflation
Origin of fluctuations
Particles form
Ordinary matter particles are coupled to light and dark matter particles start building structures
Recombination
Ordinary matter particles decouple from light and the Cosmic Microwave Background is released
Dark ages
Ordinary matter particles fall into the structures created by dark matter
First stars & galaxies
Galaxy evolution
Clusters of galaxies and superclusters form
Today
0
10-32 seconds
10-30 seconds
1 second
100 seconds
1 year
100 years
380 000 years
200 million years
1 billion years
10 billion years
13.82 billion years

Special Relativity: the concept

- $P=(t,x,y,z)$ Event : something happening in P
- S, S' Coordinate frames (in standard configuration)
- Generic linear transformation (preserve homogeneity!)

$x'=At+Bx \qquad y'=y$

$t'=Dt+Ex \qquad z'=z$

Find coefficients:

$x=0 \Rightarrow x'=At \quad t'=Dt \quad \frac{x'}{t'}\equiv v'=\frac{A}{D} \quad \boxed{A=Dv'}$

$x'=0 \Rightarrow 0=At+Bx=At+Bvt \quad (x=vt) \quad A=-Bv$

$\boxed{A=Dv'}$ and $A=-Bv$ $\Rightarrow$ $Dv'=-Bv$ $\Rightarrow$ $\boxed{D=B}$

$\boxed{x'=Dv't+Dx=D(v't+x)=D(x-vt)}$ $\qquad y'=y$

$\boxed{t'=Dt+Ex}$ $\qquad z'=z$

⊛ $\qquad D=? \quad E=?$

- Postulate: $t \overset{!}{=} t'$ (Absolute time, Galilean frame)

$t'=Dt+Ex \quad \forall t,x \iff D=1 \quad E=0 \quad \Rightarrow$

$\boxed{x'=x-vt \quad t'=t}$

- Postulate: $c \overset{!}{=}$ const $\forall$ frame (observations)

- $c=$ const $\Rightarrow$ it make sense to investigate light
- Consider a photon in 2 $\neq$ frames

$c^2t^2=x^2+y^2+z^2 \quad \Rightarrow \quad 0=-c^2t^2+x^2+y^2+z^2$

$c^2t'^2=x'^2+y'^2+z'^2 \quad \Rightarrow \quad 0=-c^2t'^2+x'^2+y'^2+z'^2$

$\Rightarrow \quad \boxed{-c^2t^2+x^2+y^2+z^2=-c^2t'^2+x'^2+y'^2+z'^2}$ (in a cartesian frame!)

Plug ⊛

$\Rightarrow$ $\boxed{\begin{aligned} ct'&=\gamma(ct-\beta x) \quad & y'&=y \\ x'&=\gamma(x-\beta ct) \quad & z'&=z \end{aligned}}$ $\qquad \gamma=(1-\beta^2)^{-1/2} \qquad \beta=\frac{v}{c}$ ← Lorentz transformations

↳ Space and time are connected "Not like the frames of a movie"

↳ The Space-Time ! $\Rightarrow$ event $P\equiv(x^\mu)\equiv(ct,x,y,z)$ (ct to have units of lengths; x^μ elements of a 4-vector)

4-interval: $d\bar{x}^2=-\underbrace{(cdt)^2}_{dx^0}+(dx^1)^2+(dx^2)^2+(dx^3)^2=\sum_i\sum_j \eta_{ij}dx^i dx^j \qquad \eta=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ Minkowski metric (!)

Some abmiguity is left

For photons $ds^2 = 0 = ds'^2$ also $\phi(r)\, ds^2 = 0$ $\phi \in \mathbb{R}$ $ds'^2 = \phi(r)\, ds^2$

$ds^2 = \phi(r)\, ds'^2 = \phi^2(r)\, ds^2$

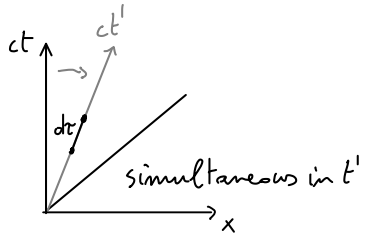
(e.g. $-c^2 dt^2 + d\vec{x}^2 = 0$ or $c^2 dt^2 - d\vec{x}^2 = 0$) $\Rightarrow$ $\boxed{\phi(r) = \pm 1}$ signature one can choose

Here we will use $(-,+,+,+)$ signature

The proper time

Time as measured with a clock at rest in a frame

$$-c^2dt^2+dx^2+dy^2+dz^2 = -c^2dt'^2+\underbrace{dx'^2+dy'^2+dz'^2}_{\text{center such that } \neq 0}$$



we call $dt' \equiv d\tau$ proper time

$$-c^2d\tau^2 = -c^2dt^2+dx^2+dy^2+dz^2 = ds^2 \quad \Rightarrow \quad \boxed{d\tau^2 = -\frac{ds^2}{c^2}} \quad (1)$$

$$= -c^2dt^2\left(1-\frac{dx^2+dy^2+dz^2}{c^2dt}\right)$$

$$\left(1-\frac{\bar{v}^2}{c^2}\right) = \left(1-\bar{\beta}^2\right) = \gamma^{-2} \quad \Rightarrow \quad \boxed{d\tau = \gamma^{-1}dt} \quad (2)$$

$dt = \gamma\, d\tau$ ↑ time dilation

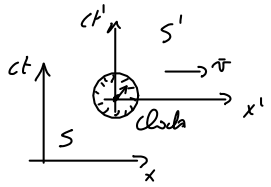
Time interval

$$\Delta\tau = \int_w d\tau = \int_w \gamma^{-1}dt \quad \leftarrow (2)$$

$$= \int_w \frac{1}{c}\sqrt{-ds^2} \quad \leftarrow (1)$$

$$= \int_w \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} \quad \leftarrow \text{the metric is "hidden" there!}$$

↑ (!)



in G.R $\eta_{\mu\nu} \to g_{\mu\nu}$ given by local properties of (curved) space-time

Implication of the Lorentz transforms

- $ct' = \gamma(ct - \frac{v}{c}x)$ $x' = \gamma(x - vt)$ $y' = y$ $z' = z$ (Standard configuration of S, S')

- **Time dilation** $t_0 \equiv t_B - t_A$ (proper time)
  $t = t_0(1-\beta^2)^{-1/2} = t_0\gamma$ $\Rightarrow$ $v\uparrow \Rightarrow t\uparrow$

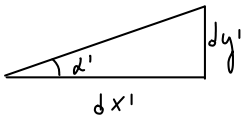
- **Length contraction** $l_0 \equiv x'_0 - x'_1$ (proper length)
  $l = l_0(1-\beta^2)^{1/2} = l_0\gamma^{-1}$ ($v\uparrow \Rightarrow l\downarrow$)
  what if $v = c$? $\Rightarrow l \to 0$
  $\Rightarrow \rho = \frac{m}{l^3} = \infty!$
  ($\Rightarrow v = c$ not possible for massive particles

- **Velocity transformation** $\boxed{u'_x \equiv \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}}$ (in lab frame: $u_x = \frac{u'_x + v}{1 + \frac{u_x v}{c^2}}$)

  $\boxed{u'_y \equiv \frac{dy'}{dt'} = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}}$ (!) $\boxed{u'_z \equiv \frac{dz'}{dt'} = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}}$ (!) because the time is affected

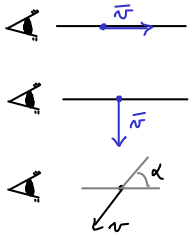
- **Accelerations** $a'_x \equiv \frac{du'_x}{dt'}$ ....

- **Angles**



$dy' = dx' \tan\alpha \Rightarrow \boxed{\tan\alpha'} = \frac{dy'}{dx'} = \frac{dy}{\gamma(dx - v\,dt)} = \frac{dy}{\gamma\,dt\left(\frac{dx}{dt} - v\right)} = \boxed{\frac{u_y}{\gamma(u_x - v)}}$

- **Doppler effect**
  - ↳ longitudinal:
  - ↳ transverse:
  - ↳ Arbitrary direction:



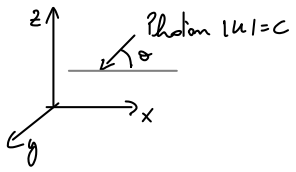
$\frac{\nu}{\nu'} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$

$\nu = \nu'\gamma^{-1}$ ($t = t'\gamma$) purely relativistic effect

$\frac{\nu}{\nu'} = \left[\gamma(1+\beta\cos\theta)\right]^{-1}$ (θ: angle seen by the observer)

$\frac{\nu}{\nu'} = \gamma(1-\beta\cos\theta')$ (θ': angle in rest frame of moving source)

• Aberration of light



Polar coordinates:

$u'_x = c\cos\theta' \quad u'_y = 0 \quad u'_z = c\sin\theta'$

$u_x = c\cos\theta \quad u_y = 0 \quad u_z = c\sin\theta$

Lorentz transf. of velocities → $u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{c\cos\theta - v}{1 - \beta\cos\theta} \Rightarrow$ (photon)

$$\boxed{\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}}$$

in lab frame: $\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$

• Beaming of light : (solid angles)

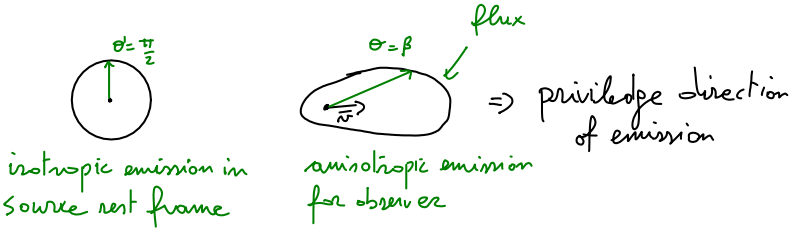
$d\Omega' = d\phi' d\cos\theta'$

$d\Omega = d\phi\, d\cos\theta$: $d\phi = d\phi'$ no change because ⊥ to velocity

$$d\cos\theta = d\left(\frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}\right) = \frac{d\cos\theta'}{\gamma^2(1 + \beta\cos\theta')^2}$$

$$= \frac{d\phi\, d\cos\theta'}{\gamma^2(1 + \beta\cos\theta')}$$

super relevant for emission processes:



eg. $\theta' = \frac{\pi}{2}$: $\cos\theta' = 0 \Rightarrow \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} = \beta$

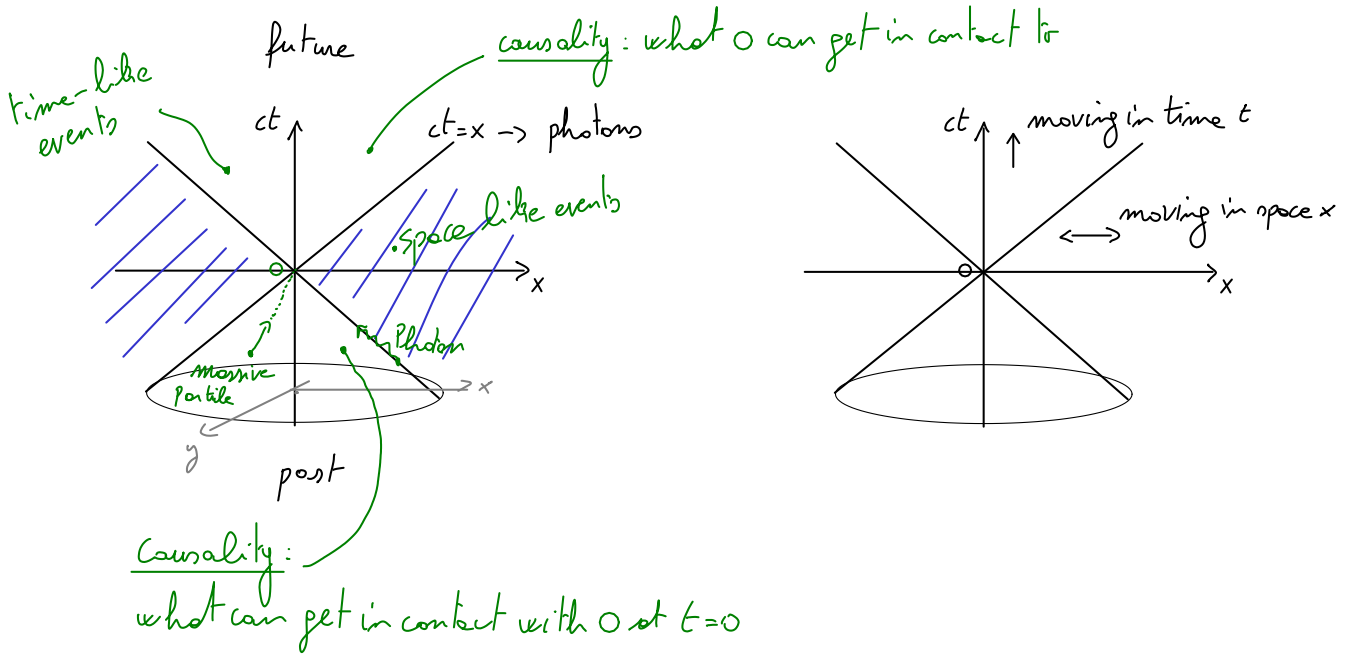
$\beta \ll 1 \Rightarrow \cos\theta \sim 0 \quad \theta \sim \pi/2$

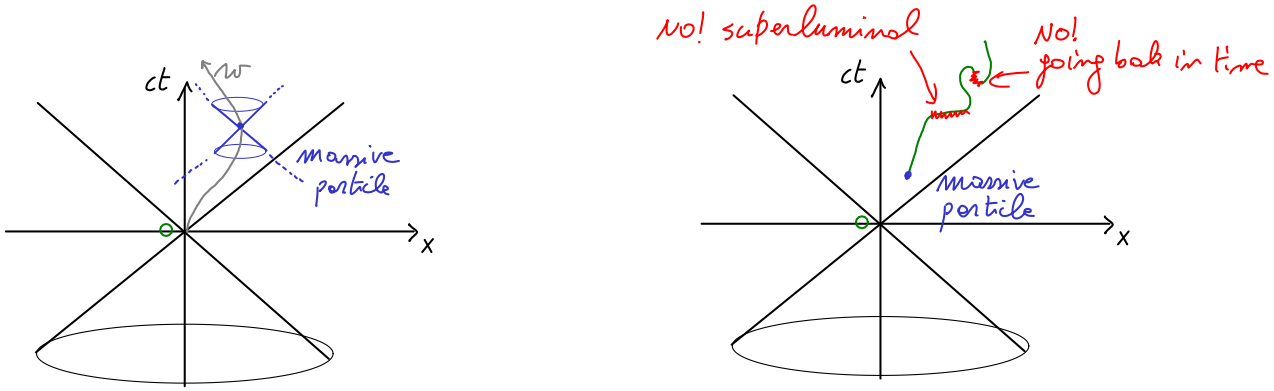
$\beta \sim 1 \Rightarrow \cos\theta \sim 1 \quad \theta \sim 0$ (beamed forward)

• Cross section

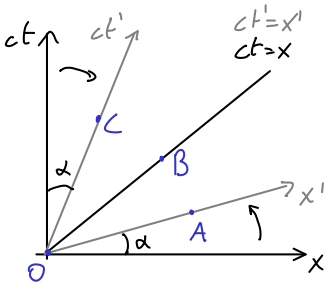
Solid angles are affected ⇒ cross sections transform as well

Space-time diagrams





Note: ds^2 is not positive definite



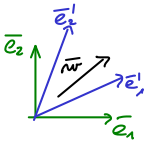
OA: Space like event $ds>0$ (frame in which $\Delta t'=0$)
OB: null $ds=0$
OC: Time like event $ds<0$ (frame in which $\Delta x'=0$)

Lorentz transf. are "rotations" (boosts) $\frac{dx}{cdt}=\frac{v}{c} \Rightarrow \alpha=\text{atan}\left(\frac{v}{c}\right)$

Because of postulate:
- null line do not change under Lorentz. transf.
- ds^2 is invariant under Lorentz. transf. (i.e. is the same $\forall$ frame)

↓

"The norm of a vector does not depend on the basis"



Lorentz geometry

Geometric interpretation of the ct,x plane

- Define a vector space
- $\bar{x} = (x^\mu) = (x^0, x^1, x^2, x^3)$ 4-vector $x^0 = ct$ $\bar{x} \in \mathbb{M}$ $\mathbb{M}$ Minkowski space
- Separations in the space-time are expressed by the 4-interval $d\bar{s}$
  ⇒ Distances are measured in terms of a metric ⇒ metric vector space

$d\bar{s}^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ 4-interval ⇒ vector space $d\bar{s} \in \mathbb{M}$ vector in Minkowski space

$= \sum_{\nu=0}^{3} \sum_{\mu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu$ Minkowski metric $\eta = \mathrm{diag}(-1,1,1,1)$ $\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$= \eta_{\mu\nu} dx^\mu dx^\nu$ ← Einstein notation

$= dx_\nu dx^\nu$ ← and index lowering

Based on a frame ⇒ components

Frameless representation

$= d\bar{x}^T \eta \, d\bar{x}$ matrix notation

$= \eta(d\bar{x}, d\bar{y})$ $\eta : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ $\eta(\bar{u}, \bar{v}) = a \in \mathbb{R}$ $\bar{u}, \bar{v} \in \mathbb{M}$ bilinear map identifying the scalar product

$= \langle d\bar{x}, d\bar{x} \rangle$ scalar product

$= d\tilde{x}(d\bar{x})$ $\eta(\bar{v}, -) \equiv \tilde{v}$ $\tilde{v} : \mathbb{M} \to \mathbb{R}$ $\tilde{v}(\bar{u}) = a \in \mathbb{R}$ $(v_\nu) \equiv \bar{v}^* \equiv \tilde{v} \in \mathbb{M}^*$ $\mathbb{M}^*$ Dual space of $\mathbb{M}$
1-form, dual vector, tangent vector ...

$= \|d\bar{s}\|^2 \in \mathbb{R}$ norm of $d\bar{s} \in \mathbb{M}$ ⇒ invariant, it can be negative!

Here we used η but this formalism is valid for any arbitrary metric g! in a grav. field $g \neq \eta$!

The metric is "hidden" in many places!

- Proper time $\tau = \frac{1}{c}\sqrt{-ds^2} = \frac{1}{c}\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = \frac{1}{c}\sqrt{-\eta(d\bar{x}, d\bar{x})}$

  ↑

  • Do you see?! The metric is hidden in crucial places!

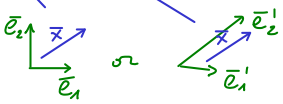
- Another example? It is even in the dear old Kinetic energy ...

$T = \frac{1}{2} m \bar{v}^2 = \frac{1}{2} m \bar{v} \bar{v} = \frac{1}{2} m \delta_{ij} v^i v^j$ $\bar{v} \in \mathbb{R}^3$

↑ $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$ (Euclidean 3D space + cartesian system)

Note: above we have choosen a specific basis set (cartesian)

$\bar{x} = x^\mu \bar{e}_\mu = x^{\mu'} \bar{e}_{\mu'}$ 4-vectors as linear combination of basis

e.g.  or different $\{\bar{e}_\mu\}$ but $\bar{x}$ is $\bar{x}$

$\{\bar{e}_\mu\}$ basis set ↑ some linearly independent vector

$\eta_{\mu\nu} = \eta(\bar{e}_\mu, \bar{e}_\nu) = \bar{e}_\mu \bar{e}_\nu$ η defines the scalar product

$\bar{e}_0 \bar{e}_0 = -1 \quad \bar{e}_i \bar{e}_i = 1 \quad \bar{e}_i \bar{e}_j = 0 \; i \neq j$ ← Cartesian frame

$\bar{e}_0 = (1,0,0,0)^T$, $\bar{e}_1 = (0,1,0,0)^T$ ....

$d\bar{s}^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$\bar{e}_\mu$ $\bar{e}_{\mu'}$ basis of S and S' frame

Minkowski metric η can also be not diagonal. Example with ≠ basis:

$$ds^2 = \underbrace{-c^2dt^2 + dx^2 + dy^2 + dz^2}_{\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} = \underbrace{(-cdt + dx)}_{\equiv du}\underbrace{(cdt + dx)}_{\equiv dv} + dy^2 + dz^2 \equiv \underbrace{du\,dv + dy^2 + dz^2}_{\eta = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Other coordinate frames: polar, Kruskal, Eddington-Finkenstein, ...

This is just linear algebra!

Frame transformations

We deal with transformation between inertial frames

$x^\mu \to x^{\mu'}$

* $\frac{dx^\mu}{d\lambda} = \text{const.}$ $\frac{dx^{\mu'}}{d\lambda} = \text{const.}'$ (shift of coordinates through a point)

** $\frac{d^2x^\mu}{d\lambda^2} = 0$ $\frac{d^2x^{\mu'}}{d\lambda^2} = 0$ (no acceleration) λ: affine parameter $x^\mu(\lambda)$

Relation between frames S, S'

$x^{\mu'} = x^{\mu'}(x^\mu(\lambda))$

↑ transformations law

transformed component

$\frac{dx^{\mu'}}{d\lambda} = \frac{\delta x^{\mu'}}{\delta x^\mu}\frac{dx^\mu}{d\lambda}$ $J^{\mu'}{}_\mu = \frac{\delta x^{\mu'}}{\delta x^\mu}$ Jacobian of the transformation

** $\neq 0$

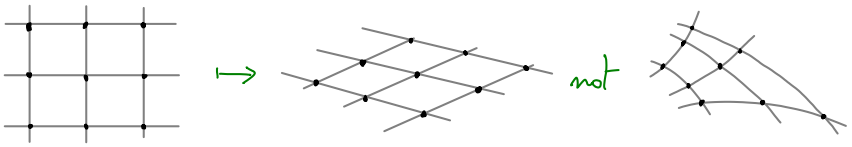
$\frac{d^2x^{\mu'}}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{dx^{\mu'}}{d\lambda}\right) = \frac{d}{d\lambda}\left(\frac{\delta x^{\mu'}}{\delta x^\mu}\frac{dx^\mu}{d\lambda}\right) = \underbrace{\frac{\delta^2 x^{\mu'}}{\delta x^\nu \delta x^\mu}}_{\text{* constant}}\frac{dx^\nu}{d\lambda}\frac{dx^\mu}{d\lambda} + \underbrace{\frac{\delta x^{\mu'}}{\delta x^\mu}}_{\text{** } \neq 0 \text{ (inertial frame)}}\frac{d^2x^\mu}{d\lambda^2} \overset{!}{=} 0 \Rightarrow \boxed{\frac{\delta^2 x^{\mu'}}{\delta x^\nu \delta x^\mu} \overset{!}{=} 0}$

Solve for $x^{\mu'}$ $\Rightarrow$ $0 = \int \frac{\delta^2 x^{\mu'}}{\delta x^\nu \delta x^\mu} x^\mu dx^\nu = \frac{\delta x^{\mu'}}{\delta x^\mu} x^\mu - \int \frac{\delta x^{\mu'}}{\delta x^\mu}\frac{\delta x^\mu}{\delta x^\nu} dx^\nu = J^{\mu'}{}_\mu x^\mu - x^{\mu'} + A^{\mu'}$

This implies a linear transfrmation of vectors between frames

$\boxed{x^{\mu'} = J^{\mu'}{}_\nu x^\nu + \cancel{\delta^{\mu'}}}$ $\bar{x}' = J\bar{x}$

origin shift (irrelevant) (matrix)



Distance between points invariant under coord. transformation

→ i.e. preserves scalar product $\boxed{\langle \bar{u}, \bar{v} \rangle = \langle \bar{u}', \bar{v}' \rangle}$ (1)

→ i.e. metric is invariant $\boxed{\eta = J^T \eta J}$ (2) i.e. J is an isometry

$\Rightarrow$ Preserves homogeneity

(1) and (2) are equivalent:

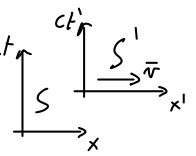
$d\bar{s}^2 = d\bar{x}^T \eta\, d\bar{x} \overset{!}{=} d\bar{x}'^T \eta\, d\bar{x}' = d\bar{x}^T J^T \eta J\, d\bar{x}$ $\Rightarrow$ $\boxed{\eta = J^T \eta J} \leftrightarrow \boxed{\eta_{\mu\nu} = J^{\alpha'}{}_\mu J^{\beta'}{}_\nu \eta_{\alpha'\beta'}}$

or $\eta_{\alpha'\beta'} = J^\mu{}_{\alpha'} J^\nu{}_{\beta'} \eta_{\mu\nu}$

For a Minkowski space:

$\eta \overset{!}{=} \eta'$ is the necessary and sufficient condition that a transformation $\Lambda^\mu{}_{\nu'} = \frac{\delta x^\mu}{\delta x^{\nu'}}$ is a Lorentz transform

$(J^{\mu'}{}_\nu) \overset{!}{=} (\Lambda^{\mu'}{}_\nu) = \left(\frac{\delta x^{\mu'}}{\delta x^\nu}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix}$ for "normal configuration"



Hence c = invariant

Lorentz invariant quantities

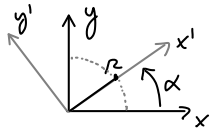
Why are there quantities that do not change under coordinate transformations?

A familiar example: rotation in 3D

i.e. same values $\forall$ frame

e.g. in 2D $r^2 = x^2 + y^2 = \delta_{ij} x^i x^j$ ↳ Invariant under rotations

$= (r\cos\alpha)^2 + (r\sin\alpha)^2$ $r^2 \geq 0$



corresponds to the statement $\cos^2\alpha + \sin^2\alpha = 1 \quad \forall \alpha \in \mathbb{R}$

Likewise, for Lorentz transforms: define a "rotation angle" $\alpha \rightarrow \psi$

- ψ = rapidity parameter (Sort of polar coord.)

$\cosh\psi \equiv \gamma \quad \in [1, \infty]$

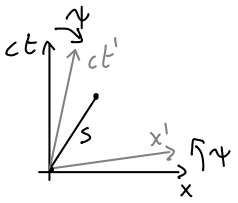
$\sinh\psi \equiv \beta\gamma \quad \in [-\infty, \infty]$

$\tanh\psi = \frac{\beta\gamma}{\gamma} = \beta$

$\cosh^2\psi - \sinh^2\psi = \gamma^2 - \beta^2\gamma^2 = \gamma^2(1-\beta^2) = 1$

$s^2 = -(dx^0)^2 + (dx^1)^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

$= -(s\sinh\psi)^2 + (s\cosh\psi)^2 = s^2(\underbrace{\cosh^2\psi - \sinh^2\psi}_{=1})$



- Lorentz transformations $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$ ~ hyperbolic rotation : boost

- Λ form the so called Lorentz group (see next page)
  the invariance is an intrinsic property of this group
  not just a property of sinh, cosh !

The Lorentz group

The Lorentz transformations form a non Abelian Lie group

$$SO(3,1,\mathbb{R}) = \{\Lambda \in \mathcal{M}(4,\mathbb{R}) \,|\, \langle \bar{u}, \bar{v}\rangle = \langle \Lambda\bar{u}, \Lambda\bar{v}\rangle \;\forall \bar{u},\bar{v} \in \mathbb{M}\}$$

($\bar{u}'$, $\bar{v}'$)

↑ orthogonal $\Lambda^T\Lambda = I_4$ ↑ 4×4 real matrices

↳ Special: $\det(\Lambda) = 1$ in general: $\det(\Lambda) = \pm 1$ (−) improper (+) proper ← we consider this subgroup

In fact:

- the product of 2 Lorentz transformations is another Lorentz transf. (addition)
- " " is associative $(\Lambda^\mu{}_\rho \Lambda^\rho{}_\gamma)\Lambda^\gamma{}_\sigma = \Lambda^\mu{}_\rho(\Lambda^\rho{}_\gamma \Lambda^\gamma{}_\sigma)$
- there is an identity transformation $\Lambda^\mu{}_\nu = \delta^\mu_\nu$ $(\nu = 0)$
- " " " inverse $\forall \Lambda$: $(\Lambda^{-1})^\mu{}_\nu \equiv \Lambda_\nu{}^\mu$

} group

Moreover:

- The group is non Abelian i.e. Λ do not commute (for rotations) $\Lambda(\bar{v}_2)\Lambda(\bar{v}_1) \neq \Lambda(\bar{v}_1)\Lambda(\bar{v}_2)$
- " " is a Lie group i.e. its elements $\Lambda(\bar{v})$ are "functions of continuous parameters v_i or ψ_i finite dimensional differentiable (i.e. smooth) manifold
  (= it is a continuous differentiable group)
- " " is a Poincaré group i.e. it has 10 parameters:
  6 free parameters (1): 3 boosts (x^i, x^0 planes), 3 rotations (x^i, x^j $i\neq j$ planes) $\dim[O(n)] = \frac{n(n-1)}{2} = 6$ ($n=4$)
  4 translations (2): shift in time and time

$$x^{\mu'} = \underset{(1)}{\Lambda^{\mu'}{}_\nu}\, x^\nu + \underset{(2)}{a^{\mu'}}$$

Proves:

- $\eta_{\mu\nu} = \Lambda^{\alpha'}{}_\mu \Lambda^{\beta'}{}_\nu \eta_{\alpha'\beta'} = \Lambda_{\beta'\mu}\Lambda^{\beta'}{}_\nu = \Lambda_{\beta'}{}^\alpha \Lambda^{\beta'}{}_\nu \eta_{\mu\alpha} \Rightarrow \Lambda_\beta{}^\alpha \Lambda^\beta{}_\nu = \delta^\alpha_\nu \Rightarrow \Lambda_\beta{}^\alpha$ is the inverse of $\Lambda^\beta{}_\alpha$
- Take infinitesimal Lorentz transf. $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \delta\omega^\mu{}_\nu$ $(|\delta\omega^\mu{}_\nu| \ll 1)$ because of $\eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta}$

$$\eta_{\mu\nu}(\delta^\mu_\alpha + \delta\omega^\mu{}_\alpha)(\delta^\nu_\beta + \delta\omega^\nu{}_\beta) = (\eta_{\alpha\nu} + \delta\omega_{\nu\alpha})(\delta^\nu_\beta + \delta\omega^\nu{}_\beta) = \eta_{\alpha\beta} + \delta\omega_{\beta\alpha} + \delta\omega_{\alpha\beta} + \cancel{\delta\omega_{\nu\alpha}\delta\omega^\nu{}_\beta}$$

(2° order)

$\Rightarrow$ $\delta\omega_{\beta\alpha} = -\delta\omega_{\alpha\beta}$ i.e. antisymmetric $\Rightarrow$ 6 free parameters

3 rotations $\delta\omega_{ij} = -\varepsilon_{ijk}n^k\delta\theta$ rotation of angle $\delta\theta$ around unit vector $\bar{n}$

3 boosts $\delta\omega_{i0} = n_i\delta\psi$ boost with rapidity $\delta\psi$ along direction $\bar{n}$

- Infinitesimal Lorentz transf. are proper

Subgroups:

- because of $(\Lambda^{-1})^{\alpha}{}_{\beta} = \Lambda_{\alpha}{}^{\beta}$: in general $\boxed{\det(\Lambda) = \pm 1}$ (-) improper Lorentz transf.
  (+) proper Lorentz transf.
  we consider this subgroup

- because of $\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$: $\eta_{\mu\nu}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} = -\Lambda^{0}{}_{0}\Lambda^{0}{}_{0} + \delta_{ij}\Lambda^{i}{}_{0}\Lambda^{j}{}_{0} = \eta_{00} = -1$

  $\Rightarrow (\Lambda^{0}{}_{0})^2 - \Lambda^{i}{}_{0}\Lambda^{i}{}_{0} = 1$ i.e. $\boxed{\begin{matrix}\Lambda^{0}{}_{0} \geq +1 \\ \Lambda^{0}{}_{0} \leq -1\end{matrix}}$ ← orthochronous Lorentz group

  ? scalar ? ↓ ?
  $\eta_{\mu\alpha}\Lambda^{\mu}{}_{0}\Lambda^{\alpha}{}_{0} = \eta_{00}\Lambda^{0}{}_{0}\Lambda^{0}{}_{0} + \eta_{ij}\Lambda^{i}{}_{0}\Lambda^{j}{}_{0}$

- infinitesimal Lorentz transf. are proper and orthochronous

- Generally, a theory is said to be Lorentz invariant for proper orthochronous sub group only
  => The Lorentz sub group we are interested in can be covered by infinitesimal transf.

- 2 discrete transf. to take us out of these sub groups

parity reversal :	$P^{\mu}{}_{\nu} = (P^{-1})^{\mu}{}_{\nu} = \mathrm{diag}(1,-1,-1,-1)$	orthochronous and improper
time reversal :	$T^{\mu}{}_{\nu} = (T^{-1})^{\mu}{}_{\nu} = \mathrm{diag}(-1,1,1,1)$	non orthochronous and improper

- What is a groups?

Set of elements $\{G_i\}$ with a connection $*$ between elements $(G, *)$ with these properties:

$*$	connection between elements	: e.g. $(a \in \mathbb{R}, \cdot)$ $1_G = 1$
$a * b = c$	addition, $\forall a,b \in G \Rightarrow c \in G$	$(a \in \mathbb{R}, +)$ $1_G = 0$
$(a * b) * c = a * (b * c)$	associativity, $\forall a,b,c \in G$	$(a \in \mathbb{R}^m \times \mathbb{R}^m, \cdot)$ $1_G = \mathbb{I}^m$
$e * a = a * e = a$	$\exists$ of identity element e, $\forall a \in G$	
$a * b = b * a = e$	$\exists$ of inverse, $\forall a,b \in G$	
extra: $a * b = b * a$	commutativity, $\forall a,b \in G$ $\Rightarrow$ Abelian group	, eg. $GL(n)$ is not abelian Rotations

- What is a Lie group?

Continuous group $\{G(\alpha_1, \dots, \alpha_m)\}$ i.e Elements are "functions" of α_j continuous parameters
Have finite dimensional differentiable (i.e. smooth) manifold

Lie-groups

- Continuous groups $\{X(\alpha_1,\dots,\alpha_m)\}$
- Elements are "functions" of α_i continuous parameters
- Have finite dimensional differentiable (i.e. smooth) manifold

e.g. collection of phase factors $e^{i\alpha}$, α = phase $U(1)=\{e^{i\alpha}\,;\,\alpha\in\mathbb{R}\}$
" " rotations $R(\alpha)$ angle α (in a plane), $R(\alpha,\beta,\gamma)$ in 3D
" " boosts $\Lambda(\gamma)$ rapidity γ (in a plane)

Lie-algebra

- Define an element of the group with a set of "basis" : generators of the group M_k
- Elements of a Lie group X can be written as

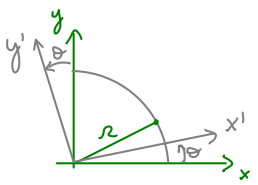
$$X = \exp\left(i\sum_k \alpha_k M_k\right) \qquad \alpha_k\in\mathbb{R} \text{ parameters} \quad k=1,\dots,r$$
$$= e^{i\alpha_1 M_1}\dots e^{i\alpha_r M_r} \qquad M_k = -i\left(\frac{\delta X}{\delta\alpha_k}\right)_{\{\alpha_k=0\}} \text{ generator}$$

$\uparrow$
X satisfies a Lie algebra if $[M_k,M_e] = i f_{kem} M^m$ ($[a,b]\equiv ab-ba$ commutator)
f_{kem} = structure constant of the group (completely anti-symmetric on kem)
it defines the connection between the elements X

- if set of generators $\{M_k\}$ $M_k \to M'_k = S M_k S^{-1}$ S $n\times n$ invertible matrix

- We have $X = e^Y \Rightarrow \det X = e^{\mathrm{Tr}\,Y}$
for $\mathrm{Tr}\,Y = 0 \Rightarrow \det X = 1$
$\uparrow$
$\mathrm{Tr}\left(\sum_k \alpha_k M_k\right) = 0$

you have conserved quantities
eg. $X=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ Rotations
$\det(X) = \cos^2\theta+\sin^2\theta = 1$



$r = r'$

eg. Lorentz boosts Λ : $ds^2 = ds'^2$

The Lorentz group is a Lie group

- Here, consider rotation in the x^0, x^1 plane $\Rightarrow$ only 1 parameter $\alpha = \psi$

$$X = \exp\left(i \sum_k \alpha_k M_k\right) \qquad k=1 \quad \alpha_1 = \psi \quad M_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\sigma^{(3)}$$

as generators we can use a Pauli matrix

$$\Lambda = \exp\left(\psi \sigma^{(3)}\right)$$

$$= \sum_n \frac{1}{n!} \left(\psi \sigma^{(3)}\right)^n \qquad \text{Taylor expansion}$$

$$\hookrightarrow \quad (\sigma^{(3)})^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^{(0)}$$

$$(\sigma^{(3)})^1 = \sigma^{(3)}$$

$$(\sigma^{(3)})^2 = \sigma^{(3)}\sigma^{(3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^{(0)}$$

$$(\sigma^{(3)})^3 = \sigma^{(0)}\sigma^{(3)} = \sigma^{(3)}$$

$$\cdots\cdots$$

$\Rightarrow$

$n=0$ even : $\sigma^{(0)}$

$n=1$ odd : $\sigma^{(3)}$

$n=2$ even : $\sigma^{(0)}$

$n=3$ odd : $\sigma^{(3)}$

$$= \sigma^{(0)} \sum_n \frac{1}{(2n)!} \psi^{2n} + \sigma^{(3)} \sum_n \frac{1}{(2n+1)!} \psi^{2n+1}$$

Taylor expansion of sinh, cosh

$$= \sigma^{(0)} \cosh(\psi) + \sigma^{(3)} \sinh(\psi)$$

$$= \begin{pmatrix} \cosh(\psi) & \sinh(\psi) \\ \sinh(\psi) & \cosh(\psi) \end{pmatrix}$$

for $x^{n'} \to x^n$; inverse transf. $x^n \to x^{n'}$ $\Rightarrow$ $\psi \to -\psi$ $\quad \Lambda = \begin{pmatrix} \cosh(\psi) & -\sinh(\psi) \\ -\sinh(\psi) & \cosh(\psi) \end{pmatrix}$

- we could have been using $\sigma^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as well (i.e. there are different possibilities for M_k)

Invariants under boosts (Lorentz invariance)

- $\Lambda = e^{(\psi \sigma^{(3)})}$: $\quad \det\Lambda = e^{\mathrm{Tr}(\psi\sigma^{(3)})} = e^0 = 1 \quad \Rightarrow \quad \boxed{\det\Lambda = 1}$

because $\sigma^{(3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow$ It provides invariant quantities $ds^2 = ds'^2$!

$\Rightarrow$ This is a property of the group rather than of sinh, cosh

$\det(\Lambda) = \cosh^2\psi - \sinh^2\psi = 1$ as we have seen before

Combination of boosts in one plane

- here the structure constant of the group $f = \varepsilon$ Levi-Civita symbol

$$\Lambda(\phi)\Lambda(\psi) = \exp(\phi\sigma^{(3)})\exp(\psi\sigma^{(3)}) = \exp[(\phi+\psi)\sigma^{(3)}] = \Lambda(\phi+\psi) \qquad (1)\ (2)$$

$$= \Lambda(\psi+\phi) = \Lambda(\psi)\Lambda(\phi) \qquad (3)$$

$\Rightarrow$

(1) Two boosts in the same plane is given by one boost with $\psi+\phi=\psi_{tot}$ (additive in ψ, not v!)
(as in a Galilean transformation with velocities $v_1+v_2=v_{tot}$)

(2) $\exists$ of inverse Λ^{-1}: $\Lambda(\psi)\Lambda(-\psi) = \Lambda(0) = I$ $\quad \Lambda(\psi)^{-1} = \Lambda(-\psi)$

(3) In one plane, boosts Λ are commutative $[\Lambda(\phi), \Lambda(\psi)] = 0$

Λ is orthogonal with respect to η not δ!

Conbination of boosts in different planes

- A, B basis generators elements generating boosts around 2 different axis

1° apply A then apply B, i.e. $\exp(A)\exp(B) \neq \exp(A+B)$ because

→ Final boost given by Baker-Hausdorf-Campbell formula

$$\exp(A)\exp(B) = \exp(A+B)\exp\left(-\tfrac{1}{2}[A,B]\right) \qquad [A,B] \equiv AB - BA \neq 0$$

As from the f of the group these generators do not commute

Rotation group $O(3,\mathbb{K})$

$\mathbb{K} \to K$
$\mathbb{I} \to L$

$$R_{\bar{m}}(\delta\phi) = \mathbb{I}_{dm} + \delta\phi\, i\bar{m}\bar{J} + O((\delta\phi)^2)$$

infinitesimal rotation around $\bar{m}$ (unit vec)

$\bar{J}$ = infinitesimal generators

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow (J_k)_{em} = -i\varepsilon_{kem}$$ Generators

↳ Rotations around the 3 axes

$[J_k, J_e] = i\varepsilon_{kem} J_m$ algebra for (e.g.) angular momenta $\hbar\bar{J}$

$= \varepsilon_{ke1}J_1 + \varepsilon_{ke2}J_2 + \varepsilon_{ke3}J_3$ → $[J_k, J_e] = J_k J_e - J_e J_k$

$$\Rightarrow R^{(\phi)} = \exp(i\phi\bar{m}\bar{J}) = \sum_{k=0}^{\infty} \frac{1}{k!}(i\phi\bar{m}\bar{J})^k = \mathbb{I}_d + i\phi\bar{m}\bar{J} - \frac{(\phi\bar{m}\bar{J})^2}{2} + i\frac{1}{3}(\phi\bar{m}\bar{J})^3 \ldots ?$$

$\phi\bar{m}\bar{J}$

$$= \mathbb{I}_d + i\sin(\phi)\,\bar{m}\bar{J} + (\cos\phi - 1)(\bar{m}\cdot\bar{J})^2$$

- You could transform J_1, J_2, J_3 as $J_i^S = S J_i S^{-1}$ to have spherical coordinates like in quantum mechanics $J_1^S = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $J_2^S = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $J_3^S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Lorentz group

$$R_{\bar{e}}(\delta\chi) = \mathbb{I}_d + \delta\chi\, i\bar{e}\bar{K} + O((\delta\chi)^2)$$

Has the generators of the spatial rotations $\begin{pmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & & \boxed{J_k} \end{pmatrix}$

mixed t,x terms →

$$K_x = \begin{pmatrix} 0 & \boxed{i} & 0 & 0 \\ \boxed{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$ Generators for the boost

↑ pure space terms (J_i)

$[J_k, J_e] = i\varepsilon_{kem}J_m \quad [J_k, K_e] = i\varepsilon_{kem}K_m \quad [K_k, K_e] = -i\varepsilon_{kem}J_m$ Lie algebra for the Lorentz group

$$\Rightarrow \boxed{L(\chi) = \exp(i\chi\bar{e}\bar{K}) = \mathbb{I}_d + i\sinh(\chi)\,\bar{e}\bar{K} + (\cosh(\chi) - 1)(\bar{e}\bar{K})^2}$$ Pure boost along $\bar{e}$

$\cosh(\chi) = \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

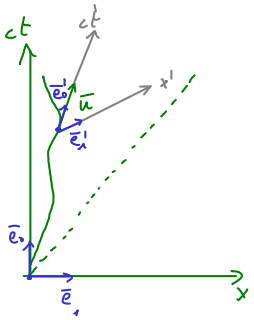
Relativistic mechanics

— Behavejour of paticles in the space-time

— 4-vector : $(x^\mu) = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$ not invariant

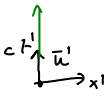
— World-line : $\boxed{x^\mu(\tau)}$ trajectory of particle in space-time τ = proper time

— 4-velocity : $\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt}}$ $(u^\mu) = \gamma\left(c\frac{dt}{dt}, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right) = \gamma(c, \bar{v}) = c\gamma(1, \bar{\beta})$



- note at which coordinates pass by an observer
- not invariant
- $\langle \bar{u}, \bar{u} \rangle = u_\mu u^\mu = \eta_{\mu\nu} u^\nu u^\mu = c^2\gamma^2(\underbrace{-1+\bar{\beta}^2}_{-\gamma^{-2}}) = -c^2 < 0$ invariant

  (this is because of ds^2: $-c^2 d\tau^2 = dx_\mu dx^\mu \Rightarrow -c^2 = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = u_\mu u^\mu$)
- time-like $\Rightarrow$ particles can only move inside the light cone
- $\bar{u}$ is tangent to the world line because it is a multiple of $d\bar{x}$: $\bar{u} = \frac{d\bar{x}}{d\tau}$ ← scalar
- In particle rest frame $\bar{u}' = (c, 0, 0, 0)^T \Rightarrow \bar{u}' = c\,\bar{e}_0$



— 4-acceleration : $\boxed{a^\mu \equiv \frac{du^\mu}{d\tau}}$ $(a^\mu) = \frac{d}{d\tau}\left[\gamma(c, \bar{v})\right] = \gamma\frac{d}{dt}\left[\gamma(c, \bar{v})\right] = \gamma(\dot{\gamma}c, \dot{\gamma}\bar{v} + \gamma\dot{\bar{v}})$ ($\gamma(v)$)

- note: $\frac{d}{d\tau}(\underbrace{u^\mu u_\mu}_{-c^2}) = \frac{du^\mu}{d\tau}u_\mu + u^\mu\frac{du_\mu}{d\tau} = \frac{du^\mu}{d\tau}u_\mu + u_\mu\frac{du^\mu}{d\tau} = 2u_\mu a^\mu = 0$

  $\Rightarrow \bar{u} \perp \bar{a}$ always! (orthogonal in 4-D)
- In instantaneous rest frame: $\gamma = 1$ $\dot{\gamma} = 0$ $(a^\mu) = (0, \bar{a})$ $\Rightarrow$

— 4-momentum : $\boxed{p^\mu \equiv m_0 u^\mu}$

$(p^\mu) = \gamma(\underbrace{m_0 c}_{(2)}, \underbrace{m_0 \bar{v}}_{(1)})$

(1) $\bar{p} = \gamma m_0 \bar{v}$

(2) $\gamma m_0 c = m_0 c(1-\bar{\beta}^2)^{-1/2} \underset{\beta \ll 1}{\simeq} m_0 c\left(1 + \frac{1}{2}\bar{\beta}^2 + \dots\right) = m_0 c + \frac{1}{2}m_0 c\frac{\bar{v}^2}{c^2} = \frac{1}{c}\left(m_0 c^2 + \frac{1}{2}m_0\bar{v}^2\right) = \frac{E}{c}$

rest energy ⇐ Kinetic energy

$\Rightarrow (p^\mu) = \gamma\left(\frac{E_0}{c}, m_0\bar{v}\right) = \left(\frac{E}{c}, \bar{p}\right)$ ← Energy-momentum (one entity!)

- $\langle p, p \rangle = -\frac{E^2}{c^2} + \bar{p}^2$

  $\langle p, p \rangle = p_\mu p^\mu = m_0^2 u_\mu u^\mu = -m_0^2 c^2$ (invariant)

  $\Rightarrow -\frac{E^2}{c^2} + \bar{p}^2 = -m_0^2 c^2$ $\boxed{E^2 = m_0^2 c^4 + c^2\bar{p}^2}$

  at rest $E = m_0 c^2$; m_0 = rest mass
- $\bar{p} = \gamma m_0 \bar{v}$ $\frac{E}{c} = \gamma m_0 c$ ↳ $\boxed{\bar{p} = \frac{E}{c^2}\bar{v}}$ ↲

Energy and momentum are components of the same entity

$\Rightarrow$ energy-momentum conservation!

· Momentum of a photon

moves along null lines $\Rightarrow p_\mu p^\mu = 0$ $\quad \bar{p} = (p^0, p^1, 0, 0) \Rightarrow$ $\quad p^0 = p^1$ i.e. $p = E/c$!

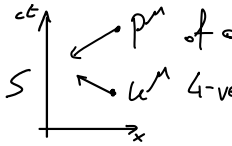
$\frac{p^1}{p^0} = 1$ alternative way to state $v = c$

· Massive particles have $0 < v < c$

$p_\mu p^\mu = -p^{0^2} + p^{1^2} = -m_0^2 c^2 \Rightarrow p^1 = \left(p^0 - m_0^2 c^2\right)^{1/2}$ $\quad \frac{p^1}{p^0} = \left(1 - \frac{m_0^2 c^2}{p^{0^2}}\right)^{1/2} \to 1$ (but not 1!) for $p^0 = \frac{E}{c} \to \infty$

$\Rightarrow$ No matter how much energy we give to the particle, $v < c$!

· A convenient expression: energy of a particle measured by a moving observer



p^μ of a particle in frame S (eg. a cosmic ray)

u^μ 4-velocity of an observer moving in S

$\Rightarrow \boxed{E' = -p_\mu u^\mu}$

$S: u = \gamma(c, \bar{v})^T \quad p = (\frac{E}{c}, \bar{p})^T$ (lab)

$S': u = (c, \bar{0})^T \quad E' = ?$ (moving observer)

$p_\mu u^\mu = p'_\mu u'^\mu = \left(\frac{E'}{c}, \bar{p}'\right)\left(\frac{c}{\bar{0}}\right) = -\frac{E'}{c}\cdot c + p'^{0}_{i} = -E'$ ✓

↑ scalar product is the same in all frames

– 4-force : · $g^\mu \equiv \frac{dp^\mu}{d\tau}$ $\quad (g^\mu) = \gamma \frac{d}{dt}\left(\frac{E}{c}, \bar{p}\right) = \gamma\left(\frac{\bar{f}\bar{v}}{c}, \bar{f}\right)$

① ② ① $\quad dE = \bar{f}\, d\bar{x} = \bar{f}\,\bar{v}\, dt \Rightarrow \dot{E} = \bar{f}\bar{v}$ Work

② $\quad \dot{\bar{p}} = \bar{f}$ 3-force

remember: a force is a change in momentum not $\bar{f} = m\bar{a}$

$\bar{f} = \frac{d\bar{p}}{dt} = \frac{d}{dt}(m\bar{v}) = m\dot{\bar{v}}$ (m = const.)

· Change in rest mass ?!

product rule (let m free)

$u^\mu g_\mu = u^\mu \frac{dp_\mu}{d\tau} = u^\mu \frac{d}{d\tau}(m u_\mu) = u^\mu\left(\frac{dm}{d\tau} u_\mu + m \frac{du_\mu}{d\tau}\right) = \underbrace{u^\mu u_\mu}_{-c^2} \frac{dm}{d\tau} + m \underbrace{u^\mu a_\mu}_{=0} = -c^2 \frac{dm}{d\tau}$

$\Rightarrow$ A force may change rest mass $\quad \boxed{\frac{dm}{d\tau} = -\frac{u^\mu g_\mu}{c^2}}$ $\quad$ if $u^\mu g_\mu \neq 0$!

· If $u^\mu g_\mu = 0 \Rightarrow \frac{dm}{d\tau} = 0$ $\quad g^\mu$ called pure force $\quad \gamma\left(\frac{\bar{f}\bar{v}}{c}, \bar{f}\right)$

↓

acts on 3-acceleration but not on rest energy

Variational approach

– Free particle : no external forces

– Action : $S \in \mathbb{R}$ must be Lorentz invariant (not dependent on motion of observer)

$\Rightarrow$ S must depend on Lorentz scalar: τ is the only one characterizing the particle

$$S = \int L\,dt = \alpha \int d\tau = \alpha \int \gamma^{-1}\,dt$$

non-relat. limit $\beta \ll 1$; no external forces

Find α: $L = \alpha\gamma^{-1} = \alpha(1-\vec{\beta}^2)^{1/2} = \alpha(1-\frac{1}{2}\vec{\beta}^2+\dots) \simeq \alpha - \frac{1}{2}\alpha\frac{\vec{v}^2}{c^2} \overset{!}{=} T + \cancel{V} = \frac{1}{2}m\vec{v}^2 \Rightarrow \alpha = -m_0c^2$

$$\boxed{L = -m_0c^2\gamma^{-1} = -m_0c^2(1-\vec{\beta}^2)^{1/2}}$$ ← Lagrangian of a free particle

· Note: S has a clear physical meaning in S.R. : interval of proper time along world line

$L \to L' = aL + b$ gives same eq. of motion ($\delta S = 0$) a = time rescaling, b = time shift

– Conjugate momentum :

$$\vec{p} = \frac{\delta L}{\delta \vec{v}} = \cancel{+}m_0c\cancel{^2}\frac{1}{\cancel{2}}(1-\vec{\beta}^2)^{-1/2}(\cancel{-1})\frac{1}{c\cancel{^2}}\cancel{2}\vec{v} = \gamma m_0\vec{v} \qquad \boxed{\vec{p} = \gamma m_0\vec{v}}$$ as seen before

– Hamiltonian :

$$H = \vec{v}\,\vec{p} - L = m_0\gamma\vec{v}^2 + m_0c^2\gamma^{-1} = \gamma m_0c^2(\vec{\beta}^2 + \underbrace{\gamma^{-2}}_{(1-\vec{\beta}^2)}) = \gamma m_0c^2 \qquad \boxed{E = \gamma m_0c^2}$$

This confirms our interpretation of the 4-momentum components $(p^\mu) = (\frac{E}{c}, \vec{p})$

– Equation of motion : (Euler-Lagrange eq.)

$$\frac{d}{dt}\frac{\delta L}{\delta \vec{v}} - \frac{\delta L}{\delta \vec{x}} = \frac{d}{dt}\underbrace{(\gamma(v) m_0\vec{v})}_{\vec{p}} = m_0\left[\dot{\gamma}\vec{v} + \gamma\dot{\vec{v}}\right]$$

$$\dot{\gamma} = +\frac{1}{2}(1-\vec{\beta}^2)^{-3/2}(\cancel{-1})\vec{\beta}\dot{\vec{\beta}} = \gamma^3\frac{\vec{v}\dot{\vec{v}}}{c^2}$$

$$= m_0\left(\gamma^3\frac{\vec{v}\dot{\vec{v}}}{c^2}\cdot\vec{v} + \gamma\dot{\vec{v}}\right)$$

$$= m_0\dot{\vec{v}}\left(\gamma^3\frac{\vec{v}^2}{c^2} + \gamma\right) \overset{!}{=} 0 \Rightarrow \boxed{\dot{\vec{v}} = 0}$$ no acceleration as expected

($\gamma^3\frac{\vec{v}^2}{c^2} + \gamma > 0$)

↳ $\vec{x} = \vec{a}t + \vec{b}$ move along straight line

(instantaneous rest frame $\vec{v} = 0$)

– Note: the metric is hidden in there!

$$-c^2d\tau^2 = ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \Rightarrow d\tau = \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} \qquad S = -m_0c\int\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$$

Very relevant in G.R. ! $\eta \to g$

- Eq. of motion but directly from least action principle

$$S = -mc^2 \int d\tau = -mc^2 \int \frac{1}{c}\left(-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2} d\tau = -mc\int \left(-\eta_{\mu\nu}u^\mu u^\nu\right)^{1/2} d\tau$$

$$\delta S = -mc \int_A^B \frac{1}{2}\left(-u_\nu u^\nu\right)^{-1/2} (-2) u_\nu \delta u^\nu \, d\tau = m\int_A^B u_\nu \delta\frac{dx^\nu}{d\tau} d\tau = m\int_A^B u_\nu \frac{d\delta x^\nu}{d\tau} d\tau \quad \leftarrow \text{by part}$$

$$= m\left[u_\nu \delta x^\nu\Big|_A^B - \int_A^B \frac{du_\nu}{d\tau}\delta x^\nu d\tau\right] \overset{!}{=} 0 \quad \forall\, \delta x^\nu \Rightarrow \text{4-velocity} \quad \boxed{a^\nu = 0}$$

$$\Rightarrow \boxed{\frac{d^2x^\mu}{d\tau^2} = 0 \rightarrow x^\mu = \alpha^\mu \tau + \beta^\mu} \Rightarrow \begin{cases} \text{no acceleration, motion along a straight line} \\ \text{no change in particle's energy } m\frac{du^0}{d\tau} = \frac{dp^0}{d\tau} = 0 \end{cases}$$

note: here $\eta_{\mu\nu} = \text{const}$!
not the case in G.R ! You will see how important this is!

- Dispersion relation for massive particles

group and phase velocities can not be the same for massive particles

$$v_{ph} = \frac{H}{p} \neq \frac{dH}{dp} = v_{gr} \quad \text{but} \quad \boxed{v_{ph}\, v_{gr}} = \frac{H}{p}\frac{dH}{dp} = \frac{cp}{\sqrt{1+p^2}}\,\frac{c\sqrt{1+p^2}}{p} = \boxed{c^2}$$

i.e. geometric mean of v_{ph} and v_{gr} is c $\Rightarrow$ $v_{ph} > c$ if $v_{gr} < c$

$$\frac{H}{p}\frac{dH}{dp} = \frac{d(H^2)}{d(p^2)} = c^2 \quad \text{integrate} \quad d(H^2) = c^2 d(p^2) \quad \Rightarrow \quad H^2 = c^2 p^2 + \text{const}$$

↑ rest mass as integration constant.

Decay of particles

Example: body of mass M splits in two parts of mass m_1, m_2

$\Rightarrow M = m_1 + m_2$? NO!

- In rest frame of body : $E = Mc^2$
- Energy conservation : $Mc^2 = E_1 + E_2$ (1) split $\Rightarrow m_1, m_2$ move apart $\Rightarrow |\bar{p}_i| > 0$

$\Downarrow$ $\Rightarrow E_i^2 = m_i^2 c^4 + c^2 \bar{p}_i^2 > m_i^2 c^4$

$M > m_1 + m_2$!

Spontaneous decay is possible

- if $M < m_1 + m_2$ $\Rightarrow$ decay is not possible (system is stable)

$\Rightarrow$ no spontaneous decay (or energy conservation would be violated)

$\Rightarrow$ to have decay, you need to give energy to the body of an amount $\Delta E = m_1 + m_2 - M$

- Momentum conservation : $0 = \bar{p}_1 + \bar{p}_2 \Rightarrow \bar{p}_1^2 = \bar{p}_2^2$ $\quad c^2 p_i^2 = E_i^2 - m_i^2 c^4$

energies of the components E_1, E_2

$E_1^2 - m_1^2 c^4 = E_2^2 - m_2^2 c^4$ $\leftarrow$ plug (1)

$\Rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2$ $\quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2$

Inverse process

at rest (lab)

m_1 $\bar{v}_1$ m_2 $\Rightarrow$ M

- $E = E_1 + E_2 = E_1 + m_2 c^2$
- $\bar{p} = \bar{p}_1 + \bar{p}_2 = \bar{p}_1$

$\Rightarrow$ Final velocity: $\bar{p} = \frac{E \bar{v}}{c^2}$ $\Rightarrow$ $\bar{v} = \frac{c^2 \bar{p}}{E} = \frac{c^2 \bar{p}_1}{E_1 + m_2 c^2}$ velocity of the merged body in the lab frame

$\Rightarrow$ Final mass M :

$E^2 = M^2 c^4 + c^2 \bar{p}^2$

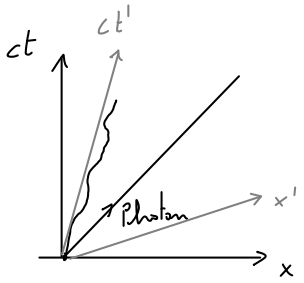
$c^4 M^2 = E^2 - c^2 \bar{p}^2$

$= (E_1 + m_2 c^2)^2 - (E_1^2 - m_1^2 c^4)$

$= E_1^2 + m_2^2 c^4 + 2 E_1 m_2 c^2 - E_1^2 + m_1^2 c^4$

$= c^4 \left(m_1^2 + m_2^2 + \frac{2 E_1 m_1}{c^2} \right)$ $\Rightarrow$ $\boxed{M^2 = m_1^2 + m_2^2 + \frac{2 E_1 m_1}{c^2}}$

Photons



Photons travel with the speed of light

$dx = c\,dt \quad 0 = -c^2dt^2 + dx^2 \Rightarrow ds^2 = 0$

null 4-interval, null gloabnic $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = 0$

moves along null lines $\Rightarrow p_\mu p^\mu = 0 \quad \bar{p} = (p^0, p^1, 0, 0) \Rightarrow \frac{p^1}{p^0} = 1$ alternative way to state $v = c$

$\Rightarrow$ must be massless! $\left(p_\mu p^\mu = -\frac{E^2}{c^2} + \bar{p}^2 \overset{!}{=} 0 \;:\; E = cp \right)$ $\quad E^2 = m^2c^4 + c^2p^2$, $m^2 = 0 \Rightarrow$ $\boxed{m = 0\,!}$

For massive particles the proper time is not identically zero: $ds^2 < 0$ (i.e. $d\tau^2 \neq 0$!)

$\boxed{ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau^2 = \eta_{\mu\nu}u^\mu u^\nu d\tau^2 < 0}$ (interval along particle world-line $\bar{x}(\tau)$)

direction of motion is given by u^μ
shift can be parameterized by τ
$\Big\}$ $dx^\mu = u^\mu d\tau$ (shift)

$\Rightarrow$ we can use $p^\mu = m u^\mu$

- **For photons** $ds^2 = 0 \quad -c^2d\tau^2 = ds^2 = 0$ proper time on the light-cone is identically zero!

$\Rightarrow u^\mu = \frac{dx^\mu}{d\tau}$ 4-velocity is not well defined!

we must use another affine parameter (not τ)

$\boxed{ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda^2 = \eta_{\mu\nu}K^\mu K^\nu d\lambda^2}$

$K^\mu \equiv \frac{dx^\mu}{d\lambda}$ wave vector (sort of 4-velocity) $\quad dx^\mu = K^\mu d\lambda$ (shift)

$K_\mu K^\mu = 0$ null-type: along the light-cone

$K_\mu K^\mu = -(K^0)^2 + (K^1)^2 + (K^2)^2 + (K^3)^2 = -(K^0)^2 + |\bar{K}|^2 \overset{!}{=} 0 \Rightarrow K^0 = \|\bar{K}\|$ $\quad \boxed{(K^\mu) = K^0(1, \bar{e})}$

- We can use K^μ to define the momentum of photons

$(p^\mu) = \left(\frac{E}{c}, \bar{p}\right)$ as for any particle, photons have a E and a $\bar{p}$

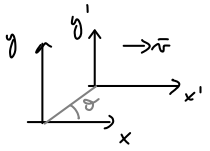
1) null vector (massless) $E = cp$
2) energy is quantized $E = h\nu = \hbar\omega$
$\Big\}$ $\boxed{p = \frac{h\nu}{c} = \frac{\hbar\omega}{c} = \frac{h}{\lambda}}$ $\quad \omega = \frac{\nu}{2\pi} \quad \lambda = \frac{c}{\nu}$

$(p^\mu) = \left(\frac{E}{c}, p\bar{e}\right) = \frac{E}{c}(1, \bar{e}) = \frac{\hbar\omega}{c}(1, \bar{e}) = \hbar(K^\mu) \Rightarrow$ $\boxed{(K^\mu) = \frac{\omega}{c}(1, \bar{e}) = \frac{2\pi}{\lambda}(1, \bar{e})} = \left(\frac{2\pi}{\lambda}, \bar{K}\right)$

"4-frequency"

– Lorentz transform of 4-frequency :

- $(k^\mu) = \left(\frac{2\pi}{\lambda}, i\vec{k}\right) = \frac{2\pi}{\lambda}(1, \cos\theta, \sin\theta, 0)$
- $(k^{\mu'}) = \left(\frac{2\pi}{\lambda'}, i\vec{k}'\right) = \frac{2\pi}{\lambda'}(1, \cos\theta', \sin\theta', 0)$



$k^{\mu'} = \Lambda^{\mu'}{}_\alpha k^\alpha$ $(\mu=0)$ time: $\frac{\lambda}{\lambda'} = \gamma(1-\beta\cos\theta)$ ← Doppler effect

$(\mu=i)$ space: $\tan\theta' = \frac{\tan\theta}{\gamma(1-\beta\sec\theta)}$ ← Abenation of light

Example:

$\mu=0 \quad k^0 = \Lambda^0{}_\alpha k^\alpha = \Lambda^0{}_0 k^0 + \Lambda^0{}_1 k^1 + \Lambda^0{}_2 k^2 + \Lambda^0{}_3 k^3$

$= \gamma k^0 - \beta\gamma k^1 + 0 + 0 = \frac{2\pi}{\lambda}(\gamma - \beta\gamma\cos\theta) \qquad \frac{2\pi}{\lambda'} = \frac{2\pi}{\lambda}\gamma(1-\beta\cos\theta)$

Energy-Momentum tensor

- Particle: characterized by $(p^\mu) = (\frac{E}{c}, \vec{p})$
- Fields: " " energy-momentum tensor $T^{\mu\nu}$ (good for covariant theory) ← Source of gravity
  - ↳ eg. matter field (continuous matter distribution), electro-magnetic field

- Consider flat geometry: $g = \eta$ $\quad q(x^\mu)$ scalar field

- Action $\quad S = \int L\, dt = \int \mathcal{L}\, dt dV = \frac{1}{c}\int \mathcal{L}\, d\Omega \quad d\Omega \equiv dx^0 dx^1 dx^2 dx^3$

- Euler-Lagrange eq. $\quad \delta S = 0 \rightarrow \frac{\delta}{\delta x^\nu}\left(\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\right) - \underbrace{\frac{\delta \mathcal{L}}{\delta q^\alpha}}_{(1)} = 0$ eq. of motion evolution of the system

- identify conserved quantities $\quad \frac{\delta \mathcal{L}}{\delta x^\mu} = 0 \Rightarrow$ corresponds to a symmetry of the system (Noether theorem)

Consider $\mathcal{L}(q^\alpha, q^\alpha_{,\nu})$: $\quad \frac{\delta \mathcal{L}}{\delta x^\mu} \overset{(2)}{=} \overbrace{\frac{\delta \mathcal{L}}{\delta q^\alpha}}^{(1)} \delta_\mu q^\alpha + \frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}} \underbrace{\delta_\mu q^\alpha_{,\nu}}_{\delta_\mu \delta_\nu \text{ commute}} = \underbrace{\frac{\delta}{\delta x^\nu}\left(\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\right)}_{\text{product rule}} \delta_\mu q^\alpha + \frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}} \delta_\nu q^\alpha_{,\mu} = \delta_\nu\left(\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}} \delta_\mu q^\alpha\right)$ ↑(b)

(a) $\delta^\nu_\mu \delta_\nu \mathcal{L} = \eta^{\sigma\nu}\eta_{\sigma\mu}\delta_\nu \mathcal{L}$

(b) $\delta_\mu q^\alpha = \eta_{\sigma\mu}\delta^\sigma q^\alpha$

$\Rightarrow \quad \delta_\nu\left(\underbrace{\frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}}^{(b)} \delta^\sigma q^\alpha - \eta^{\sigma\nu}\mathcal{L}\right)\eta_{\sigma\nu} = 0$

$\equiv T^{\sigma\mu}$ Energy-momentum tensor (this is the quantity which is conserved!)

$$\boxed{T^{\sigma\nu} \equiv \frac{\delta \mathcal{L}}{\delta q^\alpha_{,\nu}}\delta^\sigma q^\alpha - \eta^{\sigma\nu}\mathcal{L}} \quad \text{and} \quad \boxed{\delta_\mu T^{\sigma\mu} = 0} \quad \text{Energy-Momentum conservation}$$

- Note: in general T is not diagonal but it can be diagonalized

- Note: $T^{\mu\nu}$ is not unique

$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \delta_\sigma \psi^{\mu\nu\sigma}$ with $\psi^{\mu\nu\sigma} = -\psi^{\mu\sigma\nu}$ ← antisymmetric with respect to $\sigma\nu$

$\delta_\mu \tilde{T}^{\mu\nu} = 0$ still satisfied because of the antisymmetry of ψ: $\delta_\nu \delta_\sigma \psi^{\mu\nu\sigma} = 0$ (Symmetric, antisymmetric)

What is the meaning of $T^{\mu\nu}$?

- $\boxed{T^{00}} = \dot{q}\frac{\delta \mathcal{L}}{\delta \dot{q}} - \mathcal{L} = \mathcal{E}$ Hamiltonian density (local inertial frame) $\Rightarrow$ T^{00} = energy density!

- $T^{0i} = ?$

$\partial_\nu T^{\nu\mu} = \partial_0 T^{0\mu} + \partial_i T^{i\mu} = 0$ E.-M. conservation is a continuity equation for the quantity $T^{0\mu}$

$\mu = 0$: $\frac{\partial T^{00}}{c\partial t} + \partial_i T^{0i} = 0$ $\quad \frac{\partial \mathcal{E}}{\partial t} + c\partial_i T^{0i} = 0$ ↓ integrate over volume V

$\int_V \partial_t \mathcal{E}\, dV = -c\int_V \partial_i T^{0i} dV$ ↓ Gauss theorem

$\partial_t \int_V \mathcal{E}\, dV = -c\int_{\partial V} T^{0j} dA_j$

time variation of energy within volume V — variation due to energy flux across the surface A_j enclosing V

$\Rightarrow$ $cT^{0i} \equiv S^i$ energy density flux!

- $T^{ij} = ?$

$\mu = i$: $\frac{\partial T^{0i}}{c\partial t} + \partial_j T^{ij} = 0$ $\quad \frac{\partial S^i}{c^2 \partial t} + \partial_j T^{ij} = 0$ $\Rightarrow$ T^{ij} is the density flux of $\frac{S^i}{c^2} = p^i$

- what is $\frac{S^i}{c^2}$? Dimensional analysis:

for a particle: $p^i = \frac{E}{c^2} v^i$ $\Rightarrow$ $c^2 p^i = E v^i$ $\left[J\frac{m}{s}\right]$ $\Rightarrow$ $c^2 p^i$ is an energy flux

for a field: $c^2 p^i = S^i$ $\left[J\frac{m}{s}\right]$ $\Rightarrow$ $\frac{S^i}{c^2} = p^i$ is a momentum density

$\Rightarrow$ $T^{ij} \equiv \sigma^{ij}$ momentum density flux!

$$\Rightarrow (T^{\mu\nu}) = \begin{pmatrix} \mathcal{E} & S_x/c & S_y/c & S_z/c \\ \cdot & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \cdot & \cdot & \sigma_{yy} & \sigma_{yz} \\ \cdot & \cdot & \cdot & \sigma_{zz} \end{pmatrix}$$

$\vec{S}$ = energy-density flux $\left[\frac{d\mathcal{E}}{dt\,dA}\right]$

σ_{ij} = stress tensor $\left[\frac{dP}{dt\,dA}\right]$ "pressure"

- What about G.R.?!

- In a local inertial frame (free fall) the above holds

- Strong equivalence principle: $\partial_\alpha \to \nabla_\alpha$, $\eta_{\mu\nu} \to g_{\mu\nu}$ $\Rightarrow$ $\boxed{\nabla_\mu T^{\mu\nu} = 0}$ careful with interpretation

actually not just strong equiv. principle -- there is something more fundamental, more on that later ....

Energy momentum tensors of a perfect relativistic fluid

perfect = No heat conduction

$$T = \left(\rho + \frac{P}{c^2}\right)\bar{u}\otimes\bar{u} + P g^{-1} \qquad T^{ab} = \left(\rho + \frac{P}{c^2}\right)u^a u^b + P g^{ab} \quad \text{(components)}$$

ρ : density $\in \mathbb{R}$
P : pressure of the fluid $\in \mathbb{R}$
} in rest frame $\rho(x^\nu)$ $P(x^\nu)$

$\bar{u}$: time-like vector representing the 4-velocity of the fluid volume element

- note: $T^{\mu\nu}g_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u^\mu u_\mu + 4P = -\rho c^2 - P + 4P = 3P - \rho$

$\Rightarrow P = \frac{1}{3}\left(T^{\mu\nu}g_{\mu\nu} - \rho\right) \in \mathbb{R}$ because $T^{\mu\nu}g_{\mu\nu}$ is a scalar

i.e. P and ρ are scalar functions

- In a local inertial frame (isotropic) $g \to \eta$, $\bar{u} = (c, 0, 0, 0)^T$

$T^{00} = \left(\rho + \frac{P}{c^2}\right)c^2 - P = \rho c^2$ rest energy density

$T^{ii} = P$ isotropic pressure

$T = \mathrm{diag}(\rho c^2, P, P, P)$

- Pressureless fluid (dust) $P = 0$ $T^{\mu\nu} = \rho u^\mu u^\nu$

- Hydrodynamics "for free":

$\Rightarrow \nabla_a T^{ab} = 0$ hydrodynamic equations! eq. of motion of fluid

$$\begin{cases} \nabla_\mu T^{0\mu} = 0 & & \text{energy conservation} \\ \nabla_\mu T^{i\mu} = 0 & \text{Euler's eq.} & \text{momentum conservation} \\ \partial_\mu \rho = 0 & \text{continuity eq.} & \text{matter conservation} \\ P = P(\rho, T) & \text{eq. of state} & \rho = \text{scalar field} \end{cases}$$

Energy-momentum tensor of electromagnetic fields

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{16\pi}F_{\alpha\beta}F_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta} \quad \Rightarrow \quad \boxed{T^{\mu\nu} = \frac{1}{4\pi}\left[-F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right]}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ electromagnetic field A^μ = 4-potential

General Relativity in a nutshell

What is general relativity for?

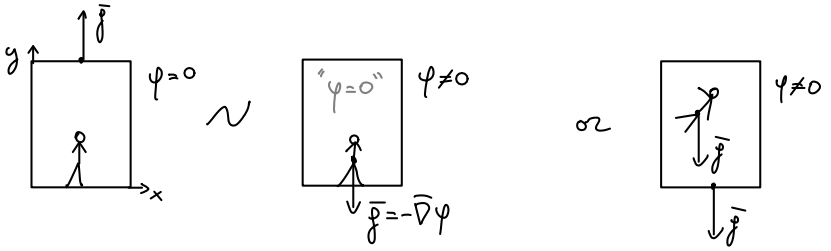
- generalization of special relativity to non-inertial frames
- it describes gravity as a manifestation of the curvature of spac-time

Why do we need it?

- we need a covariant theory: Lorentz invariance, retardation, ...
- Newton's gravity fails: perihelion shift, gravitational lensing, Grav. waves, time dilation, Lens-Thirring effe
- Newton's gravity is weird... $m_I \vec{a} = -m_G \nabla\varphi$ $m_I = m_G$! inertial mass = "gravitational charge"
  => m_G is associated to some "inertial phenomenon"

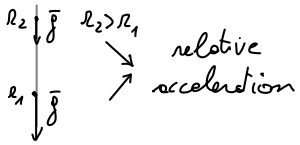
The key idea: equivalence principle

- gravity <---> non-inertial frame (Einsteil elevator)
- in local inertial frame (free fall) same tensorial expressions of special relativity (SEP)



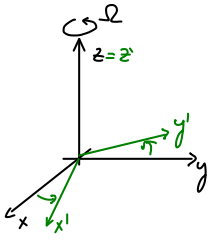
in $\varphi \neq 0$ free falling observer, feels no gravity (locally)

- real "grav. fields" can not be eliminated everyehere simultaneously by choosing a frame
- you can remove it only locally (the elevator)
- carful here: analogy, not reality. in a real "grav. field" you have tydal forces! →



Link between an accelerated frame and the metric

- Equivalence principle : accelerated frame $\leftrightarrow$ gravity
- Free particle : no field, i.e. no external force
- Inertial frame : $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$
- In an accelerated system : $\begin{cases} x = x'\cos(\Omega t) - y'\sin(\Omega t) \\ y = x'\sin(\Omega t) + y'\cos(\Omega t) \end{cases}$, $z = z'$



$$\begin{cases} dx = dx'\cos(\Omega t) + x'\sin(\Omega t)\Omega dt - dy'\sin(\Omega t) + y'\cos(\Omega t)\Omega dt \\ dy = dy'\sin(\Omega t) - x'\cos(\Omega t)\Omega dt + dy'\cos(\Omega t) + y'\sin(\Omega t)\Omega dt \end{cases}$$

$$\begin{aligned} ds^2 &= -c^2dt^2 + dx^2 + dy^2 + dz^2 \\ &= \left[c^2 - \Omega^2(x'^2+y'^2)\right]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y'dx'dt - 2\Omega x'dy'dt \\ &= g_{\mu\nu}dx^\mu dx^\nu \end{aligned}$$

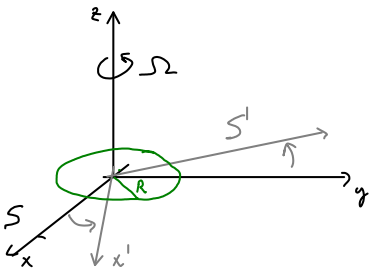
time g_{00} — space g_{ii} — off-diagonal (mixed) g_{10} g_{01} g_{20} g_{02}

equivalence principle

- Centrifugal for |---(equivalence principle)---> locally as a gravitational field
- Mathematically: associate gravity to the metric of the space-time!

Careful! Here the space is still Minkowski! (flat)

- $g = \eta$ ↔ ≠ coord system does not change the properties of space-time
- In fact non-locally, very different behavior then with a grav. field
  - ↳ in a grav. field: for $r \to \infty$ from source of field $\bar{g} = -\bar{\nabla}\varphi \to 0$
  - ↳ here $r \to \infty$ $\bar{g} \to \infty$!



Note:
measure the circonference of a circle in the x-y plane
S = inertial ($\Omega = 0$) $C = 2\pi R$
S = non-inertial ($\Omega \neq 0$) $C \neq 2\pi R$! ≠ geometry!
↑ "curved space"
because of Lorentz contraction

Connecting the metric to a gravitational field

- Take a non relativistic particle in a given fixed gravitational field
- Let's try to find the metric producing a Newtonian gravitational field

(1) Newtonian gravity

$L = -mc^2 + \frac{1}{2}m\vec{v}^2 - m\varphi$ note $m_i \stackrel{!}{=} m_G$

$S_G = \int L\,dt = -mc^2 \int \left(1 - \frac{\vec{v}^2}{2c^2} + \frac{\varphi}{c^2}\right) dt$ $\stackrel{!}{=}$ $S_f = -mc^2 \int d\tau = -mc^2 \int \frac{1}{c}\left(-g_{\mu\nu}dx^\mu dx^\nu\right)^{1/2}$

(2) Free particle with arbitrary metric

Equivalence principle

NON-free particle in flat space-time ($\eta_{\mu\nu}$) $\sim$ free particle in NON-flat space-time ($g_{\mu\nu}$)

↓

find the metric $g_{\mu\nu}$ reproducing Newtonian gravity

$\frac{1}{c}\left(-g_{\mu\nu}dx^\mu dx^\nu\right)^{1/2} \stackrel{!}{=} \left(1 - \frac{\vec{v}^2}{2c^2} + \frac{\varphi}{c^2}\right) dt \stackrel{*}{\simeq} \left(1 - \frac{\vec{v}^2}{c^2} + \frac{2\varphi}{c^2}\right)^{1/2} dt$ $\quad * \; -\frac{\vec{v}^2}{2c^2} + \frac{\varphi}{c^2} \ll 1$ i.e. non relativistic particle in weak field

$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \stackrel{!}{=} -c^2\left(1 - \frac{\vec{v}^2}{c^2} + \frac{2\varphi}{c^2}\right)dt^2 = -\left(1 + \frac{2\varphi}{c^2}\right)c^2dt^2 + \frac{d\vec{x}^2}{dt^2}dt^2$

identify $g_{\mu\nu}$

$\Rightarrow$ $g_{00} = -\left(1 + \frac{2\varphi}{c^2}\right)$, $g_{0i} = 0 = g_{i0}$, $g_{ij} = \delta_{ij}$

small correction of $\eta_{00} = -1$

In reality you have a correction of the same order of g_{00}: $g_{ii} = 1 - \frac{2\varphi}{c^2}$ (coming from Einstein eq. in weak field limit); without it you get gravitational lensing wrong!

D Reformulate Poisson equation in terms of g and T

$\begin{cases} \vec{\nabla}^2 g_{00} = -\frac{2}{c^2}\vec{\nabla}^2\varphi \\ T_{00} = -\rho c^2 \text{ (dust)} \end{cases} \Rightarrow \boxed{\vec{\nabla}^2\varphi = 4\pi G\rho}$ (Poisson eq.) $\rightarrow \boxed{\vec{\nabla}^2 g_{00} = \frac{8\pi G}{c^4}T_{00}}$ double derivatives of g!

or $\Box g_{00} = \frac{8\pi G}{c^4}T^\nu{}_\nu$

assuming: $\frac{1}{c^2}\frac{\partial^2 g_{00}}{\partial t^2} \ll \vec{\nabla}^2 x$ slow evolution

$P \ll \rho c^2$ negligible pressure

Note: $\Box \equiv \partial_\mu\partial^\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ Provides retardation

$\vec{\nabla}^2 \equiv \partial_i\partial^i = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ has no retardation

- Careful... this is not yet GR! This is just to show yuo a link between potential and metric

Link between non-inertial frames and geodesic equation

Generic coord. transformation between frames S, S'

$x^{\mu'} = x^{\mu'}(x^{\mu}(\lambda))$

↑ ↑ transformations law

↑ transformed component

$\frac{dx^{\mu'}}{d\lambda} = \frac{\delta x^{\mu'}}{\delta x^{\mu}} \frac{dx^{\mu}}{d\lambda}$ $\quad J^{\mu'}{}_{\mu} = \frac{\delta x^{\mu'}}{\delta x^{\mu}}$ Jacobian of the transformation

λ: affine parameter $x^{\nu}(\lambda)$

$$\underbrace{\frac{d^2x^{\mu'}}{d\lambda^2}}_{(c)} = \frac{d}{d\lambda}\left(\frac{dx^{\mu'}}{d\lambda}\right) = \frac{d}{d\lambda}\left(\frac{\delta x^{\mu'}}{\delta x^{\mu}}\frac{dx^{\mu}}{d\lambda}\right) \overset{\text{product rule}}{=} \left(\frac{d}{d\lambda}\frac{\delta x^{\mu'}}{\delta x^{\mu}}\right)\frac{dx^{\mu}}{d\lambda} + \frac{\delta x^{\mu'}}{\delta x^{\mu}}\left(\frac{d^2x^{\mu}}{d\lambda^2}\right) = \boxed{\underbrace{\frac{\delta^2 x^{\mu'}}{\delta x^{\mu}\delta x^{\beta}}}_{(a)}\underbrace{\frac{dx^{\beta}}{d\lambda}\frac{dx^{\mu}}{d\lambda}}_{\mu\to\nu} + \underbrace{\frac{\delta x^{\mu'}}{\delta x^{\nu}}\frac{d^2x^{\nu}}{d\lambda^2}}_{(b)}} \quad \text{(eq. 1)}$$

Special Relativity: we deal with inertial frames

- free particle: Inertial frames $\Rightarrow$ (c) $\overset{!}{=} 0$, (b) $\overset{!}{=} 0$, (a) = arbitrary $\Rightarrow$ $\boxed{\frac{\delta^2 x^{\mu'}}{\delta x^{\mu}\delta x^{\beta}} \overset{!}{=} 0}$

Solve (2) for $x^{\mu'}$: $0 = \int \frac{\delta^2 x^{\mu'}}{\delta x^{\nu}\delta x^{\mu}} x^{\mu} dx^{\nu} = \underbrace{\frac{\delta x^{\mu'}}{\delta x^{\mu}}}_{\equiv J^{\mu'}{}_{\mu}} x^{\mu} - \int \underbrace{\frac{\delta x^{\mu'}}{\delta x^{\mu}}\frac{\delta x^{\mu}}{\delta x^{\nu}}}_{=\frac{\delta x^{\mu'}}{\delta x^{\nu}}} dx^{\nu} = J^{\mu'}{}_{\mu} x^{\mu} - x^{\mu'} + a^{\mu'} = 0$

($a^{\mu'}$: integration constant)

$\boxed{x^{\mu'} = J^{\mu'}{}_{\mu} x^{\mu} + a^{\mu'}}$ i.e. a linear transformation

($a^{\mu'}$: origin shift (irrelevant))

- Distance between points invariant under coord. transformation

→ i.e. preserves scalar product $\boxed{\langle \bar{u}, \bar{v} \rangle = \langle \bar{u}', \bar{v}' \rangle}$ (1)

→ i.e. metric is invariant $\boxed{\eta = J^T \eta J}$ (2) i.e. J is an isometry

$\Rightarrow$ Preserves homogeneity

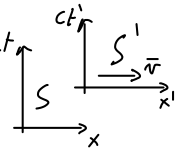
(1) and (2) are equivalent:

$$d\bar{s}^2 = d\bar{x}^T \eta\, d\bar{x} \overset{!}{=} d\bar{x}'^T \eta\, d\bar{x}' = d\bar{x}^T J^T \eta J\, d\bar{x} \quad \Rightarrow \quad \boxed{\eta = J^T \eta J} \leftrightarrow \boxed{\eta_{\mu\nu} = J^{\alpha'}{}_{\mu} J^{\beta'}{}_{\nu} \eta_{\alpha'\beta'}}$$

- Minkowski space c = invariant

$\eta \overset{!}{=} \eta'$ is the necessary and sufficient condition for transformation $\Lambda^{\mu}{}_{\nu'} \equiv \frac{\delta x^{\mu}}{\delta x^{\nu'}}$ to be a Lorentz transform

$$\left(J^{\mu'}{}_{\nu}\right) \overset{!}{=} \left(\Lambda^{\mu'}{}_{\nu}\right) = \left(\frac{\delta x^{\mu'}}{\delta x^{\nu}}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix} \text{ for "normal configuration"}$$



General Relativity: non-inertial frames



- free particle: non-inertial frames $\Rightarrow$ (a), (b) arbitrary; (c) $= \frac{d^2x^{\mu'}}{d\lambda^2} \overset{!}{=} 0$ (free to choose)

(free-falling frame)
local inertial frame
Equivalence principle

- eq. (1): $\frac{\delta^2 x^{\mu'}}{\delta x^\mu \delta x^\beta}\frac{dx^\beta}{d\lambda}\frac{dx^\mu}{d\lambda} + \frac{\delta x^{\mu'}}{\delta x^\nu}\frac{d^2x^\nu}{d\lambda^2} = 0$ $\cdot \frac{\delta x^\nu}{\delta x^{\mu'}}$ $\Rightarrow$ $\frac{d^2x^\nu}{d\lambda^2} + \frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda}\underbrace{\frac{\delta^2 x^{\mu'}}{\delta x^\beta \delta x^\mu}\cdot\frac{\delta x^\nu}{\delta x^{\mu'}}}_{\equiv \Gamma^\nu_{\beta\mu}} = 0$ ($\frac{\delta x^\nu}{\delta x^{\mu'}} \equiv \delta^\nu_{\mu'}$)

Γ affine connection

$$\boxed{\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\beta\mu}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} = 0}$$ geodesic equation: eq. of motion of free particle in a grav. field

- Note:
  - $\Gamma^\nu_{\beta\mu}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda}$ plays the role of a force (gravity!) e.g. $\lambda = \tau$ $\frac{du^\nu}{d\tau^2} + \Gamma^\nu_{\beta\mu}u^\mu u^\beta = 0$ (4-acceleration)
  - $\frac{d^2x^{\mu'}}{d\lambda^2} \overset{!}{=} 0$ in S' all laws valid in Special Relat. holds!
  - $\Gamma^\nu_{\beta\mu} \equiv \frac{\delta^2 x^\nu}{\delta x^\beta \delta x^\mu}$ symmetric with respect to lower indices (δ_μ, δ_ν commute) (Torsion free condition)

metric connection

Generic transformation of the metric

$g_{\mu'\nu'} = g_{\mu\nu}\frac{\delta x^\mu}{\delta x^{\mu'}}\frac{\delta x^\nu}{\delta x^{\nu'}}$

$\delta_\lambda g_{\mu'\nu'} = \delta_\lambda g_{\mu\nu}\frac{\delta x^\mu}{\delta x^{\mu'}}\frac{\delta x^\nu}{\delta x^{\nu'}} + g_{\mu\nu}\frac{\delta x^\beta}{\delta x^{\nu'}}\delta_\lambda\frac{\delta x^\mu}{\delta x^{\mu'}} + g_{\mu\nu}\frac{\delta x^\mu}{\delta x^{\mu'}}\delta_\lambda\frac{\delta x^\nu}{\delta x^{\nu'}} = g_{\mu\nu}\Gamma^\mu_{\lambda\mu'} + g_{\mu'\nu}\Gamma^\nu_{\lambda\nu'}$ (1)

($\overset{!}{=}0$ impose (portable))

$\delta_\lambda g_{\mu'\nu'} + \delta_{\nu'}g_{\lambda\mu'} - \delta_{\mu'}g_{\nu'\lambda} = 2\Gamma^\gamma_{\lambda\nu'}g_{\mu'\gamma}$ $\cdot g^{\mu'\gamma}$ (inverse)

(relabeling) $\lambda \to \alpha$ $\nu' \to \beta$ $\mu' \to \eta$

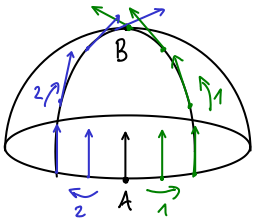
remember $g_{\alpha\beta} = g_{\beta\alpha}$

$$\Rightarrow \boxed{\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\eta}\left(g_{\beta\eta,\alpha} + g_{\alpha\eta,\beta} - g_{\alpha\beta,\eta}\right)}$$

Christoffel symbols

Note: $g_{\mu\nu}$ determines $\Gamma^\gamma_{\alpha\beta}$ $\Rightarrow$ $g_{\mu\nu}$ determines gravity!

Differential geometry, the language of GR



Curved Space-time: manifold $M = \{P\}$ relative acceleration ~ force

Tangent space M_pT vector space at point $P \in M$

Tensor $T_p : \underbrace{M_pT^* \times \dots \times M_pT^*}_{r} \times \underbrace{M_pT \times \dots \times M_pT}_{s} \to \mathbb{R}$ type $\binom{r}{s}$, rank $N = r+s$ N-linear map

Metric: $g_p : T_pM \times T_pM \to \mathbb{R}$ (depends on position!)

scalar product: $g_p(\bar{v},\bar{u}) \equiv \langle \bar{v},\bar{u}\rangle = g_{\mu\nu} v^\mu u^\nu = v_\mu u^\mu = v^\mu u_\mu = a$ ($g_{\mu\nu}$: dual vector $\in M_pT^*$)

norm squared: $g_p(\bar{v},\bar{v}) = |\bar{v}|^2 \Rightarrow$ separations

non degenerate: $g(\bar{u},\bar{v}) = 0 \ \forall \bar{v} \neq 0 \Leftrightarrow \bar{u} = 0$

4-interval: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu \Rightarrow g_{\mu\nu} = g_{\nu\mu}$ i.e. g is symmetric

Vector $\bar{v} : M_pT^* \to \mathbb{R}$ $\bar{v}(\tilde{v}) = a$ $\bar{v} = v^\mu \bar{e}_\mu$ $\{\bar{e}_\mu\}$ basis set of vectors v^μ: component, μ: label

Dual vector $\tilde{v} : M_pT \to \mathbb{R}$ $\tilde{v}(\bar{v}) = a$ $\tilde{v} = v_\mu \tilde{w}^\mu$ $\{\tilde{w}^\mu\}$ basis set of dual vectors

Covariant derivative

∂_μ is now position dependent

$$\partial_\mu \bar{v} = \partial_\mu (v^\alpha \bar{e}_\alpha) = \partial_\mu v^\alpha \bar{e}_\alpha + v^\alpha \partial_\mu \bar{e}_\alpha$$

$$= \partial_\mu v^\alpha \bar{e}_\alpha + v^\alpha \Gamma^\gamma_{\mu\alpha} \bar{e}_\gamma \quad (\alpha \leftrightarrow \gamma)$$

$$= \underbrace{\left(\partial_\mu v^\alpha + v^\gamma \Gamma^\alpha_{\mu\gamma}\right)}_{\equiv \nabla_\mu v^\alpha} \bar{e}_\alpha = \nabla_\mu v^\alpha \bar{e}_\alpha$$

← metric compatibility $\nabla_\beta g_{\mu\nu} \overset{!}{=} 0$ ⇔ assume $\partial_\mu \bar{e}_\alpha$ to be a vector: $\partial_\mu \bar{e}_\alpha = \Gamma^\gamma_{\mu\alpha} \bar{e}_\gamma$

affine connection coefficients: expresses the change of basis

Parallel transport $u^\nu \nabla_\nu v^\mu = 0$ $u^\nu \nabla_\nu \bar{v} \equiv \nabla_{\bar{u}} \bar{v} \equiv \frac{d\bar{v}}{d\lambda}$ directional derivative (change along direction given by $\bar{u}$)

Geodesic eq $u^\nu \nabla_\nu u^\mu = u^\nu \partial_\nu u^\mu + \Gamma^\mu_{\nu\gamma} u^\nu u^\gamma = 0$ generalization of a straight line on curved space; trajectories of free particles

Riemann "map" $\hat{R} : M_pT \times M_pT \times M_pT \to M_pT$ $\hat{R}(\bar{v},\bar{u},\bar{w}) \equiv \nabla_{\bar{v}} \nabla_{\bar{u}} \bar{w} - \nabla_{\bar{u}} \nabla_{\bar{v}} \bar{w} - \nabla_{[\bar{u},\bar{v}]} \bar{w}$

Riemann curvature $R : M_pT^* \times M_pT \times M_pT \times M_pT \to \mathbb{R}$ $R(\tilde{r},\bar{v},\bar{u},\bar{w}) \equiv \tilde{r}\,\hat{R}(\bar{v},\bar{u},\bar{w})$

$$R^i{}_{jke} = \partial_k \Gamma^i_{ej} - \partial_\ell \Gamma^i_{kj} + \Gamma^i_{k\gamma} \Gamma^\gamma_{ej} - \Gamma^i_{e\gamma} \Gamma^\gamma_{kj} \qquad i,j,\ell,k = 0,1,2,3$$

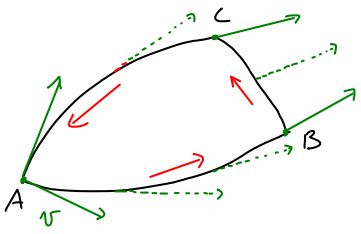
$$R^i{}_{jke} \ \frac{1}{2} g^{i\ell} \left(\partial_k \partial_j g_{\ell e} - \partial_k \partial_\gamma g_{j\ell} - \partial_e \partial_j g_{k\gamma} + \partial_e \partial_\gamma g_{jk}\right)$$ ← double derivatives of g!!

Ricci tensor and scalar $R_{\mu\nu} \equiv g^{\alpha\gamma} R^\beta{}_{\alpha\beta\nu} = R^\alpha{}_{\alpha\mu\nu}$ $R_{j\ell} = R_{\ell j}$ symmetric

$R \equiv g^{\nu\gamma} R_{\gamma\nu} = R^\nu{}_\nu$ Trace of the Ricci tensor

What des the Riemann tensor describe?

- Parallel transport of a vector along a closed loop
- Difference between initial and final vector
- Exploits Stokes theorem



1) Parallel transport along δ_α direction:

$$\nabla_{\delta_\alpha} u^\nu = 0 \qquad \delta_\alpha u^\nu + \Gamma^\nu_{\alpha\gamma} u^\gamma = 0 \qquad \delta_\alpha u^\nu = -\Gamma^\nu_{\alpha\gamma} u^\gamma \quad (1)$$

2) Change after infinitesimal shift dx^α:

$$du^\nu = \delta_\alpha u^\nu \, dx^\alpha = -\Gamma^\nu_{\alpha\gamma} u^\gamma dx^\alpha$$

3) Total change:

$$\begin{aligned}
\Delta u^\nu &\equiv \oint du^\nu \qquad \text{integral over a closed loop} \\
&= -\oint \Gamma^\nu_{\alpha\gamma} u^\gamma dx^\alpha \qquad B^\nu{}_\alpha \equiv \Gamma^\nu_{\gamma\alpha} u^\gamma \quad (2) \\
&= -\oint B^\nu{}_\alpha dx^\alpha \qquad \text{Stokes theorem!} \quad (\text{surface}) \, ds^{\alpha\beta} \\
&= -\int \delta_\beta B^\nu{}_\alpha \, ds^{\alpha\beta} \qquad (2) \\
&= -\frac{1}{2}\int \left(\delta_\alpha B^\nu_\beta - \delta_\beta B^\nu_\alpha\right) ds^{\alpha\beta} \\
&= -\frac{1}{2}\int \left[\delta_\alpha\left(\Gamma^\nu_{\beta\gamma} u^\gamma\right) - \delta_\beta\left(\Gamma^\nu_{\alpha\gamma} u^\gamma\right)\right] ds^{\alpha\beta} \\
&= -\frac{1}{2}\int \left[\Gamma^\nu_{\beta\gamma,\alpha} u^\gamma + \Gamma^\nu_{\beta\gamma} u^\gamma_{,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta} u^\gamma - \Gamma^\nu_{\alpha\gamma} u^\gamma_{,\beta}\right] ds^{\alpha\beta} \qquad (1) \\
&= -\frac{1}{2}\int \left[\Gamma^\nu_{\beta\gamma,\alpha} u^\gamma - \Gamma^\nu_{\beta\gamma}\Gamma^\gamma_{\alpha\sigma} u^\sigma - \Gamma^\nu_{\alpha\gamma,\beta} u^\gamma + \Gamma^\nu_{\alpha\gamma}\Gamma^\gamma_{\beta\sigma} u^\sigma\right] ds^{\alpha\beta} \\
&\qquad \sigma \leftrightarrow \gamma \qquad\qquad \sigma \leftrightarrow \gamma \\
&= -\frac{1}{2}\int \left(\Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\nu_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma}\right) u^\gamma \, ds^{\alpha\beta} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -R^\nu_{\gamma\alpha\beta}
\end{aligned}$$

$$\boxed{du^\nu = \frac{1}{2} R^\nu_{\gamma\alpha\beta} u^\gamma \, ds^{\alpha\beta}} \; * \qquad \boxed{R^\nu_{\gamma\alpha\beta} = \Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\nu_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} - \Gamma^\nu_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma}}$$

$\Rightarrow$ R quantifies the change of a vector when parallely transported along a closed loop

- the change depends on the path

$S^c{}_b$ tydal stress tensor (tydal forces)

- Physical meaning of *: 4-acceleration $\frac{d^2 x^\nu}{d\tau^2} = R^\nu_{\gamma\alpha\beta} u^\gamma u^\alpha dl^\beta$ geodesic deviation eq.

A theory of gravity build on that: GR

- We need something based on double derivatives of the $g_{\mu\nu}$ because we can always find a frame in which

$$g_{\mu\nu}(P) = \eta_{\mu\nu} + O(|\vec{x}-\vec{x}'|^2) \quad \text{i.e.} \quad \underbrace{g_{\alpha\beta}(P)=\eta_{\alpha\beta} \quad g_{\alpha\beta,\gamma}(P)=0}_{\text{Locally flat at 1° order}} \qquad \underbrace{g_{\alpha\beta,\gamma\mu}(P)\neq 0}_{\text{Tydal effects: second derivatives}}$$

- The strong equivalence principle: $\eta \to g$ + $\partial_\mu \to \nabla_\mu$
- The source of gravity is "matter $T_{\mu\nu}$ (symmetric, rank-2 tensor)
- Build a covariant field eq. $\boxed{G_{\mu\nu} = \alpha T_{\mu\nu}}$ with source $T_{\mu\nu}$, $G_{\mu\nu}$ geometry

$$\left\{ \begin{array}{l} G_{\mu\nu} \text{ must be symmetric} \\ \text{Tensor } G_{\mu\nu} \text{ must "respect" energy momentum conservation, } \nabla_\mu T^{\mu\nu}=0, \text{ i.e. } \nabla_\mu G^{\mu\nu}=0 \\ \text{In } G_{\mu\nu} \text{ use Riemann curvature tensor!} \end{array} \right\} *$$

* The only tensor satisfying these conditions is $\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}$ Einstein tensor

- Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \leftarrow \text{free to add } \Lambda g_{\mu\nu} \text{ Cosmological const.}$$

equivalent ↕

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) \qquad T \equiv T^\mu{}_\mu \qquad * \text{ see appendix}$$

- Lovelock theorem $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{4\pi G}{c^4} T_{\mu\nu}$ unique 4D rank 2 tensor field theory satisfying the above conditions

- Einstein-Hilbert action $S_H = \int R\sqrt{-g}\, d\Omega$ $\quad \delta(S_H + S_M) = 0 \Rightarrow$ Einstein field equations (eq of motion of the field: $g_{\mu\nu}$)

- Energy-Momentum conservation $\nabla_\nu T^{\mu\nu} = 0$

— ∘ —

Conceptually all boils down to:

$$\nabla^2 \varphi = 4\pi G \rho$$

$$\partial_\nu \partial^\nu \varphi = \frac{4\pi G}{c^2} T_\nu{}^\nu$$

$$G_{\mu\nu} = \frac{4\pi G}{c^4} T_{\mu\nu}$$

↘ $\Box \equiv \partial_\nu \partial^\nu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^1} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^3}$ Retardation

↘ $\rho \simeq \frac{T_\nu{}^\nu}{c^2}$ non relativistic pressureless ideal fluid ("$\circ_\circ$")

↘ $\eta_{\mu\nu} \to g_{\mu\nu}$ $\quad \partial_\mu \to \nabla_\mu$ Strong Equivalence Principle (SEP)

Conceptual steps and summary

$m_G = m_I$

gravity must be associated to "inertial forces" $S_G = S_{free} \Rightarrow g_{\mu\nu} \neq \eta_{\mu\nu}$

↓

The equivalence principle

accelerations due to gravity $\sim$ non inertial frame

↓

Generic coordinate transformation $x^{\mu'} = x^{\mu'}(x^\mu(\lambda))$

$$\frac{d^2x^{\mu'}}{d\lambda^2} = \frac{\delta^2 x^{\mu'}}{\delta x^\alpha \delta x^\beta}\frac{dx^\beta}{d\lambda}\frac{dx^\alpha}{d\lambda} + \frac{\delta x^{\mu'}}{\delta x^\nu}\frac{d^2x^\nu}{d\lambda^2} \Rightarrow \begin{bmatrix} \text{local inertial frame } (x^{\mu'}): \frac{d^2x^{\mu'}}{d\lambda^2} \overset{!}{=} 0 \\ \text{arbitrary frame } (x^\mu) \end{bmatrix} \qquad \boxed{\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\beta\mu}\frac{dx^\mu}{d\lambda}\frac{dx^\beta}{d\lambda} = 0}$$

geodesic equation Trajectory of a free particle in an arbitrary frame

↓

Generic transformation of the metric

$$g_{\mu'\nu'} = g_{\mu\nu}\frac{\delta x^\mu}{\delta x^{\mu'}}\frac{\delta x^\nu}{\delta x^{\nu'}} \Rightarrow \boxed{\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\eta}\left(g_{\beta\eta,\alpha} + g_{\alpha\eta,\beta} - g_{\alpha\beta,\eta}\right)}$$ Christoffel symbols

↓

Properties of the space time differ point by point Non Minkowski Space-Time

↓

Covariant derivative

$$\delta_\mu \bar{v} = \delta_\mu v^\alpha \bar{e}_\alpha + v^\alpha \delta_\mu \bar{e}_\alpha = \left(\delta_\mu v^\alpha + v^\gamma \Gamma^\alpha_{\mu\gamma}\right)\bar{e}_\alpha = \nabla_\mu v^\alpha \bar{e}_\alpha$$

↓

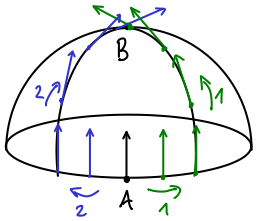
Parallel transport and Geodesic eq.

$u^\nu \nabla_\nu u^\mu = 0$ generalization of a straight line on an arbitrary manifold

↓

Riemann curvature Tensor

$$\Delta u^\nu \equiv \oint \delta u^\nu \qquad R^\nu_{\gamma\alpha\beta} = \Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\nu_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} - \Gamma^\nu_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma}$$



↓

Einstein field equations

$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$: Space-time tells matter how to move, matter tells S-T how to "curve"

Particle in a gravitational field → Free particle in a curved space time

Appendix

Einstein eq.s can be writte an:

$$g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}R\,g^{\mu\nu}g_{\mu\nu} = \kappa\, g^{\mu\nu}T_{\mu\nu} \qquad R - \frac{1}{2}4R = \kappa T \qquad \boxed{R = -\kappa T}$$

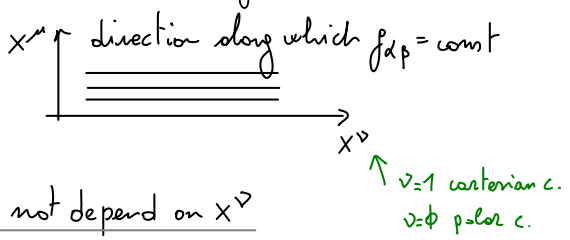
$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = R_{\mu\nu} + \frac{\kappa}{2}T g_{\mu\nu} = T_{\mu\nu} \quad \Rightarrow \quad \boxed{R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)}$$

- Convenient for vacuum ($T_{\mu\nu}=0$) solutions where they reduce to $R_{\mu\nu}=0$
- " in cosmology where T can be easily found

A glimps in Energy-momentum "conservation" in GR

$\frac{du_\nu}{d\tau} - \frac{1}{2}\delta_\nu g_{\alpha\gamma}\,u^\alpha u^\gamma = 0$ Geodesic eq. Torsion free + Metric compatibility (see previous notes)

$p^\alpha \equiv m u^\alpha \quad \Rightarrow \quad \boxed{\frac{dP_\nu}{d\tau} = \frac{m}{2}\delta_\nu g_{\alpha\beta}\,u^\alpha u^\beta}$



$\Rightarrow$ $g_{\alpha\beta,\nu} = 0 \Rightarrow P_\nu$ is conserved ! i.e. metric does not depend on x^ν

such conservation is due to the symmetries of the system (i.e. in the metric)

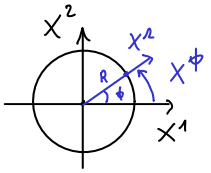
- This is just a glimps, for more look for
  Killing vector fields, Killing equations, Lie derivatives, Isometries

local inertial rest frame

e.g.: $g_{\alpha\beta,0} = 0$ (no time dependency, i.e. g is stationary) $P_0 = g_{0\nu}P^\nu = \text{const} = -\frac{E}{c}$ ($\not\Rightarrow P^\nu = \text{const}$)
universe expands $\Leftrightarrow$ g evolves with time $\Rightarrow$ energy of photons is not conserved: cosmolo. redshift

e.g.: $g_{\alpha\beta,\phi} = 0$ (in spherical coordinates) $\Rightarrow P_\phi = g_{\phi\phi}P^\phi = r^2 m\frac{d\phi}{d\tau} = \text{const}$. angular momentum!

$\delta_\phi g_{\alpha\beta} = 0$ $g_{\alpha\beta} = \text{const}$ for a fixed r (spherical symmetry)



(x^1, x^2) cartesian coord.
(x^r, x^ϕ) spherical coord.

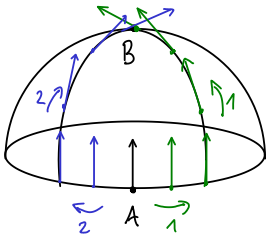
The Curvature of space

- Geodesic eq.: line of free falling particles
  rate of change of $\dot{x}$ (i.e. $\frac{d\dot{x}}{d\tau}$) across the particle
  this is completely independent from the presence or not of curvature in space-time
  also $g_{\mu\nu}$, $\Gamma^{\alpha}_{\beta\gamma}$, ∇_a contains no information about gravity
  build only to deal with arbitrary coordinate choices B36

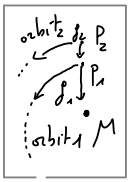
- Gravitational fields: "do not exist" in one single point because you can always obtain $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\alpha}_{\beta\gamma} = 0$ locally

  => To quantify gravit you need a non local "measure" to be sensitive to $\delta^2 g$

  eg. "going around" with parallel transport:
  going from A to B along different paths gives ≠ results



  eg. 2 points, they are both free falling
  but their relative acceleration is different
  => detect presence of gravity
  → tydal forces, geodesic deviation equation



$$\frac{d^2 x_2}{d\tau^2} \neq \frac{d^2 x_1}{d\tau^2}$$

- Mathematically, this is expressed by the Riemann (curvature) tensor $R^{\alpha}{}_{\nu\beta}$

- Intrinsic properties of space-time are identified by tensors

  ↳ If a tensor A is $A \neq 0$ in one frame => $A \neq 0$ $\forall$ frame
  i.e. it can not be made vanish with a suited coord transformation

  => we need tensors to express intrinsic properties of the space-time
  e.g. Γ is not good because not a tensor

Curvature map

commutation of the directions

$$\hat{R} : TM \times TM \times TM \to TM \qquad \hat{R}(\bar{v},\bar{u})\bar{s} \equiv \nabla_{\bar{v}}\nabla_{\bar{u}}\bar{s} - \nabla_{\bar{u}}\nabla_{\bar{v}}\bar{s} - \nabla_{[\bar{v},\bar{u}]}\bar{s}$$

$[\nabla_{\bar{v}}, \nabla_{\bar{u}}]\bar{s}$

commutator of double covariant derivatives

Meaning 1:

- $\hat{R}$ tells you by how much covariant derivatives do not commute $[\nabla_{\bar{v}}, \nabla_{\bar{u}}]$
- "Double derivatives" like in functional analysis $\leftrightarrow$ info about curvature

Meaning 2:

- P with coords $\{x^\mu\}$

vectors with infinitesimal components

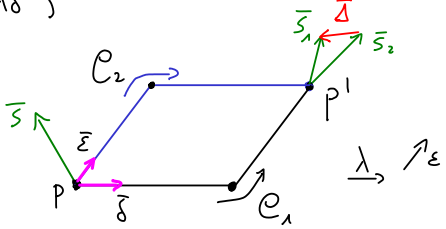
- 2 curves:

$$\begin{cases} \mathcal{C}_1 = \mathcal{C}_x \cup \mathcal{C}_{xy} \\ \mathcal{C}_2 = \mathcal{C}_y \cup \mathcal{C}_{yx} \end{cases}$$

$$\mathcal{C}_x = \{P \in M \mid x = h(P),\ x^\mu_P + \lambda\delta^\mu\}$$
$$\mathcal{C}_{xy} = \{\ " \mid " ,\ x^\mu_P + \delta^\mu + \lambda\varepsilon^\mu\}$$
$$\mathcal{C}_y = \{\ " \mid " ,\ x^\mu_P + \lambda\varepsilon^\mu\}$$
$$\mathcal{C}_{yx} = \{\ " \mid " ,\ x_P + \varepsilon^\mu + \lambda\delta^\mu\}$$

$\lambda, \varepsilon \in [0,1]$ $\varepsilon^\mu, \delta^\mu \ll 1$

- $\bar{v} \in M_P T$
- parallel transport it to P' along $\mathcal{C}_1$ $\Rightarrow \bar{s}_1 \in M_{P'}T$
- " " " " " $\mathcal{C}_2$ $\Rightarrow \bar{s}_2 \in M_{P'}T$



$$\Rightarrow \boxed{\Delta^\mu = s_1^\mu - s_2^\mu = R^\mu{}_{\gamma\alpha\beta}\delta^\alpha\delta^\beta s^{\bar{\gamma}}}$$

- quantifies the difference in $\bar{s}$ when parallely transported along 2 different infinitesimal paths
- "input vectors $\bar{v}, \bar{u}$ give the direction of the path,
  e.g. $\nabla_{\bar{v}}\nabla_{\bar{u}}\bar{s}$ $\to$ parallel transport of $\bar{s}$ along $\bar{u}$ and then along $\bar{v}$
  (α,β): directions along which to sum; "2 vectors $\bar{v}, \bar{u}$"; antisymmetric pair
- indeces (μ): component of the output vector
  (γ): dummy

Basic properties: $\hat{R}(\bar{v},\bar{u})\bar{s} = -\hat{R}(\bar{u},\bar{v})\bar{s}$ antisymmetric

$\hat{R}(f\bar{v}, g\bar{u})h\bar{s} = fgh\hat{R}(\bar{v},\bar{u})\bar{s}$ $f,g,h \in \mathcal{F}$ C^∞ functions on M $\Rightarrow$ trilinear map

$\Rightarrow$ implies a type $\binom{1}{3}$ tensor $R^d{}_{cab} \equiv \langle \tilde{\omega}^d, \hat{R}(\bar{e}_a, \bar{e}_b, \bar{e}_c)\rangle \in \mathcal{T}(1,3)$
curvature tensor

Curvature / Riemann tensor

$\hat{R} : TM \times TM \times TM \to TM \qquad \hat{R}(\bar{v},\bar{u},\bar{s}) \equiv \nabla_{\bar{v}}\nabla_{\bar{u}}\bar{s} - \nabla_{\bar{u}}\nabla_{\bar{v}}\bar{s} - \nabla_{[\bar{v},\bar{u}]}\bar{s}$ curvature map

$T^*M \times TM \times TM \times TM \to \mathbb{R} \qquad R(\tilde{w},\bar{v},\bar{u},\bar{s}) \equiv \tilde{w}\,\hat{R}(\bar{v},\bar{u},\bar{s}) \quad \tilde{w}\in T^*M \quad \bar{v},\bar{u},\bar{s}\in TM \quad R\in\mathcal{T}(1,3)$

- Components on $\{\tilde{dx^i}\}\,\{\bar{\delta}_j\}$ $\nabla_{\bar{\delta}_k} \equiv \nabla_k$ (with abstract notation you should use greek letters)

$R^d{}_{cab} = \langle \tilde{\omega}^d, \hat{R}(\bar{e}_a,\bar{e}_b,\bar{e}_c)\rangle$

$$R^i{}_{jke} = \tilde{dx^i}\left[R(\bar{\delta}_k,\bar{\delta}_e)\bar{\delta}_j\right]$$

c2) $[\bar{\delta}_a,\bar{\delta}_b] = \delta_a\delta_b - \delta_a\delta_b = 0$ the directions commute and $[\bar{s}^{=\bar{0}}_k,\bar{\delta}_e]^* \nabla_m = 0$

$$= dx^i_\alpha\left[\underbrace{\nabla_k\nabla_e\delta^\alpha_j}_{(1)} - \nabla_e\nabla_k\delta^\alpha_j - \underbrace{\nabla_{[\bar{\delta}_k,\bar{\delta}_e]}}_{=0}\delta^\alpha_j\right]$$

(1) $\nabla_k\nabla_e\delta^\alpha_j = \nabla_k\left(\underbrace{\delta_e\delta^\alpha_j}_{=0} + \Gamma^\alpha_{ep}\delta^p_j\right) = \nabla_k(\Gamma^\alpha_{ej}) = \delta_k\Gamma^\alpha_{ej} + \Gamma^\alpha_{k\gamma}\Gamma^\gamma_{ej}$ $\qquad \delta^\alpha_j = \begin{cases}0 & \alpha\neq j\\ 1 & \alpha = j\end{cases}$

$$= dx^i_\alpha\left[\delta_k\Gamma^\alpha_{ej} + \Gamma^\alpha_{k\gamma}\Gamma^\gamma_{ej} - \delta_e\Gamma^\alpha_{kj} - \Gamma^\alpha_{e\gamma}\Gamma^\gamma_{kj}\right]$$

$$= \delta_k\Gamma^i_{ej} - \delta_e\Gamma^i_{kj} + \Gamma^i_{k\gamma}\Gamma^\gamma_{ej} - \Gamma^i_{e\gamma}\Gamma^\gamma_{kj}$$

- Comments:

- All symmetries of R (such as $R^i{}_{cab} = -R^i{}_{cba}$) will be discussed later

- Warning: $R^i{}_{jke}$ the order of the indexes is conventional and there is no agreement!!

- Being a tensor, if $R\neq 0$ in one frame $\Rightarrow$ $R\neq 0$ $\forall$ frame
  it is an intrinsic property of space-time

- Flatness of space and $R^d{}_{cab}$

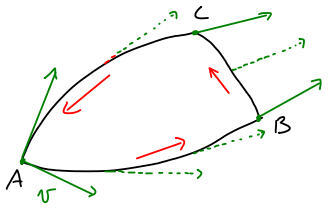
$R^a{}_{bcd} = 0$ covariant eq. $\Rightarrow$ characterizes an intrinsic property of manifolds
sufficient and necessary condition for a flat manifold
if $\neq 0$ $\Rightarrow$ curved space = gravity

↑

$R^a{}_{bcd}$ contains $\delta\Gamma$, i.e. double derivatives of g: $\delta\delta g$, the famous 2° order that can not be made vanish with a simple choice of frame

"Alternative" derivation of R

(4)



- Parallel transport of a vector along a closed loop
- Difference between initial and final vector
- Exploit Stokes theorem

1) Parallel transport along δ_α direction:

$$\nabla_{\delta_\alpha} u^\nu = 0 \qquad \delta_\alpha u^\nu + \Gamma^\nu_{\alpha\gamma} u^\gamma = 0 \qquad \delta_\alpha u^\nu = -\Gamma^\nu_{\alpha\gamma} u^\gamma \quad (1)$$

2) Change after infinitesimal shift dx^α:

$$du^\nu = \delta_\alpha u^\nu \, dx^\alpha = -\Gamma^\nu_{\alpha\gamma} u^\gamma dx^\alpha$$

3) Total change:

$$\begin{aligned}
\Delta u^\nu &\equiv \oint du^\nu \qquad \text{integral over a closed loop} \\
&= -\oint \Gamma^\nu_{\alpha\gamma} u^\gamma dx^\alpha \qquad B^\nu_\alpha \equiv \Gamma^\nu_{\gamma\alpha} u^\gamma \quad (2) \\
&= -\oint B^\nu_\alpha dx^\alpha \qquad \text{Stokes theorem! (surface) } ds^{\alpha\beta} \\
&= -\int \delta_\beta B^\nu_\alpha \, ds^{\alpha\beta} \qquad (2) \\
&= -\frac{1}{2}\int \left(\delta_\alpha B^\nu_\beta - \delta_\beta B^\nu_\alpha\right) ds^{\alpha\beta} \\
&= -\frac{1}{2}\int \left[\delta_\alpha\left(\Gamma^\nu_{\beta\gamma} u^\gamma\right) - \delta_\beta\left(\Gamma^\nu_{\alpha\gamma} u^\gamma\right)\right] ds^{\alpha\beta} \\
&= -\frac{1}{2}\int \left[\Gamma^\nu_{\beta\gamma,\alpha} u^\gamma + \Gamma^\nu_{\beta\gamma} u^\gamma_{,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta} u^\gamma - \Gamma^\nu_{\alpha\gamma} u^\gamma_{,\beta}\right] ds^{\alpha\beta} \qquad (1) \\
&= -\frac{1}{2}\int \left[\Gamma^\nu_{\beta\gamma,\alpha} u^\gamma - \Gamma^\nu_{\beta\gamma}\Gamma^{\bar\gamma}_{\alpha\bar\sigma} u^{\bar\sigma} - \Gamma^\nu_{\alpha\gamma,\beta} u^\gamma + \Gamma^\nu_{\alpha\gamma}\Gamma^{\bar\gamma}_{\beta\bar\sigma} u^{\bar\sigma}\right] ds^{\alpha\beta} \\
&\qquad \sigma \leftrightarrow \gamma \qquad\qquad \sigma \leftrightarrow \gamma \\
&= -\frac{1}{2}\int \left(\Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\nu_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma}\right) u^\gamma \, ds^{\alpha\beta} \\
&\qquad\qquad -R^\nu_{\gamma\alpha\beta}
\end{aligned}$$

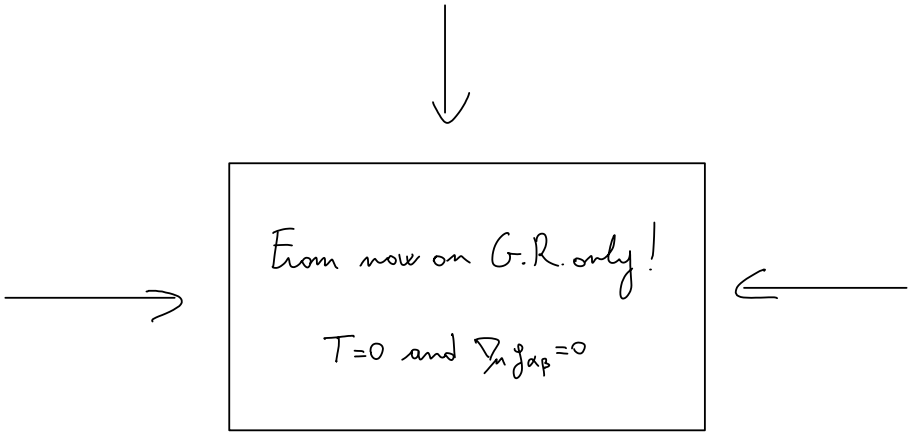
$$\boxed{du^\nu = \frac{1}{2} R^\nu_{\gamma\alpha\beta} u^\gamma \, ds^{\alpha\beta}} \qquad \boxed{R^\nu_{\gamma\alpha\beta} = \Gamma^\nu_{\beta\gamma,\alpha} - \Gamma^\nu_{\alpha\gamma,\beta} + \Gamma^\nu_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma} - \Gamma^\nu_{\beta\sigma}\Gamma^\sigma_{\alpha\gamma}}$$

$\Rightarrow$ R quantifies the change of a vector when parallely transported along a closed loop

- the change depends on the path

$ds^{\alpha\beta} = dx^\alpha dx^\beta$

$\div d\tau$ $\quad \frac{du^\nu}{d\tau} \equiv a^\nu$ $\rightarrow$ acceleration $\propto$ R geodesic eq. deviation $\quad a^\nu = \frac{1}{2} R^\nu{}_{\gamma\alpha\beta} u^\gamma u^\alpha dx^\beta$



$\Rightarrow$ Riemannian connection

$R^i{}_{jke}$ with a Riemann connection

- both $R^i{}_{jke}$ and T^i_{ke} are based on connection Γ only! No metric was used! no reference to metric compatibility or Torsion free, super general!
- In G.R. we assume $T=0$ and $\nabla_\gamma g_{\alpha\beta}=0$ $\Rightarrow$ Γ^i_{jk} = Christoffel symbol (Riemann connection) (!)
- Explicit expression:

coord system such that in P $\Gamma=0$ and the 1° order $\delta_k g^{ir}=0$ (local inertial frame)

$$R^i{}_{jke} = \delta_k \underbrace{\Gamma^i_{ej}}_{=0} - \delta_e \underbrace{\Gamma^i_{kj}}_{=0} + \Gamma^i_{kr}\Gamma^r_{ej} - \Gamma^i_{er}\Gamma^r_{kj}$$

$$= \frac{1}{2}\delta_k\left[g^{ir}\left(\delta_e g_{rj}+\delta_j g_{er}-\delta_r g_{je}\right)\right] - \frac{1}{2}\delta_e\left[g^{ir}\left(\delta_k g_{rj}+\delta_j g_{kr}-\delta_r g_{jk}\right)\right]$$

$$= \frac{1}{2}g^{ir}\left(\delta_k\delta_e g_{rj}+\delta_k\delta_j g_{er}-\delta_k\delta_r g_{je}-\delta_e\delta_k g_{rj}-\delta_e\delta_j g_{kr}+\delta_e\delta_r g_{jk}\right) + \frac{1}{2}\delta_k g^{ir}\underbrace{(\ldots)}_{=0}$$

$$= \boxed{\frac{1}{2}g^{ir}\left(\delta_k\delta_j g_{er}-\delta_k\delta_r g_{je}-\delta_e\delta_j g_{kr}+\delta_e\delta_r g_{jk}\right)}$$

in local inertial frame

(1) $R^i{}_{jke} = -R^i{}_{jek}$ last 2 indexes

$$R_{ijke} = g_{ir}R^r{}_{jke} = \frac{1}{2}g_{ir}g^{\sigma r}\left(\delta_k\delta_j g_{e\sigma}-\delta_k\delta_\sigma g_{je}+\delta_e\delta_j g_{k\sigma}-\delta_e\delta_\sigma g_{jk}\right)$$

$$= \boxed{\frac{1}{2}\left(\delta_k\delta_j g_{ei}-\delta_k\delta_i g_{je}-\delta_e\delta_j g_{ki}+\delta_e\delta_i g_{jk}\right)}$$

Just swap the indexes to see it

(2) $R_{ijke} = R_{keij}$ swap first 2 with last 2 indexes

(3) $R_{ijke} = -R_{jike} = -R_{ijek}$ swap first/last two indexes

(4) $R_{ijke} + R_{ikej} + R_{iejk} = 0$ cyclic indexes (anti-symmetric part of last 3 indexes)

(5) $R_{[ijke]} = 0$ Total antisymmetric part

(1) (2) (3)

$\rightarrow$ R_{ijke} has $4^4 = 256$ components

$\rightarrow$ The for conditions (1) (2) (3) (4) are not independent (they are linked by g_{ir}) $\Rightarrow$ 3 "constraints"

$\Rightarrow$ because of (2), (3) and (4): only 20 components are independent

(like a 6×6 symmetric matrix (21 elements) + condition (4) = 20 independent elements)

- **Bianchi identity**

choose Riemannian coordinates (locally geodetic, $\Gamma=0$) ⇒ $\nabla_\beta R_{ijke} = \partial_\beta R_{ijke}$

$$\nabla_{[\beta} R_{ij]ke} = \nabla_\beta R_{ijke} + \nabla_i R_{j\beta ke} + \nabla_j R_{\beta ike}$$

$$= \frac{1}{2}\partial_\beta\left(\delta_k\delta_j g_{\ell i} - \delta_k\delta_i g_{j\ell} + \delta_\ell\delta_j g_{ki} - \delta_\ell\delta_i g_{jk}\right)$$

$$+ \frac{1}{2}\partial_i\left(\delta_k\delta_\beta g_{\ell j} - \delta_k\delta_j g_{\beta\ell} + \delta_\ell\delta_\beta g_{kj} - \delta_\ell\delta_j g_{\beta k}\right)$$

$$+ \frac{1}{2}\partial_j\left(\delta_k\delta_i g_{\ell\beta} - \delta_k\delta_\beta g_{i\ell} + \delta_\ell\delta_i g_{k\beta} - \delta_\ell\delta_\beta g_{ik}\right) = 0$$

(cancellations because of the symmetries of $g_{\alpha\beta}$ and anti-symmetry (3))

(with torsion you have additional terms)

it holds in all frames

if a tensor is null in one frame it is null in all frames ⇒ general validity

↳ Related to Jacobi identity $\left[[\nabla_\alpha,\nabla_\beta],\nabla_\gamma\right] + \left[[\nabla_\beta,\nabla_\gamma],\nabla_\alpha\right] + \left[[\nabla_\gamma,\nabla_\alpha],\nabla_\beta\right] = 0$

Appendix

Homology and homology group

• Loop

A closed curve passing through a point $P \in M$ $\quad \mathcal{C}: S^1 \to M$

$\lambda \in [0,1] \to P = \mathcal{C}(\lambda) \quad \mathcal{C}(0) = \mathcal{C}(1)$



• Homology

- Given a connection, any $v \in M_p T$ can be parallely transported along $\mathcal{C}$ to give a new vector $v' \in M_p T$
- This defines a linear transformation $M_{\mathcal{C}}: M_p T \to M_p T \quad M_{\mathcal{C}}: \bar{v} \to \bar{v}' \quad v'^{\beta} = (M_{\mathcal{C}})_{\alpha}^{\ \beta} v^{\alpha}$ called homology at p of the connection for the loop $\mathcal{C}$
- Going around the loop in the opposite direction → inverse transformation

  General, Linear group ↙ dimension

  $\Rightarrow M_{\mathcal{C}}$ is invertible $M_{\mathcal{C}} \in GL(m, \mathbb{R})$
- $M_{\mathcal{C}_3} = M_{\mathcal{C}_2} M_{\mathcal{C}_1}$ is an homology (first you go along $\mathcal{C}_1$ and then $\mathcal{C}_2$, Two loops trough same P)

• Homology group

$\{M_{\mathcal{C}}\}$ for all curves $\mathcal{C}$ through P form a group H_p = homology group at P

is a sub group of $GL(m, \mathbb{R})$

Existence of the curvature tensor

$f \in \mathcal{F}$ $\tilde{w} \in M_p T^*$ $\nabla_a \nabla_b (f w_c) \in \mathcal{T}\binom{0}{3}$

$$\nabla_a \nabla_b (f w_c) = \nabla_a (\nabla_b f \cdot w_c + f \nabla_b w_c) = (\nabla_a \nabla_b f) w_c + \nabla_b f \nabla_a w_c + \nabla_a f \nabla_b w_c + f \nabla_a \nabla_b w_c$$

$$\begin{aligned}\nabla_a \nabla_b (f w_c) - \nabla_b \nabla_a (f w_c) = &(\nabla_a \nabla_b f) w_c + \nabla_b f \nabla_a w_c + \nabla_a f \nabla_b w_c + f \nabla_a \nabla_b w_c - \\ &(\nabla_b \nabla_a f) w_c + \nabla_a f \nabla_b w_c + \nabla_b f \nabla_a w_c + f \nabla_b \nabla_a w_c \\ = &f (\nabla_a \nabla_b - \nabla_b \nabla_a) w_c\end{aligned}$$

$(\nabla_a \nabla_b - \nabla_b \nabla_a) w_c \big|_p$ depends only on value of w_c in $P \in M$ $\Rightarrow$ linear map

$R : M_p T^* \to \mathcal{T}\binom{0}{3}$ $\quad R(\tilde{w}) = R_{abc}{}^{d} w_d \equiv (\nabla_a \nabla_b - \nabla_b \nabla_a) w_c$ Rieman curvature tensor

- Expresses the commutation of double derivatives
- you can do the same by applying it to a vector

$$\begin{aligned}\nabla_a \nabla_b (f v^c) - \nabla_b \nabla_a (f v^c) = &(\nabla_a \nabla_b f) w_c + \nabla_b f \nabla_a w_c + \nabla_a f \nabla_b w_c + f \nabla_a \nabla_b w_c - \\ &(\nabla_b \nabla_a f) w_c + \nabla_a f \nabla_b w_c + \nabla_b f \nabla_a w_c + f \nabla_b \nabla_a w_c \\ = &f (\nabla_a \nabla_b - \nabla_b \nabla_a) v^c\end{aligned}$$

$R : M_p T^* \to \mathcal{T}\binom{1}{2}$ $\quad R(\tilde{w}) = R_{abc}{}^{d} v^a \equiv (\nabla_b \nabla_c - \nabla_c \nabla_b) v^d$ Rieman curvature tensor

Relativistic cosmological model

- A model to describe the general behavejour of the universe, i.e. of its content

- Forces in play

	Theory	Interaction	Behavior	Typical scale	
1)	QCD	Strong	$1/r^7$	1.4×10^{-15} m	irrelevant
2)	Electroweak	Weak	$1/r^5$ to $1/r^7$	10^{-18} m	irrelevant
3)	QED	Electromagnetic	$1/r^2$	∞	irrelevant
4)	GR	Gravity	$1/r^2$	∞	yes!

{ The universe is neutral on large scales

$\Rightarrow$ General Relativity: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$

(2) Geometry (1) "matter" source of gravity

(1) The "matter" term $T_{\mu\nu}$

- Radiation photons
- Matter { baryons $P \neq 0$ ithos pressure; dark matter $P = 0$ (dust) only gravitational effect

↳ evidence from: velocity dispersion of galaxies in galaxy clusters
temperature of intra-cluster gas " " "
rotation curves of individual galaxies
gravitational lensing of galaxies and galaxy clusters
cosmic structure formation
dynamic of the universe

gravitational interactions

↳ direct detection: so far not successful

↳ production at LHC: so far not successful

- Cosmological const. OR Dark energy: needed to explain accelerated expansion of the universe
  Λ DE

$\Rightarrow$ ΛCDM model or wCDM model

- Treat all of them as ideal fluids, yes also Λ.... $T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu}$

Cosmological constant

- We associate an energy-momentum tensor to Λ

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} + T^{\Lambda}_{\mu\nu}\right) \quad \Rightarrow \quad \boxed{T^{\Lambda}_{\mu\nu} \equiv \frac{c^4 \Lambda}{8\pi G} g_{\mu\nu}}$$

- We interpret Λ as a fluid :

$$T^{\Lambda}_{\mu\nu} = \frac{c^4\Lambda}{8\pi G} g_{\mu\nu} \overset{!}{=} \underbrace{\overset{!}{=}0}_{*} \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p\, g_{\mu\nu} \quad \Rightarrow \quad \boxed{c^2 \rho_\Lambda = -p_\Lambda}$$ ← not to have $\bar{u}$ dependency

Eq. of state

$$= p\, g_{\mu\nu}$$

$$= -\rho c^2 g_{\mu\nu} \quad \Rightarrow \quad \boxed{\rho_\Lambda = \frac{c^2\Lambda}{8\pi G}}$$

constant density !

Parametric eq. of state : $p = w\rho c^2$ $\Rightarrow$ $\boxed{w_\Lambda = -1}$ eq. of state parameter

- $T^{\Lambda}_{\mu\nu}$ has no dependency on observer motion (there is no $\bar{u}$) *2

↳ characteristic of empty space $\Rightarrow$ T^{Λ} associated to vacuum energy

↳ (there is nothing $\Rightarrow$ there are no velocities of something with respect to something else)

- But... if this is so, we get a problem....

(a) from cosmological observations : $|\Lambda| < 10^{-56}\,cm^{-2}$ or equivalently $m_\Lambda < 10^{-32}\,eV$

(eg. photon with $\lambda = R_H = \frac{c}{H_0} \approx 3\,Gpc/h$ (i.e. reddest possible photon) would have $E_\gamma \sim 10^{-27}\,eV$)

↳ fine tuning problem, why so "unnaturally small" but not zero?

(b) From an elementary particle perspective : vacuum energy is $\sim 10^{121}$ times larger !

- This is why people set $\Lambda = 0$ and introduce a new dynamical field : Dark-Energy (DE)

not a constant, it can evolve with time : $w(a)$

in any case, behavejour of DE must be very close to Λ : so for observations are compatible with Λ!

Evolution of the components

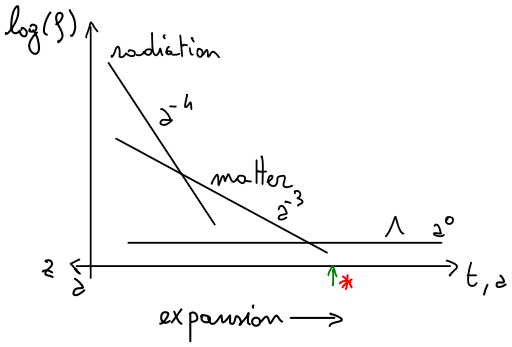
- Assuming components are not interacting $\Rightarrow \nabla_\mu (T^{\mu\nu})_i = 0$
  $\Rightarrow$ independent evolution of densities (conservation laws holds for each of them independently)
  $\nabla_\mu T^{\mu 0} = 0$: Adiabatic condition *2

- Convenient to use a parametric eq. of state

$p_i = w_i \rho c^2$: $* \quad \frac{d(c^2 \rho a^3)}{dt} + p \frac{da^3}{dt} = 0 \qquad d(\rho a^3) + w \rho a^2 da^3 = 0 \qquad a^3 d\rho + 3\rho a^2 da + w\rho 3 a^2 da = 0$

$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \qquad \log\rho = -3(1+w)\log a + C \qquad \boxed{\rho = \rho_0 a^{-3(1+w)}}$ (C: const)

$w = 0$	dust	$\rho_m = \rho_{m0} a^{-3}$	matter
$w = 1/3$	relativistic matter	$\rho_r = \rho_{r0} a^{-4}$	photons, neutrinos
$w = -1$	cosmo. const.	$\rho_\Lambda = \text{const}$	vacuum



3 epocs : radiation domination era
matter domination era
Λ domination era (de Sitter universe)

You are here *

Multi-component cosmological fluid

We can define an effective $T_{\mu\nu}$ containing all species

$T_{\mu\nu} = \sum_i (T_{\mu\nu})_i$ i-th component i = Baryons, Dark-matter, Radiation, Λ (or Dark Energy)

$= \sum_i \left[\left(\rho_i + \frac{p_i}{c^2}\right) u_\mu u_\nu + p_i g_{\mu\nu} \right]$

$= u_\mu u_\nu \left(\sum_i \rho_i + \sum_i \frac{p_i}{c^2}\right) + g_{\mu\nu} \sum_i p_i$ $\Rightarrow$ behaves as one single fluid with $\boxed{\rho = \sum_i \rho_i \quad p = \sum_i p_i}$

$\Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$ Contains all!

Assomptions: the cosmological principle

Assumptions: homogeneity + isotropy on large scales = cosmological principle (observations)

$\Rightarrow$ matter remains at rest in comoving coordinates : i.e. fixed r, θ, ϕ $\Rightarrow$ $\bar{u} = (c, 0, 0, 0)^T$

$\Rightarrow T = \text{diag}(\rho c^2, p, p, p)$ $\quad T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p g_{\mu\nu}$

(2) The "geometry" term $G_{\mu\nu}$

To get analytical solution for Einstein eq.s impose symmetries on $g_{\mu\nu}$ based on these assumptions

- Assumptions: homogeneity + isotropy on large scales = cosmological principle (observations)
- Non static universe (observations)

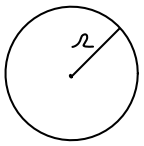
1) Fundamental observer (free fall): $ds^2 = -c^2dt^2 = g_{00}(dx^0)^2 \Rightarrow g_{00} = -1$
this sets the time unit

2) Isotropy: $g_{0i} = 0$ $g_{ij} = 0$ $i \neq j$ No privileged direction * we can sincronize all clocks

3) Isotropy: Spherical coordinates: $\vec{x} = (r, \theta, \phi)^T$ 3D polar coordinates

- flat space, observer in the center of symmetry:

$dl^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)$ 3D part of interval



$2\pi r$ circonference
$4\pi r^2$ surface of sphere
$dA = r^2 \sin\theta\, d\theta\, d\phi$ surface element of sphere

- Curved space time is different! distinct "radiuses": R, r

$dl^2 = dR^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)$, with $d\Omega^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2$

dR: radious for a fixed R (r = in "angular" coordinates)
θ, ϕ: " " " ($dR = f(r)dr$, $f(r)$ some function)

Here a sphere has radious R but circonference $= 2\pi r$ and surface $= 4\pi r^2$

$\Rightarrow g_{11} = f^2(r)$ $g_{22} = r^2$ $g_{33} = r^2\sin(\theta)$ $(h_{ij}) = (f^2(r), r^2, r^2\sin(\theta))$

4) Same expansion everywhere: (isotropy)

$dl^2 = a^2(t)(h_{ij}\, dx^i dx^j)$ $a(t)$ = scale factor
rescaled by a time dependent factor only $\Rightarrow$ dynamics!
set distance units such that $a(t_{today}) \doteq 1$
expansion $\Rightarrow a(t) < 1$ for $t < t_{today}$

5) homogeneity:

$R^i_i = \text{const}$ $\overset{*}{\Leftrightarrow}$ $G^i_i = \text{const} \equiv 3K$ on $t=\text{const}$ hypersurfaces $(i=1,2,3)$

i.e. Same space curvature on the time slice : foliation

$G^i_i = g^{ij} G_{ij} = f^2(r) G_{rr} + r^{-2} G_{\theta\theta} + r^{-2}\sin^{-2}\theta\, G_{\phi\phi} = -\frac{1}{r^2}\left[r\left(1-f^2(r)\right)\right]' \overset{!}{=} 3K$

integrate: $r\left(1-f^2(r)\right) = -3K\frac{r^3}{3} + C$ $\qquad f^2(r) = g_{rr} = \left(1+Kr^2\right)^{-1}$

(initial condition: flatness at $r=0 \Rightarrow C=0$ "on small scales you see no curvature")

* $\left(G^i_i = R^i_i - \frac{1}{2}R\,g^i_i = -R^i_i\right)$ ($R=4$)

☆ See appendix $g_{\mu\nu} \to \Gamma \to R_{\mu\nu}$

$\Rightarrow$ Friedman-Lamaitre-Robertson-Walker metric (FLRW)

$$\boxed{ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right)}$$

r, θ, ϕ = comoving coordinates

- $K \in \mathbb{R}$ $(K<0\ K=0\ K>0)$ but r can always be rescaled such that:

$\left(K \to \frac{K}{|K|}\ ,\ r \to \sqrt{|K|}\,r\ ,\ a \to \frac{a}{\sqrt{|K|}}\right)$ leave the 4 interval the same

$$K = \begin{cases} 0 & \text{flat} \\ 1 & \text{closed / spherical} \\ -1 & \text{open / hyperbolic} \end{cases}$$

names are from geometry of space surfaces with $t=\text{const}$

For convenience change the units of length:

- $K=1$: define parameter χ: $(dr)^2 \equiv \frac{(dr)^2}{1-r^2}$ $\Rightarrow$ $r = \sin\chi$ (spherical)

$(\chi=0 \Leftrightarrow r=0)$ $\left(\chi=\frac{\pi}{2} \Leftrightarrow r=1\right)$ $(\chi=\pi \Leftrightarrow r=0)$ $\Rightarrow$ $r\in[0,1]$ closed universe

$d\tilde{\ell}^2 = d\chi^2 + \sin^2\chi\, d\Omega^2$ surface element of 3D sphere embedded in 4D space

3D volume element: $V = \int \underbrace{\sqrt{g_{(3)}}}_{\text{3D det.}} dx^3 = \int \sin^2\chi\, d\Omega^2 d\chi = 4\pi\int_0^\pi \sin^2\chi\, d\chi = 2\pi^2$ (finite)

- $K=-1$: as above: $(dr)^2 \equiv \frac{(dr)^2}{1+r^2}$ $\Rightarrow$ $r = \sinh\chi$ (hyperbolic)

$(\chi=0 \Leftrightarrow r=0)$ $(\chi\to\infty \Leftrightarrow r\to\infty)$ $\Rightarrow$ $r\in[0,\infty)$ open universe

$V = 4\pi\sinh^2\chi \to \infty$ for $\chi\to\infty$ (infinite)

$$\Rightarrow \boxed{ds^2 = -c^2dt^2 + a^2(t)\left[d\chi^2 + f_K^2(\chi)\left(d\theta^2 + \sin\theta\, d\phi^2\right)\right]} \qquad f_K(\chi) = \begin{cases} \chi = r & K=0 \\ \sin\chi & K=1 \\ \sinh\chi & K=-1 \end{cases}$$

$$ds^2 = a^2(t)\left[-c^2d\tau^2 + d\chi^2 + f_K^2(\chi)\left(d\theta^2 + \sin\theta\, d\phi^2\right)\right] \qquad d\tau \equiv \frac{dt}{a(t)}$$

τ = conformal time, not the proper time

- Plug the metric to find the "geometry" terms: (see Appendix)

1) $\Gamma^0_{11} = \frac{a\dot{a}}{(1-kr^2)c}$ $\quad \Gamma^0_{22} = a\dot{a}\frac{r^2}{c}$ $\quad \Gamma^0_{33} = \Gamma^0_{22}\sin^2\theta$

$\Gamma^i_{0i} = \Gamma^i_{i0} = \frac{\dot{a}}{ac}$ $\quad \Gamma^1_{11} = \frac{kr}{(1-kr^2)}$ $\quad \Gamma^1_{22} = -r(1-kr^2)$ $\quad \Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta$

$\Gamma^2_{12} = \Gamma^3_{13} = \Gamma^2_{21} = \Gamma^3_{31} = \frac{1}{r}$ $\quad \Gamma^2_{33} = -\sin\theta\cos\theta$

$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$ $\quad$ factors $\frac{1}{c}$ come from $\delta_0 = \frac{\delta}{c\delta t}$ $\dot{a} = \frac{da}{dt}$

2) $R_{00} = -3\frac{\ddot{a}}{ac^2}$ $\quad R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{(1-kr^2)c^2}$ $\quad R_{22} = r^2(a\frac{\ddot{a}}{c^2} + 2\frac{\dot{a}^2}{c^2} + 2k)$ $\quad R_{33} = r^2(a\frac{\ddot{a}}{c^2} + 2\frac{\dot{a}^2}{c^2} + 2k)\sin^2\theta$

3) $R = \frac{6}{a^2}(a\frac{\ddot{a}}{c^2} + \frac{\dot{a}^2}{c^2} + k)$ not needed here...

Solve the field euquations to get the eq. of motion of g, i.e. of a(t)

- Note: Cosmological principle $\Rightarrow$ $(T_{\mu\nu}) = \mathrm{diag}(\rho c^2, P, P, P)$ $\Rightarrow$ $T^\mu{}_\mu = \boxed{T = -\rho c^2 + 3P}$

- Convenient to write Einstein field eq.s in this form to solve them $\boxed{R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)}$

the only non trivial solutions are: ($g_{00} = -1$)

(1) $\mu\nu = 00$ $\quad -3\frac{\ddot{a}}{c^2 a} = \frac{8\pi G}{c^4}\left(\rho c^2 - \frac{\rho c^2}{2} + \frac{3}{2}P\right) = \frac{4\pi G}{c^2}\left(\rho + 3\frac{P}{c^2}\right)$ $\quad \frac{\ddot{a}}{ac^2} = -\frac{4\pi G}{3c^2}\left(\rho + 3\frac{P}{c^2}\right)$

(2) $\mu\nu = ii$ $\quad$ all the same because of isotropy (eq. $\mu\nu = 22$) $\quad \frac{\ddot{a}}{ac^2} + 2\left(\frac{\dot{a}}{ac}\right)^2 + \frac{2k}{a^2} = \frac{4\pi G}{c^2}\left(\rho - \frac{P}{c^2}\right)$

plug (1) in (2) $\Rightarrow$

$$\boxed{\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) \quad (1) \\ H^2 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \quad (2) \end{aligned}}$$

Friedmann eq.s. $\quad \left(H \equiv \frac{\dot{a}}{a}\ \text{Hubble function}\right)$

$$= \frac{8\pi G}{3}(\rho_M + \rho_R) - \frac{k}{a^2} + \frac{\Lambda}{3} \quad \text{for } \rho = \rho_M + \rho_R + \rho_\Lambda$$

- Solve conservation low:

$\nabla_\mu T^{\mu 0} = \boxed{\dot{\rho}c^2 + 3H(\rho c^2 + P) = 0}$ (3) see appendix $\quad \Leftrightarrow$ (equivalent) $\quad \frac{d(c^2\rho a^3)}{dt} + P\frac{da^3}{dt} = 0$

$E = \rho c^2 V$, $V = V_0 a^3$ $\quad$ i.e. $dE + pdV = 0$

1° law of thermodynamics

i.e. Adiabatic condition

With (3) you can obtain Friedmann eq. (1) from Friedmann eq. (2)

$\Rightarrow$ the 2 Friedmann eq.s are not independent!

Important quantities and condition for flat universe:

- Friedmann (2), Set $K \overset{!}{=} 0 \Rightarrow \boxed{\rho_c \equiv \frac{3H^2}{8\pi G}}$ critical density: density for which the universe is flat

- $\frac{\text{Friedmann (2)}}{H^2}$: $\boxed{1} = \frac{8\pi G}{3H^2}\rho - \frac{Kc^2}{a^2H^2} = \boxed{\Omega + \Omega_K}$ with $\boxed{\Omega \equiv \rho/\rho_c}$ density contrast, $\boxed{\Omega_K \equiv -\frac{Kc^2}{a^2H^2}}$ density contrast of curvature
must hold $\forall$ time

$\Rightarrow$ condition for flat universe: $K=0 \Rightarrow \Omega_K = 0 \Leftrightarrow \Omega = 1$

$M_{200} = 200 \frac{4}{3}\pi \rho_c R_{200}^3$ $\qquad R_{200}^3 = \frac{3}{800\pi}\frac{M_{200}}{\rho_c} = \frac{3}{800\pi}\frac{M_{200}\,8\pi G}{3H^2} = \frac{G}{100H^2}M_{200}$ (100)

General Friedmann eq.

$$H^2 = \frac{8\pi G}{3c^2}\left(\rho_{m_0}a^{-3} + \rho_{r_0}a^{-4} + \rho_\Lambda\right) - Ka^{-2} = H_0^2\left(\Omega_{m_0}a^{-3} + \Omega_{r_0}a^{-4} + \Omega_\Lambda - \Omega_{K_0}a^{-2}\right) = H_0^2 E(a)$$

Measurements at $a=1$, i.e. today: $H_0 \simeq 70$ $\Omega_{m_0} \sim 0{,}3$ $\Omega_\Lambda \sim 0{,}7$ $\Omega_{K_0} \sim 0$ flat

Cosmological redshift

- Redshift: $z \equiv \frac{\lambda_0 - \lambda}{\lambda} = \frac{\lambda_0}{\lambda} - 1 = \frac{\nu}{\nu_0} - 1$ $\qquad 0$ = observed (ws, t_0 = today $a_0 = 1$)

$ds^2 = 0 = -c^2dt^2 + a^2(t)\left(\frac{dR^2}{1-kR^2} + R^2 d\Omega^2\right) = -c^2dt^2 + a^2(t)R$ $\qquad R$ = const. (comoving coord.)^2

$\Rightarrow 1+z = \frac{\nu}{\nu_0} = \frac{dt_0}{dt} = \frac{a(t_0)}{a(t)} = a^{-1}(t)$ ($a(t_0) = 1$) $\Rightarrow$ $\boxed{a(z) = (1+z)^{-1}}$ a = direct observable!

- Energy of photons decreases with time because of the time evolution of g!
$\Rightarrow$ Redshift is due to time evolution of the metric!

$D_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu r} T^{r 0} + \Gamma^0_{\mu r} T^{\mu r} = 0$ $\qquad \partial_\mu T^{\mu 0} = -\Gamma^\mu_{\mu r} T^{r 0} - \Gamma^0_{\mu r} T^{\mu r}$

(!) No "strict" energy conservation $\partial_\mu T^{\mu 0} = 0$
See also appendix in GR notes

The Hubble constant H_0

$D = \frac{c}{H_0} z$ Mpc
- at low z linear relationship between D and z
- H_0 set the expansion rate of the universe, currently $H_0 \simeq 70$ Km/s/Mpc $[H_0] = s^{-1}$
- You will also find $H_0 = 100\, h$ Km/s/Mpc with $h = 0,7$ for example
- Carefull, expansion changes as a function of time $H(z)$ Hubble function

Hubble time $t_H \equiv H_0^{-1} = 13.9$ Gyr

Hubble distance: $R_H = c \cdot H_0^{-1} = 4,2$ Gpc

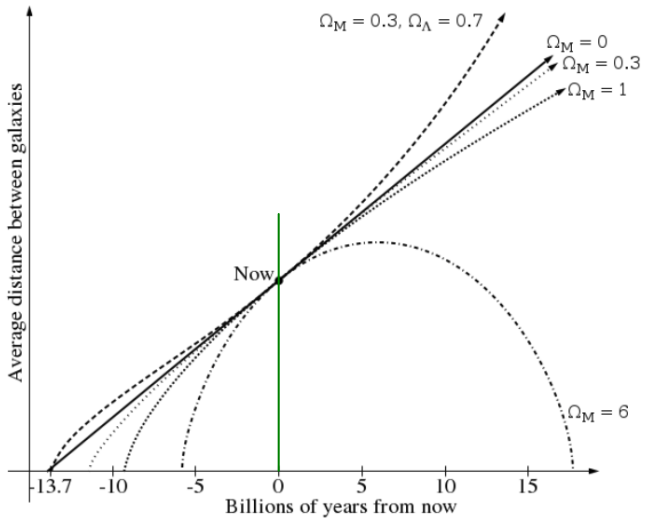


Figure 3.8.1: Evolution of the scale factor in various cosmological models (from WikiCommons, author BenRG, public domain).

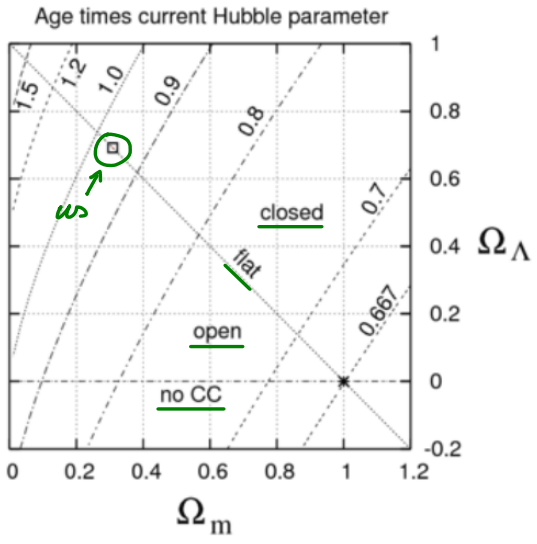


Figure 3.6.1: Age of the Universe as a function of matter and cosmological constant fractions. Notice how for

Appendix: explicit computations

- Adiabatic condition (page 3)

$\nabla_\mu T^{\mu\nu} = 0$ $\quad T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu + P g^{\mu\nu}$ $\quad$ ↓ for metric compatibility $\nabla_\mu g^{\mu\nu} = 0$

$u^\mu u^\nu \nabla_\mu \left(\rho + \frac{P}{c^2}\right) + \left(\rho + \frac{P}{c^2}\right)\left[u^\nu \nabla_\mu u^\mu + u^\mu \nabla_\mu u^\nu\right] + g^{\mu\nu} \nabla_\mu P = 0$ $\quad \rho(t)$ $\nabla_\mu P = 0$

$-c^2 u^\mu \nabla_\mu \left(\rho + \frac{P}{c^2}\right) + \left(\rho + \frac{P}{c^2}\right)\left[-c^2 \nabla_\mu u^\mu + 0\right] + u_\nu g^{\mu\nu} \nabla_\mu P = 0$ $\quad$ (*1) $\quad \times u_\nu$

$-u^\mu \nabla_\mu (\rho c^2 + P) - (\rho c^2 + P) \nabla_\mu u^\mu + u^\mu \nabla_\mu P = 0$

$u^\mu \partial_\mu (\rho c^2) + (\rho c^2 + P)\left(\partial_\mu u^\mu + \Gamma^\mu_{\mu\gamma} u^\gamma\right) = 0$ $\quad$ (ρ = scalar) $\quad u^\mu = \partial_0 x^\mu = \delta^\mu_0$ $\quad \Gamma^\mu_{\mu\gamma} \delta^\gamma_0 = \Gamma^\mu_{\mu 0} = \frac{\dot{a}}{a}$

$\partial_0 (\rho c^2) + (\rho c^2 + P)\left(0 + 3\frac{\dot{a}}{a}\right) = 0$

$\dot{\rho} c^2 + 3\frac{\dot{a}}{a}(\rho c^2 + P) = 0$ $\quad \boxed{\dot{\rho} c^2 + 3H(\rho c^2 + P) = 0}$

(*1) $u_\nu \nabla_\mu u^\nu + u^\nu \nabla_\mu u_\nu = 2 u_\nu \nabla_\mu u^\nu = \nabla_\mu (u^\nu u_\nu) = \nabla_\mu (-c^2) = 0$

- Einstein eq.s can be written as:

$g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = \kappa g^{\mu\nu} T_{\mu\nu}$ $\quad R - \frac{1}{2} 4R = \kappa T$ $\quad \boxed{R = -\kappa T}$

$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = R_{\mu\nu} + \frac{\kappa}{2} T g_{\mu\nu} = T_{\mu\nu}$ $\quad \Rightarrow \quad \boxed{R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)}$

- Most general spherical metric in a time slice t = fixed (rest frame at center of frame)

- For convenience use: $g_{00} \equiv -e^{2A(r)}$ $\quad g_{rr} \equiv e^{2B(r,t)}$

$\boxed{ds^2} = g_{00} c^2 dt^2 + g_{rr} dr^2 + r^2 d\Omega^2 = \boxed{-e^{2A(r)} c^2 dt^2 + e^{2B(r,t)} dr^2 + r^2 d\Omega^2}$

  - "exp" to have a positive value
  - e^2 for convenience (square)
  - For from the same $\Rightarrow$ Minkowski metric $\lim_{r\to\infty} g_{00} = -1$, $\lim_{r\to\infty} g_{rr} = 1$ for $\lim_{r\to\infty} A(r) = 0$, $\lim_{r\to\infty} B(r,t) = 0$

- Einstein tensor (non zero components)

$$G_{00} = \frac{1}{r^2} e^{2A} \frac{\partial}{\partial r}\left[r\left(1-e^{-2B}\right)\right]$$
$$G_{rr} = -\frac{1}{r^2} e^{2B}\left(1-e^{-2B}\right) + \frac{2}{r}\frac{\partial A}{\partial r}$$
$$G_{\theta\theta} = r^2 e^{-2B}\left(A'' + A'^2 + \frac{A'}{r} - A'B' - \frac{B'}{r}\right) \qquad \frac{\partial}{\partial r} = "'"$$
$$G_{\phi\phi} = G_{\theta\theta}\sin^2\theta$$

other components $= 0$

- Constrain on spatial curvature

$$G^i_i = g^{ij} G_{ij} = e^{-2B} G_{rr} + r^{-2} G_{\theta\theta} + r^{-2}\sin^{-2}\theta\, G_{\phi\phi} \qquad i = 1,2,3$$

G same as in Schwarzschild but not r dependency in $g_{00}(t)$

$$= e^{-2B} G_{rr} + 2 r^{-2} G_{\theta\theta}$$
$$= -e^{-2B}\left(\frac{1}{r^2} e^{2B}\left(1-e^{-2B}\right) + \frac{2}{r}\frac{\partial A}{\partial r}\right) + r^{-2} 2 r^2 e^{-2B}\left(A'' + A'^2 + \frac{A'}{r} - A'B' - \frac{B'}{r}\right) \qquad \frac{\partial}{\partial r} = "'"$$
$$= -\frac{1}{r^2}\left(1-e^{-2B}\right) - 2e^{-2B}\frac{B'}{r} = \boxed{-\frac{1}{r^2}\left[r\left(1-e^{-2B}\right)\right]' \overset{!}{=} 3K} \quad \text{const.}$$

- Christoffel symbols:

$$\Gamma^a_{bc} = \frac{1}{2} g^{a\gamma}\left(\partial_b g_{\gamma c} + \partial_c g_{b\gamma} - \partial_\gamma g_{cb}\right) \overset{g \text{ diag.}}{=} \frac{1}{2} g^{aa}\left(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{cb}\right) \qquad i,j = 1,2,3 \qquad \mu,\nu = 0,1,2,3$$

$$\Gamma^0_{ij} = \frac{1}{2} g^{0\gamma}\left(\partial_i g_{\gamma j} + \partial_j g_{i\gamma} - \partial_\gamma g_{ij}\right) = \frac{1}{2} g^{00}\left(\partial_i g_{0j} + \partial_j g_{i0} - \partial_0 g_{ij}\right) = +\frac{1}{2} g^{00}\partial_0 g_{ij}$$

($\partial_i g_{0j} = 0$, $\partial_j g_{i0} = 0$, $g^{00} = -1$)

$$\Gamma^0_{ij} = 0 \quad i \neq j \qquad \Gamma^0_{11} = \frac{1}{2}\frac{\partial}{c\partial t}\left(\frac{a^2}{(1-kr^2)}\right) = \frac{2a\dot{a}}{2c(1-kr^2)} \qquad \Gamma^0_{22} = \frac{1}{2}\partial_0 a^2 = \frac{a\dot{a}}{c} r^2 \qquad \Gamma^0_{33} = \Gamma^0_{22}\sin^2\theta$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) = \frac{1}{2}\frac{(1-kr^2)}{a^2}\frac{\partial}{\partial r}\left[\frac{a^2}{(1-kr^2)}\right] = \frac{1}{2}\frac{(1-kr^2)}{a^2}\left[a^2\frac{-2kr}{(1-kr^2)^2}\right] = \frac{-kr}{(1-kr^2)}$$

- - - - -

Cosmological distances

There are various way to define distances

- Note: it is convenient to keep in mind: $H \equiv \frac{\dot{a}}{a}$ $\quad \dot{a} = a H(a)$

  $\dot{a} = \frac{da}{dt}$ $\quad dt = \frac{da}{\dot{a}} = \frac{da}{aH(a)}$ integrals $dt \to da$

  $z+1 = \frac{1}{a}$ $\quad da = -a^2 dz = -\frac{dz}{(1+z)^2}$ " $da \to dz$

- Proper distance: distance covered by a photon in dt ($\nearrow c dt$)

  $dD_p \equiv c\,dt = \frac{c\,da}{a\,H(a)}$ $\quad \boxed{D_p = \frac{c}{H_0}\int \frac{da}{a E(a)}}$

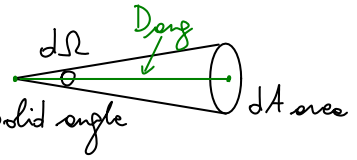
- Comoving distance: distance between 2 hypersurfaces for t ($\nearrow D_c$)

  $ds^2 = -c^2 dt^2 + a^2 d\vec{x}^2 = 0$ $\quad d\chi = c\frac{dt}{a} = \frac{c\,da}{a^2 H(a)} = \frac{dD_p}{a}$ $\quad \boxed{D_c = \frac{c}{H_0}\int \frac{da}{a^2 E(a)}}$

  analogy to conformal time $d\tau = \frac{dt}{a}$

- Angular diameter distance: distance obtained by measuring angles

  $dA \equiv d\Omega\, D_{ang}^2$

  $\Rightarrow \quad D_{ang} = \left(\frac{dA}{d\Omega}\right)^{1/2}$

  d solid angle $\to \frac{d\Omega}{4\pi} = \frac{dA}{4\pi a^2 D_c^2}$ $\leftarrow$ d area

  solid angle of sphere $\to$ $\leftarrow$ area of sphere

  $\Rightarrow \frac{dA}{d\Omega} = a^2 D_c^2$

  $\boxed{D_{ang} = a D_c}$

- Luminosity distance: distance obtained by measuring a flux F

  $F \equiv \frac{L}{4\pi D_L^2}\left[\frac{J}{s\,m^2}\right]$ (related to angles)

  redshift, "stretching of λ": $\left(\frac{a_1}{a_2}\right)$

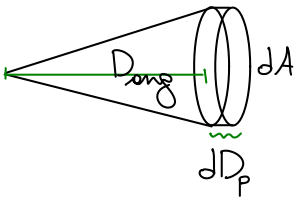
  spatial dilution: $\left(\frac{a_1}{a_2}\right)^2$ $\Rightarrow \propto \left(\frac{a_1}{a_2}\right)^4$ on F

  delayed arrival time γ_s: $\left(\frac{a_1}{a_2}\right)$

  $\boxed{D_L = \left(\frac{a_1}{a_2}\right)^2 D_{ang}(a_1, a_2)}$ $\neq D_{ang}$ between a_1 and a_2

Volumes

- Proper volume



$$dV_p = dA(z)\, dD_p = D_{ang}^2(z)\, d\Omega\, \frac{c\, dz}{H_0\, z\, E(z)} = D_z^2(z)\, \frac{c}{H_0}\, \frac{dz\, d\Omega}{(1+z)\, E(z)}$$

(area)

- Comoving volume

$$dV_c(z) = \frac{dV_p(z)}{a^3} = (1+z)\, dV_p(z) = \frac{c}{H_0}\, \frac{(1+z)^2 D_A^2(z)}{E(z)}\, dz\, d\Omega$$

Why are distances and volumes relevant?

- D_{lum}: observation of standard candles, eg. cepheids, Type 1a Supernovae
- D_{ang}: " " " rooters, eg. CMB, BAO
- Number counts: Redshift surveys of galaxies or clusters

Assume a population of objects with comoving density $n_c(z)$

Number of objects: $dN = n_c(z)\, dV_c \Rightarrow \frac{dN}{dz\, d\Omega} = n_c(z)\, \frac{dV_c}{dz\, d\Omega} = n_c(z)\, \frac{c}{H_0}\, \frac{(1+z)^2 D_A^2(z)}{E(z)}$

- structor formation
- astrophysics
- selection function (observations)

geometry

$\Rightarrow$
- Cosmology: knowing a population, you constrain the properties of the universe (geometry)
  eg. non evolving population $n_c(z) = \text{const.} \Rightarrow$ geometry only
- Astrophysics: for a fixed cosmology, study evolution of population

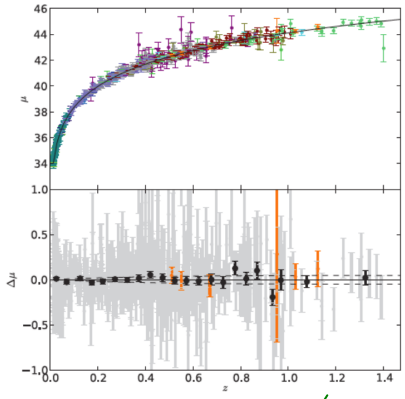


Figure 6.1.2: Union2 combined catalog of 557 SNIa. Here $\mu = 5 \log d_L + const$. From Amanullah et al. 2010 ApJ...716..712A *(© AAS. Reproduced with permission)*. Bottom panel: residuals.

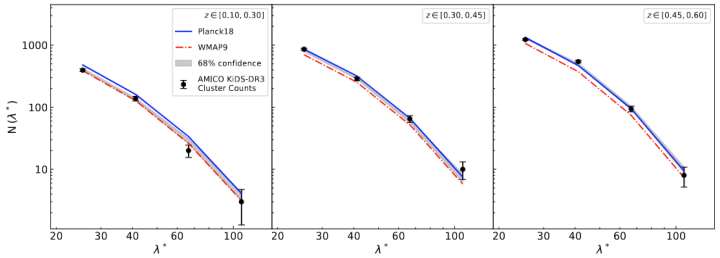


Fig. 4. Number counts from the AMICO KiDS-DR3 cluster catalogue as a function of the intrinsic richness λ^*, in the redshift bins $z \in [0.10, 0.30]$, $z \in [0.30, 0.45]$, and $z \in [0.45, 0.60]$, *from left to right*. The black dots represent the counts directly retrieved from the catalogue, where the error bars are given by the Poissonian noise. The solid blue lines represent the model computed by assuming the cosmological parameters obtained by Planck Collaboration VI (2020) (Table 2, TT, TE, and EE+lowE), while the red dashed lines show the results based on the WMAP cosmological parameters (Hinshaw et al. 2013) (Table 3, WMAP-only Nine-year). Both in the Planck and WMAP cases, the scaling relation parameters and the intrinsic scatter have been fixed to the median values listed in Table 2, retrieved from the modelling. The grey bands represent the 68% confidence level derived from the multivariate posterior of all the free parameters considered in the cosmological analysis.

Cosmological horizons

Related to distances, the concept of horizon

Particle horizon

$ds = 0 \quad c\,dt = a\,dr$ $\quad \boxed{R = \frac{c}{H_0}\int_0^a \frac{da'}{a'^2 E(a')}}$

Causally connected points / Retrive information from the past

Set furthest object we can see, observable universe

Hubble radious

proper

$\boxed{R_H \equiv \frac{c}{H(t)}}$ (Hubble's law $v = rH : r = \frac{v}{H}$) ($v \to c$)

- "distance at which particles would have $v = c$"
- it ignores expansion hystory (is not an integral function)
- if $r > R_H$ to points can not talk any more
- $R_{H_0} = \frac{c}{H_0} = 4{,}1\ Gp$ often used as a length scale

Dynamical Dark Energy

- Λ explains very well observations, why do we need DE?

- $T^{\wedge}_{\mu\nu}$ has no dependency on observer motion (there is no $\vec{u}$) *2
  ↳ characteristic of empty space ⇒ $T^{\wedge}$ associated to vacuum energy
  ↳ (there is nothing ⇒ there are no velocities of something with respect to something else)

- But... if this is so, we get a problem....

Planck mass $m_p = \sqrt{\frac{\hbar c}{G}} = 10^{19}\,GeV$

(a) from cosmological observations: $|\Lambda| < 10^{-56}\,cm^{-2}$

- Friedmann eq. $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$ ⇒ $\Lambda \sim H_0^2$ observations: $H_0 \simeq 70\ km/s\,Mpc$

$$\rho_\Lambda = \frac{c^2\Lambda}{8\pi G} = \frac{c^2\Lambda m_p^2}{8\pi\hbar^2 c^2} = \frac{\Lambda m_p^2}{8\pi\hbar^2} \approx 10^{-47}\,GeV^4 \approx 10^{-123}\,m_p^4$$

- As a reference, to see how small this is, consider a photon with the smallest possible λ i.e. the size of the universe: $\lambda = R_H = \frac{c}{H_0} \approx 3\,Gpc/h$ ⇒ $E_\gamma = h\nu \sim 10^{-27}\,eV$

↳ fine tuning problem, why so "unnaturally small" but not zero?!

(b) From an elementary particle perspective: vacuum energy is $\sim 10^{121}$ times larger!

- vacuum energy $\langle\rho\rangle$ of empty space ($\hbar = 1 = c$)
- zero energy of a field of mass m with momentum k and frequency ω $E = \frac{\omega}{2} = \sqrt{k^2+m^2}/2$
- sum zero-point energies of the field up to a cut-off k_{max} ($\gg m$)

$$\rho_{vac} = \int_0^{k_{max}} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} = \int_0^{k_{max}} \frac{2\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2} \simeq \int_0^{k_{max}} \frac{2\pi k^3 dk}{(2\pi)^3 2}\,dk = \frac{k_{max}^4}{16\pi^2} \simeq 10^{74}\,GeV^4$$

↑ on k shells (isotropy) ↑ $k \gg m$ dominates ↑ GR valid up to Planck scale ⇒ take k_{max} to l_p, m_p

Even using other energy scales the problem remains: QCD scale $k_{max} \approx 0{,}1\,GeV$ ⇒ $\rho_{vac} \approx 10^{-3}\,GeV^4$

- This is why people set $\Lambda = 0$ and introduce a new dynamical field: Dark-Energy (DE)
  not a constant, it can evolve with time: $w(a)$ ←

  · $\Lambda = 0$ ⇒ ∃ of some symmetry (supersymmetric theories)
  even if symmetry is broken still possible to have $\Lambda = 0$

  · In any case, behavior of DE must be very close to Λ: so far observations were compatible with Λ!

A simple model for Dark Energy

What is it for?

- Λ is problematic => set $\Lambda = 0$
- substitute it with a fluid with w such to have accelerated expansion

Simplest model:

- self interacting scalar field ϕ (spin 0 particle like Higgs field)

Lagrangian density $\mathcal{L} = -\frac{1}{2}c^2 \partial_\alpha\phi\,\partial^\alpha\phi - U(\phi)$

energy-momentum $T_{\mu\nu} = c^2 \partial_\mu\phi\,\partial_\nu\phi + \mathcal{L} g_{\mu\nu} = c^2 \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}c^2 \partial_\alpha\phi\,\partial^\alpha\phi\, g_{\mu\nu} - U(\phi) g_{\mu\nu}$

- Isotropy: only diagonal components of T are $\neq 0$

Homogeneity: $\partial_i\phi = 0$ * i.e. $\phi(t)$ only as well as isotropy (vector $\vec{\nabla}\phi$ = privileged direction)

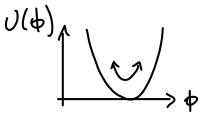
$$\begin{cases} T_{00} = \frac{1}{2}\dot\phi^2 + U(\phi) + \frac{1}{2}c^2(\vec\nabla\phi)^2 \overset{*}{=} \overset{!}{=} \rho_\phi c^2 & \text{energy density} \\ T_{ii} = \frac{1}{2}\dot\phi^2 - U(\phi) - \frac{1}{6}c^2(\vec\nabla\phi)^2 \overset{!}{=} P_\phi & \text{pressure} \end{cases}$$

ideal fluid $T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu}$ "$\cdot$" $\equiv \frac{d}{c\,dt}$ conformal time

$P = w\rho c^2$: $w = \frac{P}{\rho c^2} = \frac{\frac{1}{2}\dot\phi^2 - U}{\frac{1}{2}\dot\phi^2 + U}$ effective eq. of state => w not constant in general!

Different solutions

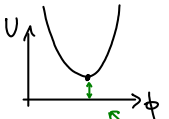
A) if ϕ oscillates harmonically around a zero-energy potential



=> kinetic energy = potential energy ($\frac{1}{2}\dot\phi^2 = U$) when averaging over several cycles => $P = 0$ i.e. $w = 0$

=> Dust!

B) Static field ($\dot\phi = 0$) at non-zero energy minimum of U

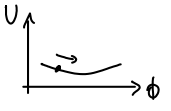


$\neq 0$ or $\rho c^2 = U = 0$ = nothing

=> $\boxed{w = -1} = \frac{P}{\rho c^2} = \frac{-U}{U}$ => field represents the cosmological const. Λ!

state in which the field has non zero vacuum energy

C) Dynamical field ($\dot\phi \neq 0$) but not oscillating:



- For DE observations says that we need $w \sim -1$ and nearly const. (slow evolution, no oscillations!)

=> slow roll limit: $\boxed{\dot\phi^2 \ll U(\phi)}$ => $w = -1 + \varepsilon$ ($\varepsilon = \frac{\dot\phi}{U} \ll 1$) (* just 1° order for) w_{DE} slightly larger than -1

$w < -1$ you get ghosts (theories are unstable)

- as field evolves $\dot\phi \uparrow$ => $w \uparrow$ => necessary dynamic! key difference from Λ Dynamical DE!

Eq. of motion of DE

- Find evolution of the field ϕ

- Adiabatic condition: $\dot{\rho}c^2 + 3H(\rho c^2 + p) = 0$ $\quad \frac{1}{2}\dot{\phi}\ddot{\phi} + \frac{\delta U}{\delta \phi}\dot{\phi} + 3H\left(\frac{1}{2}\dot{\phi}^2 + U(\phi) + \frac{1}{2}\dot{\phi}^2 - U(\phi)\right) = 0$

$\Rightarrow$ $\boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{\delta U}{\delta \phi} = 0}$ Klein-Gordon eq. in an expanding universe

scalar field mass $m^2 = \frac{d^2U}{d\phi^2}$ in general is a func. of time

Dynamics depends on U, choose the model of your like

get inspiration from supersymmetry, extra dimensions, ...

In fact KG eq.:

$(\Box + m^2)\psi(t,x) = 0$ with $m^2 = \frac{\delta U}{\delta \phi}$ in conformal flat metric $\dot{} = \frac{d}{cd\tau}$

$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right) = a^{-4}\partial_\mu\left(a^4 a^{-2}\eta^{\mu\nu}\partial_\nu\phi\right) = a^{-4}\left(2a\partial_\mu a\,\eta^{\mu\nu}\partial_\nu\phi + a^2\partial_\mu\partial^\mu\phi\right) = a^{-2}\left(-2H\dot{\phi} - \ddot{\phi}\right)$

$\sqrt{-g} = \sqrt{-(-a^2)a^2a^2a^2} = a^4$ $\quad \partial_\mu a\partial^\mu\phi = \partial_0 a\partial^0\phi = -\dot{a}\dot{\phi}$ ($a(t)$ only) $\quad$ isotropy $\Rightarrow \partial_i\phi = 0$ $\partial_\mu\partial^\mu\phi = -\ddot{\phi}$

Coupling of dark energy with matter

- Scalar field (like Higgs field) $\rightarrow$ can couple to any other matter component (not only self interaction)

$\hookrightarrow$ if no, DE might carry a new force: fifth force

- New force is unscreened $\Rightarrow$ extend to large distances

$\Rightarrow$ it appears as a modification of gravity

$\boxed{\nabla^2\phi = -4\pi G f(a,x)\, a^2\hat{\rho}\delta}$ modified Poisson eq.

f: some function of time and space

- coupling, i.e. the fift force, affects cosmic structure formation

growth eq. $\delta_m'' + \frac{1}{2}(1 - 3w_{eff})\delta_m' - \frac{3}{2}f(a,x)\Omega_m\delta_m = 0$ (we will look in to that later...)

$\Rightarrow$ could be characterized by observations (e.g. galaxy clustering, CMB, BAOs, Clusters, ...)

DE in the Friedmann equations

Condition on w to have acceleration

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3} \overset{!}{>} 0 \quad \Rightarrow \quad \left(\rho + \frac{3p}{c^2}\right) < 0 \quad \Rightarrow \quad p > -\frac{1}{3}\rho c^2 \quad \Rightarrow \quad w < -\frac{1}{3} \quad \text{and} \quad \dot{\phi}^2 < V(\phi)$$

(w_{max})

DE $\boxed{-1 \leq w \leq -\frac{1}{3}}$ for $w < -1$ most fields are unstable : ghosts

Density ρ_ϕ evolution

- Adiabatic condition $\dot{\rho}c^2 + 3H(\rho c^2 + p) = \dot{\rho}c^2 + 3H\rho c^2(1+w) = 0$ ($p = w\rho c^2$)

$$\Rightarrow \quad \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \qquad \ln\frac{\rho}{\rho_0} = -3\int_1^a (1+w(a))\frac{da}{a}$$

$$\boxed{\rho = \rho_0 e^{-3\int_1^a (1+w(a))\frac{da}{a}}}$$

Chevallier-Polarski-Linder parametrization

There are many DE models and $w(a)$ evolves slowly over a large range of cosmic time

$\Rightarrow$ Assume linear model for w_{DE}: $\boxed{w_{DE} = w_0 + w_a \cdot (1-a)} = w_0 + w_a \frac{z}{1+z}$

$w_0 = w_{DE}(1)$ today

i.e. as Taylor expansion of $w(a)$ at 1° order

$$\Rightarrow d\rho = -3(1+w)\frac{da}{a} = -3(1+w_0 + w_a\cdot(1-a))\frac{da}{a} \quad \text{integrate}$$

$$\int\frac{d\rho}{\rho} = -3(1+w_0+w_a)\int\frac{da}{a} + 3\int w_a\, da \qquad \ln\rho'\Big|_\rho^{\rho_0} = -3(1+w_0+w_a)\ln a'\Big|_a^1 + 3w_a\cdot a'\Big|_a^1$$

$$\ln\rho_0 - \ln\rho = +3(1+w_0+w_a)\ln a + 3w_a\cdot(1-a) \quad \Rightarrow \quad \boxed{\rho_{DE} = \rho_0 e^{-3(1+w_0+w_a)a}\cdot e^{3w_a(a-1)}}$$

Friedmann eq.

$$\boxed{H^2(a) = H_0^2\left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\phi e^{-3\int_1^a (1+w(a))\frac{da}{a}}\right]}$$

$$H^2(a) = H_0^2\left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\phi e^{-3(1+w_0+w_a)a}\cdot e^{3w_a(a-1)}\right]$$

Appendix

Dark energy as a self interacting fluid

signature $(-1,1,1,1)$

$$\mathcal{L} = -\frac{1}{2}c^2\partial_\alpha\phi\partial^\alpha\phi - U(\phi) \qquad T_{\mu\nu} = c^2\partial_\mu\phi\partial_\nu\phi + \mathcal{L}g_{\mu\nu} = c^2\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}c^2\partial_\alpha\phi\partial^\alpha\phi\, g_{\mu\nu} - U(\phi)g_{\mu\nu}$$

$$T_{00} = c^2(\partial_0\phi)^2 + \frac{1}{2}c^2\left[-(\partial_0\phi)^2 + (\vec{\nabla}\phi)^2\right](-1) - U(\phi)(-1) = \dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}c^2(\vec{\nabla}\phi)^2 + U(\phi) = \frac{1}{2}\dot{\phi}^2 + U(\phi) + \frac{1}{2}c^2(\vec{\nabla}\phi)^2$$

$$T_{ii} = c^2(\partial_i\phi)^2 - \frac{1}{2}c^2\left[-(\partial_0\phi)^2 + (\vec{\nabla}\phi)^2\right]\cdot 1 - U(\phi)\cdot 1 = c^2(\partial_i\phi)^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}c^2(\vec{\nabla}\phi)^2 - U(\phi) = \frac{1}{2}\dot{\phi}^2 - U(\phi) - c^2\frac{1}{6}(\vec{\nabla}\phi)^2$$

$$* \quad (\vec{\nabla}\phi)^2 = \vec{\nabla}\phi\cdot\vec{\nabla}\phi = (\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 = 3(\partial_i\phi)^2$$

here we defined "$\cdot$" $\equiv \frac{\partial}{c\,d\tau}$ dot "$\cdot$" derivative with respect to conformal time $\qquad \frac{\partial}{\partial x^0} = \frac{\partial}{c\,dt} \equiv \frac{\partial}{a c\, d\tau}$

Recall: $ds^2 = a^2(t)\left[-c^2d\tau^2 + dx^2 + f_k^2(x)\left(d\theta^2 + \sin\theta\, d\phi^2\right)\right]$ $\quad d\tau \equiv \frac{dt}{a(t)}$ $\quad \tau$ = conformal time

- Example of constraint coming from type 1a supernova data

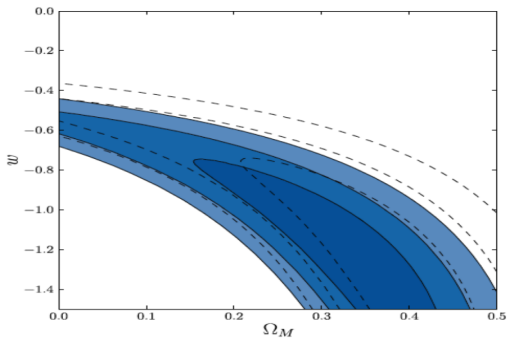


Figure 6.1.3: 68.3 %, 95.4 % and 99.7 % confidence level contours on $(\Omega_m^{(0)}, w_{\rm DE})$ from the SN Ia observations (denoted as Ω_M and w in the figure) compiled in Amanullah et al. 2010ApJ...716..712A *(© AAS. Reproduced with permission).*

Cosmological Inflation

Problems of the standard model

Friedmann model works very well but...

1) The horizon problem
   - Universe is uniform on scales larger than the cosmological horizon
   - There are coherent structures larger than the horizon (CMB)

   } how can it be?!

2) The flatness problem:
   - we see that today $\Omega_0 \sim 1$, why?!
   - Friedmann model: $\Omega = 1$ ($k=0$) is very unstable
     i.e. small deviation from $\Omega = 1$ rapid departure: → collapse $\Omega \to \infty$, → expansion $\Omega \to 0$
     fine-tuning $|\Omega_{tot} - 1| \ll 1$

3) The relics problem:
   - fundamental theories → e.g. monopoles... but they are not here nowadays, why?

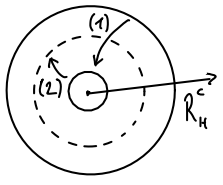
4) The origin of cosmic structures:
   - Structure formation arise because of initial energy-density fluctuation
   - Origin of these seeds?

One idea to fix everything: cosmic inflation

$\ddot{a} > 0$: phase of accelerated expansion of several e-folds at early times $t \ll t_{eq}$

- The horizon problem:

need phase in which the comoving horizon shrinks, i.e. $\dot{R}_H < 0$



(0) R_H^c is "large", within R_H^c causal connection => homogeneity, coherency
(1) R_H^c shrinks => scale of connected structures larger than R_H^c
(2) R_H^c normal growth

$R_H^c \equiv \frac{c}{aH} = \frac{c}{\dot{a}}$ if $\ddot{a} > 0$ accelerated expansion $\Rightarrow \frac{dR_H^c}{dt} = -\frac{c\ddot{a}}{\dot{a}^2} < 0 \Rightarrow \boxed{\ddot{a} > 0}$!!! ✓

- The flatness problem: $|\Omega_{tot} - 1| = \frac{Kc^2}{\dot{a}^2 H^2} = \frac{Kc^2}{(aH)^2} = \frac{Kc^2}{(R_H^c)^2} \to 0$ (Friedmann; shrinks) :)))))...| $\Omega_K \propto \frac{1}{K}$ ✓

- The relics problem: $\rho_{relic} \to 0$ because $a \uparrow$ (i.e. volume $\uparrow$)

- The origin of cosmic structures: primordial quantum fluctuations δ_q stretched and amplified

How much inflation do we need?

$|\Omega_{tot} - 1| = \frac{K c^2}{\dot{a}^2} \frac{R_H^{c^2}}{H^2} \to 0$ (Friedmann)

- # of e-foldings to fix all problems $N \equiv \frac{\log(a_{end})}{\log(a_{in})} \sim 60$ at least

1) Curvature K: Current age of the universe $t_0 \approx 10^{17}$ s

Planck time $t_p \approx 10^{-43}$ s

$\frac{K_0 c^2}{\dot{a}_0^2 H_0^2} \cdot \frac{\dot{a}_e^2 H_e^2}{K_e c^2} = \frac{K_0 \dot{a}_e^2 H_e^2}{K_e \dot{a}_0^2 H_0^2}$

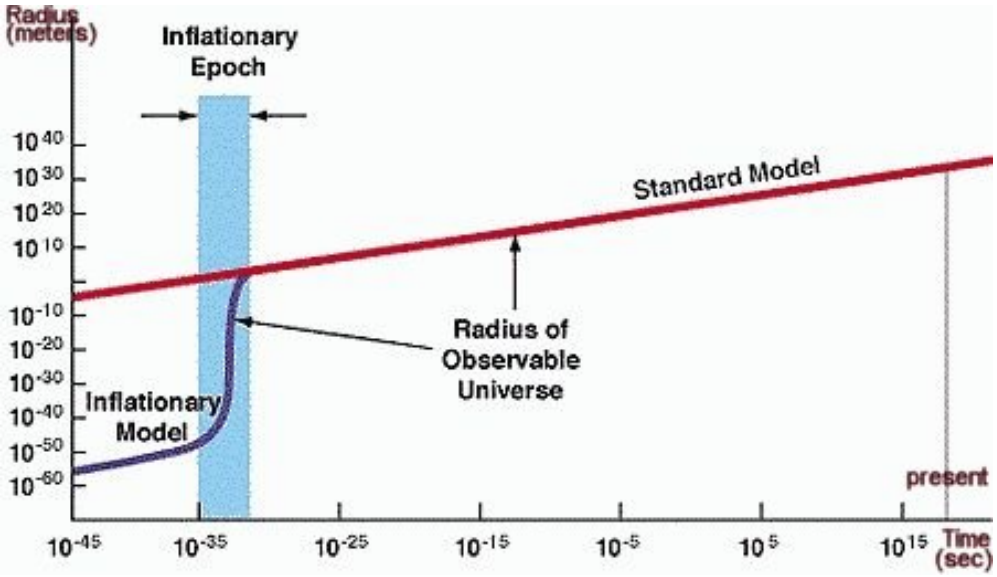
At radiation era $|\Omega_{tot} - 1| \propto t$ $\quad \frac{|\Omega_{tot}(a_0) - 1|}{|\Omega_{tot}(a_{end}) - 1|} \approx \frac{t_0}{t_{end}} \approx \frac{10^{17}\,s}{10^{-43}\,s} = 10^{60}$

$\Rightarrow$ to have $|\Omega_{tot}(a=1) - 1| \simeq 0{,}01$ (observational uncertainty, upper bound)

2) Horizon: $R_H^c = \frac{\dot{a} H}{c}$ need to shrink by a factor of 30 corresponding to approx 60 orders in a

3) Relics: due to this large values of N this problem is solved for free

$\rho_m \propto a^{-3}$ given $N = 60$, ρ_m drops by a factor of $10^{-3 \cdot 60} = 10^{-180}$



Which fluid provides $\ddot{a}>0$?

$P = w\rho c^2$

- $\frac{\ddot{a}}{c} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \overset{!}{>} 0 \quad \Rightarrow \quad \rho + \frac{3P}{c^2} < 0 \quad \Rightarrow \quad P < -\frac{1}{3}\rho c^2 \quad \Rightarrow \quad \boxed{w < -\frac{1}{3}}$

same as dark energy

- This fluid must dominate the expansion, i.e. all other species ($\Omega_m, \Omega_r, \dots$) are negligible, and get an exponential growth, eg. Λ dominated universe:
  from Friedmann equations it follows $a \propto \exp(\sqrt{\Omega_\Lambda} H_0 t)$ (de-Sitter model)

- It has to last long enough (60 e-foldings) to have enough expansion to fix all four problems

- It must end to become negligible to get the "normal" expansion. How?
  Λ would be constant $\Rightarrow$ No $\Rightarrow$ need something else
  Dynamical scalar field: the inflaton! Yes!

The simplest inflaton model

- self interacting scalar field ϕ (spin 0 particle like Higgs field)

  Lagrangian density $\mathcal{L} = -\frac{1}{2}c^2 \partial_\mu\phi\partial^\mu\phi - V(\phi)$

  energy-momentum $T_{\mu\nu} = c^2\partial_\mu\phi\partial_\nu\phi + \mathcal{L} g_{\mu\nu} = c^2\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}c^2\partial_\alpha\phi\partial^\alpha\phi\, g_{\mu\nu} - V(\phi) g_{\mu\nu}$

- Isotropy: only diagonal components of T are $\neq 0$
  Homogeneity: $\partial_i\phi = 0$ * i.e. $\phi(t)$ only as well as isotropy (vector $\vec{\nabla}\phi$ = privileged direction)

$$\begin{cases} T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}c^2(\vec{\nabla}\phi)^2 \overset{!}{=} \rho_\phi c^2 & \text{energy density} \\ T_{ii} = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{1}{6}c^2(\vec{\nabla}\phi)^2 \overset{!}{=} P_\phi & \text{pressure} \end{cases} \Big\} \text{ ideal fluid } T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u_\mu u_\nu + P g_{\mu\nu}$$

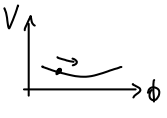
"$\cdot$" $\equiv \frac{d}{c\,dt}$ conformal time

$P = w\rho c^2$: $w = \frac{P}{\rho c^2} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}$ effective eq. of state $\Rightarrow$ w not constant in general!

- Same "set up" of dynamical dark energy but inflaton has a V such to fulfill certain behaviours!
  i.e. dominant at very early times, slow roll ends at a certain point (inflation must end)

- Evolution (eq. of motion): Klein-Gordon eq. for an expanding universe $\boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{\delta V}{\delta\phi} = 0}$

  with specific conditions we have a dynamical "slow rolling" field



How we want it to behave?

1) Accelerated expansion, i.e. $w < -\frac{1}{3}$: $p < w \rho c^2 \Rightarrow \frac{1}{2}\dot\phi^2 - V(\phi) < -\frac{1}{3}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right) \Rightarrow \boxed{\dot\phi^2 < V(\phi)}$ (kinetic, potential)

we can also obtain a constrain on $\ddot\phi$: $\frac{d}{dt}$ $\boxed{2\dot\phi\ddot\phi < \frac{dV}{d\phi}\dot\phi}$

$\frac{d}{dt}(\dot\phi^2) = 2\dot\phi\ddot\phi < \frac{dV}{d\phi}\dot\phi$

2) Exponential expansion, i.e. similar to Λ, and it must last long enough

(1) $\begin{cases} \dot\phi < V(\phi) \\ 2\dot\phi\ddot\phi < V'(\phi)\dot\phi \end{cases}$ $\xrightarrow{!}$ Slow roll approximation $\boxed{\begin{array}{ll} \dot\phi \ll V(\phi) & (a) \\ \ddot\phi \ll V'(\phi) & (b) \end{array}}$ $V' \equiv dV/d\phi$

- Ensures large number of e-foldings $N \equiv \frac{\log(a_{end})}{\log(a_{init})} \gg 1$ (~ 60) it must last long enough
- It simplifies eq.s, from 2° to 1° order diff. eq. (but lose generality, need of an attractor behavejour)
- For some simple potential V, there are analytical solutions
- It naturally provides end of inflation: when $\dot\phi$ and $\ddot\phi$ are not any more negligible
  $\Rightarrow$ ϕ starts to oscillate (damped oscillation) see eq. K-G eq.
  $\Rightarrow$ the inflaton energy goes in to creation of particles (reheating)

Apply approximation to the eq.s of motion

- Field: $\begin{cases} \rho_\phi c^2 = \frac{1}{2}\dot\phi^2 + V \overset{(a)}{\simeq} V \\ P_\phi = \frac{1}{2}\dot\phi^2 - V \overset{(a)}{\simeq} -V \end{cases}$ $V(\phi) \sim$ const. because $\dot\phi \ll V$

- Dynamics $\begin{cases} H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3} \approx \frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right) & (1) \\ \ddot\phi + 3H\dot\phi + V' = 0 \end{cases}$ ($K \to 1$)

$\dot\phi = -\frac{V'}{3H}$ $\dot\phi^2 = \frac{V'^2}{9H^2} = \frac{V'^2}{9}\frac{3c^2}{8\pi G V} = \frac{c^2}{24\pi G}\frac{V'^2}{V} \overset{(a)}{\ll} V \Rightarrow \boxed{\varepsilon(\phi) \equiv \frac{c^2}{24\pi G}\left(\frac{V'}{V}\right)^2 \ll 1}$

from $\varepsilon'(\phi) = \frac{c^2}{24\pi G}\,2\,\frac{V'}{V}\left(\frac{V''}{V} - \frac{V'^2}{V^2}\right) \to \boxed{\eta(\phi) \equiv \frac{c^2}{8\pi G}\frac{V''}{V} \ll 1}$

Slow-Roll parameters

- Whenever $\varepsilon \ll 1$, $\eta \ll 1$: slow roll is ensured $\Rightarrow$ we have inflation
- As soon as $\varepsilon \sim 1$, $\eta \sim 1$: $\ddot\phi$ is "active" dumped oscillations with damping factor $3H$
  $\Rightarrow$ Reheating, Inflation ends naturally

Example of potential

free massive scalar field: $V(\phi) = \frac{1}{2}m^2\phi^2$ ("straight"), $V' = \frac{1}{2}m^2 \not{2}\phi$ Chaotic inflation Linde (1983)

around global minima at $\phi = 0$ $V(\phi)$ approximated as a parabola

Time duration of inflation

- Solve (2) for ϕ:

$3H\dot\phi + V' = 0$ $\quad \dot\phi = -\frac{V'}{3H} = -\frac{V'}{3}\sqrt{\frac{3}{8\pi G V}} = -\frac{V'}{\sqrt{24\pi G V}}$ $\quad \phi = \phi_i - \frac{V'}{\sqrt{24\pi G V}}t = \phi_i\left(1 - \frac{t}{\tau}\right)$ $\quad \phi_i$: initial value at $t=0$, τ: inflation time scale

$$\tau \equiv \frac{\phi_i\sqrt{24\pi G V}}{V'} = \frac{\phi_i\sqrt{12\cdot 2\pi G \frac{1}{2}m^2\phi^2}}{m^2\phi} = \frac{\phi_i\sqrt{12\pi G}}{m}$$

- first time field reaches minima $\phi = 0$ all potential energy is converted in potential energy
  - $\Rightarrow$ slow roll conditions are broken $\dot\phi \not\ll V$ and oscillations start
  - $\Rightarrow$ $\phi = 0 = \phi_i\left(1 - \frac{t}{\tau}\right)$ $\quad t = \tau$ τ tells your duration of inflation

Initial conditions to have sufficient inflation

- Number of e-foldings α: $\alpha \equiv \ln\frac{a}{a_i} \overset{*}{=} \int H dt \Rightarrow d\alpha = H dt$ * exponential expansion
- Use eq. (2) and integrate in $d\alpha$: $3H\dot\phi \overset{(2)}{=} -\frac{dV}{d\phi}$ $\quad 3H\dot\phi = 3H\frac{d\alpha}{dt}\frac{d\phi}{d\alpha} = 3H^2\phi' = 3\cdot\frac{4}{3}8\pi G\frac{1}{2}m^2\phi^2\frac{d\phi}{d\alpha} = -m^2\phi$ ($H \overset{(1)}{=} \sqrt{\frac{8\pi G}{3}\frac{1}{2}m^2\phi^2}$, $-V' = -m^2\phi$)

$\Rightarrow$ $4\pi G\left(\phi^2 - \phi_i^2\right) = -\alpha = -\ln\frac{a}{a_i}$ $\quad \left(a = a_i\exp\left[2\pi G(\phi_i^2 - \phi^2)\right]\right)$

inflation ends when $\phi \to 0$ $\Rightarrow$ $\alpha = 2\pi G\phi_i^2$ $\mapsto$ $\boxed{\phi_i^2 > \frac{\alpha}{2\pi G}}$ $\Rightarrow$ # of e-foldings α constrain ϕ_i !

- for minimum needed of $\alpha \geq 60$: $\phi_i > 3m_p$ ϕ has units of a mass
- Init. condition ϕ_i randomly distributed in space (this is why "chaotic inflation")
  if a region meets this condition $\Rightarrow$ produces sufficient inflation to explain our universe
- We can also generalize for a potential $V = \frac{\lambda}{2n}\phi^{2n}$ $\Rightarrow$ $a = a_i\exp\left[2\pi G(\phi_i^2 - \phi^2)/n\right]$, $\phi_i^2 > 60n/2\pi G$
- Can we reproduce the conditions at time of inflation? NO

  $m_\phi c^2 = 6.5\cdot 10^{13}$ TeV max of LHC ~ 13 TeV $\Rightarrow$ out of reach

What happens to all species already present before inflation?

- Clearly, during inflation, energy density of inflaton ρ_ϕ remains relevant despite strong volume increase

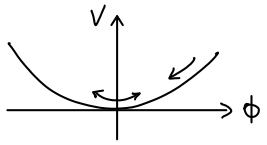
in fact: { V evolves very slowly (slow roll!) i.e. $V \sim$ "const." $\dot\phi \ll V$ $V(\phi)$
{ Energy density of the field $\rho_\phi c^2 = \frac{1}{2}\dot\phi - V \overset{*}{\simeq} V$ $\Rightarrow$ $\rho_\phi \simeq$ const.

- But all other species are "diluted away": $\rho_r \propto a^{-4}$, $\rho_m \propto a^{-3}$ given $N=60$, ρ_{ij} drops by a factor of $10^{-4\cdot 60} = 10^{-240}$, $10^{-3\cdot 60} = 10^{-180}$

$\Rightarrow$ Only the inflaton field is left but today we have radiation and matter
where are they coming from?

End of inflation and reheating

Slow roll but eventually $\varepsilon \approx 1$, $\eta \approx 1$ $\Rightarrow$ $\dot\phi$ and $\ddot\phi$ become relevant $\Rightarrow$ $\ddot\phi + 3H\dot\phi + V' = 0$ (!! friction)



$\Rightarrow$ dumped oscillations of ϕ

$\Rightarrow$ inflation ends and ϕ decays into particles *

↳ How? some coupling between ϕ and matter (?)

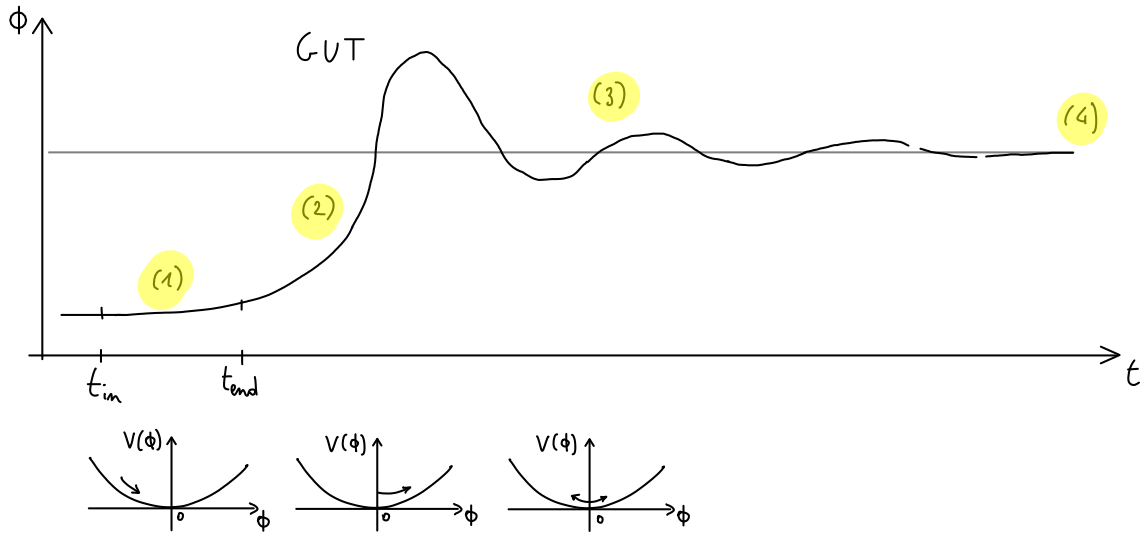
$\Rightarrow$ $\boxed{\rho_\phi \to \rho_{particles}}$ Reheating $T \uparrow$

↓

Fluids in "hot state" in thermal equilibrium
to allow a subsequent nucleosynthesis

* ϕ stores kinetic energy $\dot\phi^2$

An overview of inflaton time evolution



(1) Slow roll → inflation ($3H\dot{\phi} + V' = 0$)
- Slow rolling ends when ϕ first crosses the potential minimum at $\phi = 0$
- At this moment, all potential energy is converted in kinet. energy $\dot{\phi}$ and slow roll conditions are broken

(2) Reheating:
- fast evolution towards the true vacuum
- release of the latent heat (phase transition)

=> T stays high => still in GUT

=> you have violation of baryonic number for baryogenesis

(3) Dumped oscillations ($\ddot{\phi} + 3H\dot{\phi} + V' = 0$) => energy goes in to particle creation $\rho_\phi \rightarrow \rho_{particles}$ energy and entropy (???)

(4) Friedmann evolution (radiation dominated era)

Appendix

How does ϕ evolve? Eq. of motion

(1) Friedmann eq.: $H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}$ $\rightarrow$ $\begin{cases} H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right) \\ \ddot\phi + 3H\dot\phi + \frac{dV}{d\phi} = 0 \end{cases}$

$k=0$ very fast, negligible at early times

(2) Continuity eq.: $\frac{d}{dt}(a^3\rho c^2) + p\frac{d}{dt}(a^3) = 0$ $\rightarrow$

acceleration, friction, force (source)

- $3a^2\dot a\rho c^2 + a^3\dot\rho c^2 + p\,3a^2\dot a = 0 \rightarrow 3H\left(\rho c^2 + p\right) + \dot\rho c^2 = 0 \rightarrow \dot\rho = -\frac{3H\dot\phi^2}{c^2}$
- $\rho c^2 = \frac{1}{2}\dot\phi^2 + V(\phi)$ $\quad \dot\rho c^2 = \frac{1}{2}2\dot\phi\ddot\phi + V'(\phi)\dot\phi$ $\quad -3H\dot\phi^2 = \dot\phi\ddot\phi + V'(\phi)\dot\phi$ $\quad /V$

Slow-Roll parameters

Dynamics: $\begin{cases} H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3} \approx \frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right) & (1) \\ \ddot\phi + 3H\dot\phi + V' = 0 & (b) \end{cases}$ $\quad \dot\phi = -\frac{V'}{3H}$ $\quad \dot\phi^2 = \frac{V'^2}{9H^2} = \frac{V'^2}{9}\frac{3}{8\pi G V}c^2 = \frac{c^2}{24\pi G}\frac{V'^2}{V} \ll V$ (a) $\Rightarrow$

$$\boxed{\varepsilon(\phi) \equiv \frac{c^2}{24\pi G}\left(\frac{V'}{V}\right)^2 \ll 1}$$

Additionally: $\dot\phi = -\frac{V'}{3H}$ $\quad \ddot\phi = -\frac{d}{dt}\left(\frac{V'}{3H}\right) = -\frac{V''\dot\phi}{3H} + \frac{V'\dot H}{3H^2} = -\frac{V''\dot\phi}{3H} + \frac{V'^2\dot\phi}{6VH} \ll \frac{V'}{2} = -\frac{3H\dot\phi}{2}$ (b), (2)

(2) $2H\dot H = -\frac{8\pi G}{3c^2}V'\dot\phi$ $\quad \frac{\dot H}{H} = -\frac{8\pi G}{3c^2H^2}V'\dot\phi = \frac{\dot\phi}{2}\frac{V'}{V}$

$\Rightarrow$ $\frac{V''}{3H^2} - \frac{V'^2}{6VH^2} = \frac{3}{8\pi G}\left(\frac{V''}{3V} - \frac{V'^2}{6V^2}\right) \equiv \eta(\phi) - \frac{3}{2}\varepsilon(\phi) \ll \frac{3}{2} \sim 1$

$$\boxed{\eta(\phi) \equiv \frac{c^2}{8\pi G}\frac{V''}{V} \ll 1}$$

Origin of the fluctuation seeds

- At $t \approx t_{Planck}$ (Bouch-Davis) vacuum quantum fluctuations
- $\langle 0|\phi_k|^2|0\rangle$ expectation value of the ground level of the inflaton field at wave number k
  solve eq. for field amplitude $\{ \frac{H^2}{\dot\phi} + \dots$
  r.m.s. of fluctuations: $\sqrt{\langle \delta\varphi^2 \rangle} \propto \frac{H^2}{\dot\phi} \sim$ const because of slow roll ($\delta\varphi \sim$ metric fluctuations)
- Creation/Annihilation of virtual particles
- During inflation: R_H^c shrinks and pairs (fluctuations) are no more in causal contact $\Rightarrow$ "Freeze in"
- Being a random process, all scales, k, are equally affected

A qualitative but complete approach for scalar fluctuations:

- Fluctuations of the grav. potential caused by a spherical overdensity:

$$\delta\varphi = \frac{G\delta M}{R} = \frac{G}{R}\,\delta\rho\,\frac{4\pi}{3}R^3 = \frac{4\pi G}{3}\,\frac{\delta\rho_0}{a^2}(aL)^2 \propto L^2 = \text{const.}$$

comoving rad. $\quad R = aL$ physical radius

$\Rightarrow \delta\varphi$ remains const. during inflation

Friedmann eq.:

$$H^2(a) = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} \qquad \rho = \frac{3H^2}{8\pi G} + \frac{3Kc^2}{8\pi G a^2} = \rho_a + \frac{3Kc^2}{8\pi G a^2} = \frac{\rho}{\Omega} + \frac{3Kc^2}{8\pi G a^2} \qquad \rho a^2\left(1-\frac{1}{\Omega}\right) = \frac{3Kc^2}{8\pi G} = \text{const.}$$

$$\rho a^2\left(\frac{\Omega-1}{\Omega}\right) = \rho a^2\left(\frac{1+\delta\Omega-1}{\Omega}\right) \simeq \rho a^2\delta\Omega = a^2\delta\rho = \text{const.} \Rightarrow \boxed{\delta\rho \propto a^{-2}}$$

($\Omega \sim 1$)

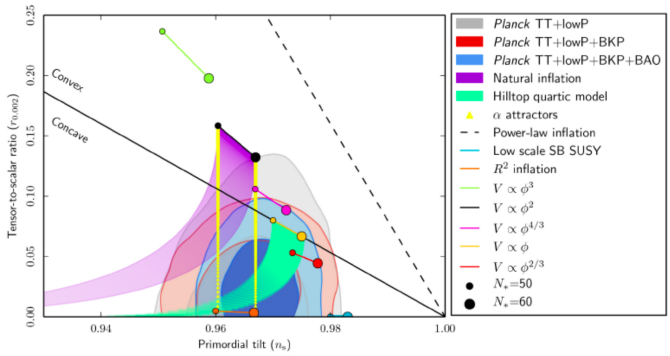
For a linear overdensity $\Omega = 1+\delta\Omega \quad \delta\Omega \ll 1 \quad \delta\rho = \rho\,\delta\Omega$

- $R = aL$ changes by ~ 30 orders of magnitude during inflation $\Rightarrow$ all physical scales are involved (also macroscopic ones)
- Fluctuations power spectrum:
  - Poisson eq. to link $\delta\varphi$ to $\delta\rho$: $\vec\nabla^2\delta\varphi = -4\pi G\delta\rho$ $\xrightarrow{\text{Fourier space}}$ $k^2\hat{\delta\varphi}(k) \propto -\hat{\delta\rho}(k)$
  - fluctuations power spectrum $k^3 P_i(k) \propto |\hat{\delta\rho}|^2 \propto k^4|\hat{\delta\varphi}|^2 \sim k^4\,\text{const.}$
    $\Rightarrow P_i(k) \propto k$ Harrison-Zel'dovich-Peebles spectrum

    Precise computations $\boxed{P_i(k) \propto k^{n_s} \quad n_s \simeq 0{,}96}$ (!) n_s: scalar spectral index observable

- We have superposition of a very large number of independent vacuum fluctuations (central limit theorem) $\Rightarrow$ expected to be Gaussian random field (!) observable

Deviations from that are called non Gaussianities



Planck 2015 (XX) Ade et.al

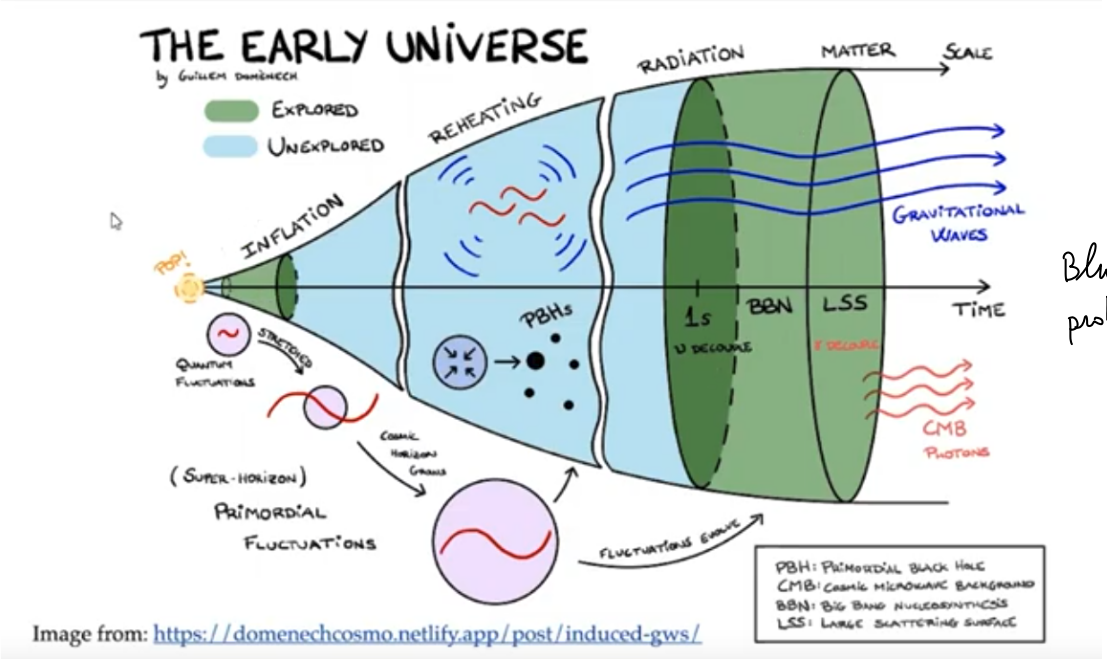
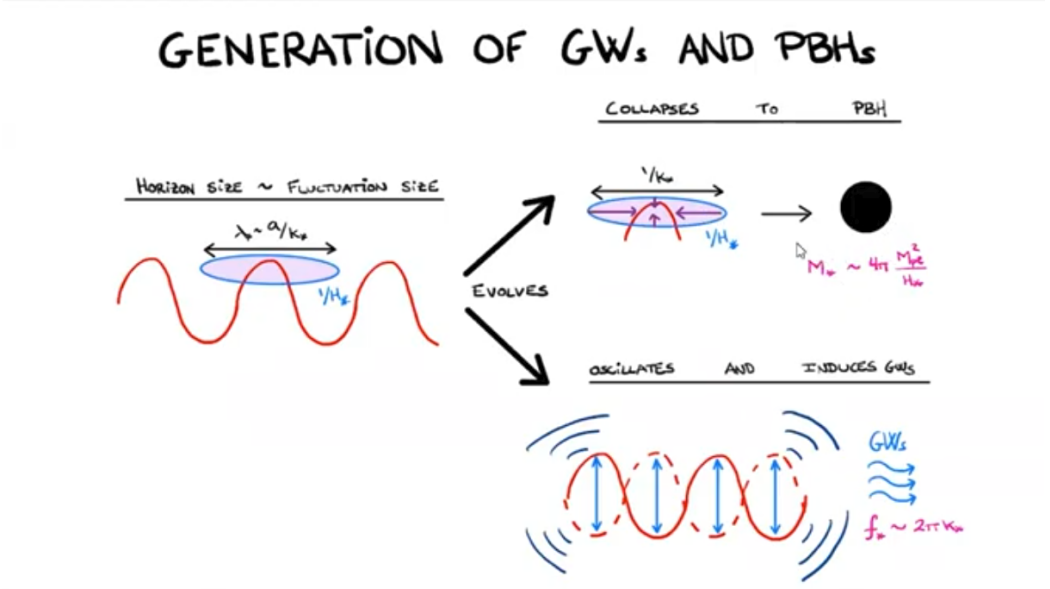


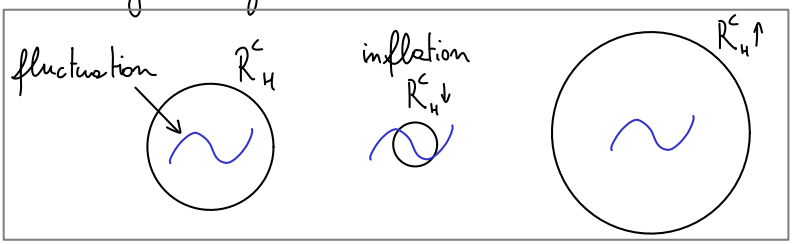
Image from: https://domenechcosmo.netlify.app/post/induced-gws/

Blue uncertain, probe it with GW



Primordial quantum fluctuations

- Bottom line:
  - Cosmic structures grow from initial primordial (Bunch-Davis) vacuum quantum fluctuations at $t \approx t_{Planck}$
  - Inflation "stretches" these fluctuations to a macroscopic scale
  - they freeze when they are larger than the horizon which shrinks because of inflation



  - There are 3 types of metric perturbations: scalar, vectorial, tensorial

- Quantum fluctuations can in general excite all 3 types

  Metric decomposition Scalar-Vector-Tensor decomposition (SVT)

  1) vector ones decay rapidly ⇒ neglect
  2) scalar ones, difficult derivation: need to perturbe simultaneously metric and scalar field
  3) tensorial easier: vacuum solution of linearized Einstein eq.s → harmonic oscillations
     ⇒ straightforward quantization, easier derivation
     ↳ we will describe these to show the idea behind all of these

Quantum harmonic oscillator, reminder

- Quantization of a scalar field — Spin 0 particles — here $t=1$
- Hamiltonian of a free particle: $H=\int d^3x\, \hat{a}^\dagger(\vec{x})\left(-\frac{1}{2m}\vec{\nabla}^2\right)\hat{a}(\vec{x}) = \int d^3p\, \frac{\vec{p}^2}{2m}\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})$ $\quad \hat{a}(\vec{p}) = \int \frac{d^3x}{(2\pi)^{3/2}}\hat{a}(\vec{x})e^{-i\vec{p}\vec{x}}$ #13a
- Harmonic oscillator: $\ddot{x}+\omega^2 x=0$ — solutions → $v(\omega,t)\sim e^{-i\omega t}$
- After quantization, solution: $\hat{x}=v(\omega,t)\hat{a}+v^*(\omega,t)\hat{a}^\dagger$
- Operators: $\hat{a}$ annihilates the vacuum state; $\hat{a}^\dagger$ creates a state of definite momentum $\vec{p}$
- Commutation rules: $[\hat{a}(\vec{p}),\hat{a}^\dagger(\vec{p}')]=\delta(\vec{p}-\vec{p}')$ $\quad [\hat{a},\hat{a}]=0$ $\quad [\hat{a}^\dagger,\hat{a}^\dagger]=0$
- $\hat{a}$ annihilates the vacuum state: $\hat{a}|0\rangle=0 \Rightarrow |0\rangle$ has 0 energy eigenvalue
- other eigenvalues of H: $\hat{a}^\dagger|\vec{p}_1\rangle \dots \hat{a}^\dagger|\vec{p}_m\rangle$
- corresponding energy eigenvalues: $E(\vec{p}_1)+\dots+E(\vec{p}_m)$ with $E(\vec{p})=\frac{1}{2m}\vec{p}^2$

$\Rightarrow [\hat{x},\hat{p}]=i$ if $\hat{p}=\frac{d\hat{x}}{dt}=-i\omega v\hat{a}+i\omega v^*\hat{a}^\dagger$ momentum operator and v normalized such $v(\omega,t)=\frac{e^{-i\omega t}}{\sqrt{2\omega}}$

Variance of $\hat{x}$: $\langle|\hat{x}|^2\rangle = \langle 0|\hat{x}^\dagger\hat{x}|0\rangle = |v^2|\langle 0|[\hat{a},\hat{a}^\dagger]+\hat{a}^\dagger\hat{a}|0\rangle = |v^2|$

Tensor perturbations

$\Box h^{\mu\nu}=0$ — Vacuum solution of linearized Einstein equation (1° order perturbation of metric)

TT gauge — Tensor perturbation: 2 modes h_+, $h_\times$ $\Rightarrow$ massless spin 2 field

Background metric is the Perturbation of the metric around FLRW (background)

Here, we consider a flat universe $\Rightarrow ds^2=a^2(-d\tau^2+\delta_{ij}dx^i dx^j)$ conformal time τ ($c=1$)

$\ddot{h}+2\frac{\dot{a}}{a}\dot{h}+k^2 h=0$ — harmonic at 1° order in h (in conformal metric) — "$\cdot$" $=\frac{d}{d\tau}$ conformal time

$\ddot{\tilde{h}}+\left(k^2-\frac{\ddot{a}}{a}\right)\tilde{h}=0$ — having defined $\tilde{h}\equiv\frac{ah}{\sqrt{16\pi G}}$] GR

$\hat{\tilde{h}}=v(k,\tau)\hat{a}+v^*(k,\tau)\hat{a}^\dagger$ — general solution, v satisfies

$[\hat{\tilde{h}}^\dagger,\hat{\tilde{h}}]=|v(k,\tau)|^2(2\pi)^3\delta_D(\vec{k}-\vec{k}')$ — commutation rules for continuous field] canonical quantization

$\langle\hat{h}\rangle=0$ — no relevant info on oscillations — average around zero = 0

$\langle\hat{\tilde{h}}^\dagger\hat{\tilde{h}}\rangle=\frac{16\pi G}{a^2}v^2(2\pi)^3\delta_D(\vec{k}-\vec{k}') \overset{*}{\equiv} P_h(k)(2\pi)^3\delta_D(\vec{k}-\vec{k})$ — Variance — * definition of $P(k)$

$P_h=16\pi G\frac{|v(k,t)|^2}{a^2}$ — Power spectrum — (volume factors, neglected by convention)

* The $16\pi G$ is there only for Lagrangian to second order in h
↳ necessary to have a canonical kinetic term
↳ needed to adopt the the standard commutation relation

Explicit solution for $v \Rightarrow P$

- During inflation, assume $H \sim$ const. $\Rightarrow$ conformal time:

$$\tau = \int_{a_e}^{a} \frac{dt}{a} \overset{(1)}{=} -\frac{1}{aH} + \frac{1}{a_e H} \overset{(2)}{\approx} -\frac{1}{aH} \overset{(4)}{=} -\frac{a^2}{a\dot{a}}$$

(1) $a = Hs$ (2) much before the end $a_e \gg a$ (3) "$-$" to have a growing time (4) $H = \frac{\dot{a}}{a^2}$ because "$\cdot$" conformal

$$\dot{a} = -\frac{a}{\tau} \qquad \ddot{a} \overset{**}{=} -\frac{\dot{a}}{\tau} + \frac{a}{\tau^2} = 2\frac{a}{\tau^2}$$

$$\tau = -H^{-1} \ *$$

$$\Rightarrow \ddot{\tilde{h}} + \left(k^2 - \frac{2}{\tau^2}\right)\tilde{h} = 0 \ (**) \rightarrow \ddot{v} + \left(k^2 - \frac{2}{\tau^2}\right)v = 0 \rightarrow \boxed{v = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)}$$

solved by means of Bessel functions

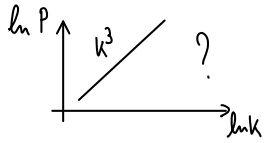
init condition of 1 outgoing wave $v \sim e^{-ik\tau}$

same normalization $\sqrt{2\pi}$ used in the harmonic oscillator

- When wave exit horizon ($|k| \ll |\tau|^{-1}$) $\Rightarrow$ oscillation freezes (there is no more causal connection)

$$v \simeq -\frac{e^{-ik\tau}}{\sqrt{2k}}\frac{i}{k\tau} \quad (\leftarrow R = R_H = \tau) \quad \Rightarrow \quad \boxed{P_h} = \frac{k^3}{2\pi^2}\frac{8}{16\pi G}\frac{1}{2k(k\tau)^2} = \boxed{8\pi G \frac{H^2}{k^3}}$$

(P_h: Horizon; $*$)



$R_H = c\tau = \tau \quad (c \doteq 1)$

Observational constrain: CMB $\frac{\Delta T}{T} \approx 10^{-4}$

No growth of spectrum amplitude $\Rightarrow$ these scales have same spectrum when we see them imprinted on the CMB

$$\sigma_R^2 \approx \int P_h W^2(k)\, d^3k \approx 8\pi G H^2 \int_{k_{min}}^{\infty} k^{-3} W^2(k,R)\, d^3k \approx G H^2$$

GW Fluctuations must be smaller or equal to the observed ones (some mode grows)

$$\Rightarrow G H^2 = G^2 V = M_p^{-4} E_{infl}^4 \leq 10^{-8} \quad \Rightarrow E_{infl} \leq 10^{-2} M_p \ll M_p$$

- Slope of spectrum: $n_T - 3 \equiv \frac{d \log P_h}{d \ln k}$ T = tensorial $\rightarrow n_T = -2\varepsilon$!

$$n_T = 3 + \frac{d\log(H^2 k^{-3})}{d\log k} = 3 + \frac{d\log k^{-3}}{d\log k} + \frac{d\log H^2}{d\log k} = 3 - 3 + 2\frac{\log H}{\log k} = 2\frac{k}{H}\frac{dH}{dk} = 2\frac{k}{H}\frac{dH}{d\tau}\frac{d\tau}{dk}\Big|_{k=aH} \text{ at Horizon}$$

$\Rightarrow n_T$ depends on $\dot{H}$, i.e. how H changes with time

$\Rightarrow$ can not assume $H \sim$ const during inflation or $n_T = 0 \Rightarrow$ work at 1° order correction ("1" $\equiv 0/\delta\phi$)

Define dimensionless slow roll parameters: $\varepsilon \equiv \frac{dH^{-1}}{dt} = -\frac{\dot{H}}{aH^2}$ $\qquad \delta \equiv \frac{1}{H}\frac{d^2\phi/dt^2}{d\phi/dt} = -\frac{a}{H\dot{\phi}}\left(3H\frac{\dot{\phi}}{a} + V'\right)$

..... $\varepsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$ $\delta = \varepsilon - \frac{1}{8\pi G}\frac{V''}{V}$ $\dot{H} = -aH^2\varepsilon$ $\tau = -1/aH$ .... $n_T = -2\varepsilon$

$\Rightarrow$ measure n_T = measure ε property of inflation

Scalar perturbations

Same approach as above but first perturb field φ relate φ to metric perturbations : ϕ, ψ

$P_\psi = P_\phi = \frac{8\pi G}{9k^3}\frac{H^2}{|\varepsilon|}\Big|_{k=aH}$ now $|\varepsilon|^{-1}$ term, small $\Rightarrow$ $P_S \gg P_T$

$P_{\delta_m} \sim k^4 P_\psi$ because density contrast $\delta_m \propto k^2 \psi$ (from Poisson eq. for sub horizon modes)

$n_S - 4 \equiv \frac{d\ln P_\psi}{d\ln k}$ slope of scalar mode spectrum

$n_S - 1 = \frac{d\log H^2}{d\log k} - \frac{d\log\varepsilon}{d\log k} = (1) - (2)$

$(1) = -2\varepsilon$ as for n_T

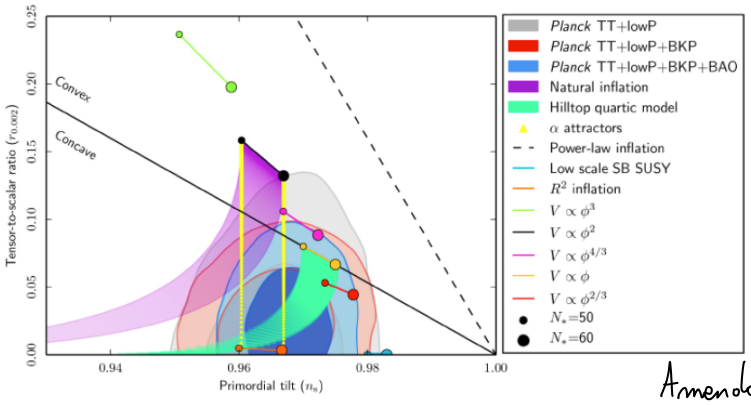
$(2) \approx \frac{d\log\varepsilon}{d\log a} = \frac{d\log\varepsilon}{d\phi}\frac{d\phi}{d\log a} = -\frac{1}{4\pi G}\left(\frac{V''}{V'} - \frac{V'}{V}\right)\frac{V'}{V} = 2(\varepsilon+\delta)$ ($k=aH$)

$\frac{d\phi}{d\log a} = -\frac{V'}{3H^2} = -\frac{V'}{8\pi G V}$

$\Rightarrow n_S = 1 - 4\varepsilon - 2\delta$ in many models $\varepsilon \gg \delta$ $\Rightarrow$ n_T a bit less than 1 (!)

CMB measurements : $n_T \simeq 0{,}96 \pm 0{,}01$ Planck (2015) :)

An investigate parameter is $r_p \equiv \frac{P_h}{P_\psi} = 16|\varepsilon|$ (Horizon (Tensorial)) exact value depends on the chosen value



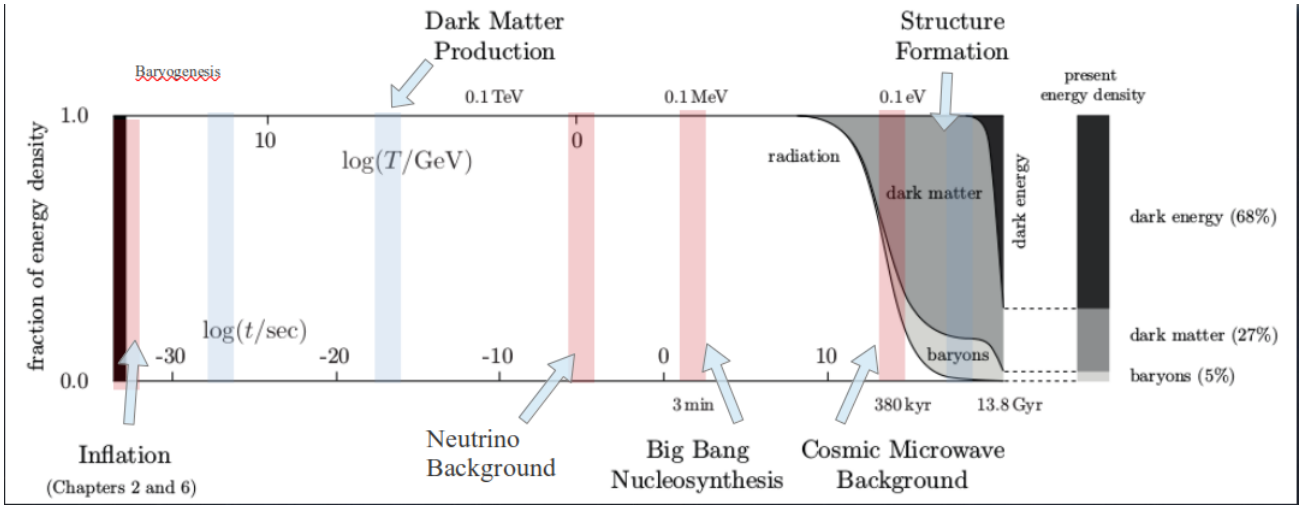
Amendola's script

Figure 10.2.1: Tensor-to-scalar ratio r and spectral slope n_S estimated by the Planck satellite (2015). From arXiv:1502.02114, Planck 2015 (XX) Ade et al..

$\Rightarrow$ P_T amplitude : direct measure of inflationary energy scale $H^2 \approx \frac{8\pi G}{3}V$

$\Rightarrow$ P_S amplitude and n_S : measure of potential derivatives through ε, δ

Thermal hystory of the universe



Event	time t	redshift z	temperature T	
Inflation	10^{-34} s (?)	–	–	> Inflaton field
Baryogenesis	?	?	?	> Particle anti-particle annihilation, small asymmetry
EW phase transition	20 ps	10^{15}	100 GeV	> Particles get mass (Higgs mechanism)
QCD phase transition	20 μs	10^{12}	150 MeV	> Quarks and gluons combine → Baryons , Quarks and anti-quarks combine → Mesons
Dark matter freeze-out	?	?	?	> Quite early to explain its weak interaction (t_coll >>t_exp)
Neutrino decoupling	1 s	6×10^9	1 MeV	> Weak inteaction $e^+ + e^- \longleftrightarrow$ nu + nu-
Electron-positron annihilation	6 s	2×10^9	500 keV	> $e^+ + e^- \leftrightarrow \gamma + \gamma$
Big Bang nucleosynthesis	3 min	4×10^8	100 keV	> p, n → D, H, He, Li, (Be)
Matter-radiation equality	60 kyr	3400	0.75 eV	
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV	> $e^- + p^+ \longleftrightarrow$ H + gamma
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV	> CMB
Reionization	100–400 Myr	11–30	2.6–7.0 meV	> H + gamma $\longleftrightarrow e^- + p^+$
Dark energy-matter equality	9 Gyr	0.4	0.33 meV	> Acceleration "start"
Present	13.8 Gyr	0	0.24 meV	> Life

These phases occours when certain processes are no more reversible due to the temperature drop

- e.g.. neutrinos freez-out : $e^+ + e^- \leftrightarrow \gamma + \gamma$ (reversible process) Adiabatic process
  $e^+ + e^- \rightarrow \gamma + \gamma$ become irreversible when $h\nu < m_e c^2$

Key concepts

- $P_1 + P_2 \overset{\text{interaction}}{\longleftrightarrow} P_3 + P_4$ Reaction
  - e.g. $e^+ + e^- \to \gamma + \gamma$ annihilation
    $e^- + \mathcal{Z}^+ \to e^- + \mathcal{Z}^+$ scattering process
  - 3 or 4 body interactions are rare

- Reaction is governed by Γ = rate of collisions $t_{coll} \equiv 1/\Gamma$
  H = " " expansion $t_{exp} \equiv 1/H$

- if $\begin{cases} \Gamma \gg H \Rightarrow \text{particles interact} \Rightarrow \text{Thermal equilibrium} \\ \Gamma \ll H \Rightarrow \text{no interactions} \Rightarrow \text{Particles decouple / Freeze-out} \end{cases}$

- The decoupling time depends on the interaction "$\longleftrightarrow$"

- $\Gamma = n \langle \sigma v \rangle$ $\left[\frac{1}{m^3} m^2 \frac{m}{s}\right] = [s^{-1}]$
  n = number density $dN = n\langle\sigma v\rangle dt$ $\Gamma \equiv \frac{dN}{dt}$
  σ = interaction cross section
  v = relative velocity $\Rightarrow v(T)$

$\Rightarrow$ We need to study n, T in a cosmological context

A "geometric" approach: $P = h/\lambda$ de-Broglie
$E = k_B T$ Thermal energy $\quad \lambda \propto a$

Non relativistic: $E = k_B T = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} \Rightarrow \boxed{T \propto a^{-2}}$

Relativistic: $E = k_B T = cp = \frac{h}{\lambda} \Rightarrow \boxed{T \propto a^{-1}}$

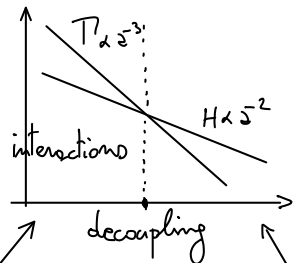
Condition for equilibrium at early times

- Early times: radiation domination, $K=0$, Λ or DE negligible $\Rightarrow H = \sqrt{\frac{8\pi G}{3}\rho}$ Friedmann eq.

$$\begin{cases} H = \sqrt{\frac{8\pi G}{3}\rho} \approx (G\rho)^{1/2} \propto a^{-2} & \uparrow a^{-4} \text{ radiation} \\ \Gamma = (n\langle\sigma v\rangle) \propto a^{-3} & \uparrow a^{-3} \text{ Bosons} \end{cases}$$

$n_B \propto T^3$ $T \propto a^{-1} \Rightarrow n_B \propto a^{-3}$

$\frac{\Gamma}{H} \propto \frac{a^{-3}}{a^{-2}} \propto a^{-1} \xrightarrow{\text{for } a \to 0} \infty$



$a(t) \uparrow \quad T(t) \downarrow$

Equilibrium is guaranteed at early times! freeze-out

Conditions for freez-out

- With no collisions : continuity eq. $\Leftrightarrow$ number conservation n_s = const.

$$\dot{n}_s + \vec{\nabla}(n_s \vec{v}) = 0 \qquad \vec{v} = H\vec{r} \qquad \vec{\nabla}(n_s H \vec{r}) = n_s H \vec{\nabla}\vec{r} = 3Hn_s$$
$$\dot{n}_s + 3Hn_s = 0 \quad \text{(expansion)} \qquad \frac{\delta x}{\delta x}+\frac{\delta y}{\delta y}+\frac{\delta z}{\delta z} = 3$$

- With collisions : source term

$$\dot{n}_s + 3Hn_s = j - \alpha n_s$$

α: destruction rate of particles
j: creation " " "

$P_1 + P_2 \underset{j}{\overset{\alpha}{\rightleftharpoons}} P_3 + P_4$ (destruction α, creation j)

$j - \alpha n_{sT} = 0 \qquad n_{sT} = \frac{j}{\alpha}$ at equilibrium

$$= \alpha n_s \left(\frac{j}{\alpha n_s} - 1\right) = \alpha n_s \left(\frac{n_{sT}}{n_s} - 1\right)$$

$j = \langle \sigma v \rangle n_T^2$
$\alpha = -\Gamma$

- Go comoving for convenience

$$a^3(\dot{n}_s + 3Hn_s) = \alpha n_s a^3 \left(\frac{n_{sT} a^3}{n_s a^3} - 1\right)$$

comooving: $N_s \equiv n_s a^3$

$$\dot{N}_s = \dot{n}_s a^3 + n_s 3a^2\dot{a} = a^3(\dot{n}_s + 3Hn_s)$$

$$\dot{N}_s = \alpha N_s \left(\frac{N_{sT}}{N_s} - 1\right)$$

$$\frac{d}{dt} = \frac{da}{dt}\frac{d}{da} = aH\frac{d}{da} = H\frac{d}{d\ln a} \qquad \dot{N}_s = H\frac{dN_s}{d\ln a}$$

$$\boxed{\frac{d\ln N_s}{d\ln a} = \frac{\alpha}{H}\left(\frac{N_{sT}}{N_s} - 1\right)}$$

Freez-out equation / Boltzmann eq. it tells you how the reaction proceeds

- Consequences :

1) if equilibrium : $N_s = N_{sT}$ $\Rightarrow$ N_s does not change

2) if off equilibrium $\Rightarrow$ N_s changes by moving toward equilibrium
if $N_s > N_{st} \Rightarrow \dot{N}_s < 0$ destruction $\longrightarrow$
if $N_s < N_{st} \Rightarrow \dot{N}_s > 0$ creation $\longleftarrow$

3) if $\alpha \ll H$ ($\alpha = -\Gamma$) $\Rightarrow \dot{N}_s \to 0$ Freez-out (their number is conserved) reaction stops $\not\longleftrightarrow$
particles do not participate any more to the reaction, they leave a background (relics)
e.g. CMB photons, neutrino background

- $\Gamma \ll H$ equivalent to : mean free path of interaction $\lambda \ll cH^{-1}$ cosmic horizon

- Relativistic particles (Fermions and Bosons) $\begin{cases} n_s \propto T^3 \\ T \propto a^{-1} \end{cases} \Rightarrow n_s \propto a^{-3} \Rightarrow N_s \equiv a^3 n = \text{const.}$

$\frac{d\ln N_s}{d\ln a} \overset{\text{const}}{=} 0 \overset{\downarrow}{=} \frac{\alpha}{H}\left(\frac{N_{sT}}{N_s} - 1\right) = 0 \Rightarrow$ i.e. $N_s = N_{sT}$ remains in T. equilibrium regardless $\frac{\alpha}{H}$!

- Non relativistic particles $\begin{cases} n_s \propto T^{3/2} e^{-mc^2/kT} \\ T \propto a^{-2} \end{cases} \Rightarrow N_s = \cancel{a^3}\,\cancel{a^{-3}}\, e^{-\frac{mc^2}{kT}}$

$\Rightarrow$ for $kT \ll mc^2$ "cold", exponential drop $\Rightarrow N_s \ll N_{sT}$: $\frac{d\ln N_s}{d\ln a} = -\frac{\alpha}{H} \to 0$

- to find out $n_s(T)$ we need to study quantum gases .... Next topic!

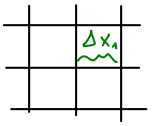
(Another kind of freez-out
- e.g. neutrinos freez-out : $e^+ + e^- \leftrightarrow \gamma + \gamma$ (reversible process) Adiabatic process
$e^+ + e^- \to \gamma + \gamma$ become irreversible when $h\nu < m_e c^2$
see later....)

Quantum statistics

Basic concepts

- Particle : · $(\bar{x}, \bar{p})$ point in a 6-dimensional space = phase-space
  · it occupies a quantum energy state with energy ε_α α: label of the state
  · phase-space is quantized (also for free particles because of indeterminination principle)

  $\Delta x \Delta p > h$ => phase-space "cells"

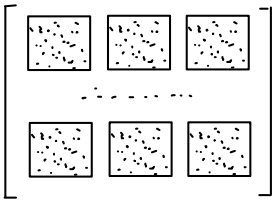


=> cube of h^3

- Statistical approach :
  · we do not care about each single particle, we care about ensamble properties
  · Phase space distribution $f(\bar{x}, \bar{p}, t)$: probability of having a particle in a certain $(\bar{x}, \bar{p})$ cell
    it can evolve as a function of time (dynamical system)
  · We are interested probability of having a certain number of particles in a given state

- System : collection of N particles 

- Ensamble : very large collection of identical systems
  (different realizations with the same macro properties)
  We consider 3 types of ensambles : Microcanonical
  Canonical
  Grand canonical



- Surrounding : what the system is in contact with, the thermal bath

Our universe: Assumptions

adiathermal => adiabatic <= reversible <= no S generation

- Adiabatic expansion : no heat flux because of isotropy + entropy dominated by photons (CMB)
- Thermal equilibrium : despite expansion, the system has the time to thermalize $\frac{\alpha}{H} \ll 1$
- Ideal fluids : no long scale interactions, direct collisions

Micro-canonical ensamble

Thermodynamical isolated/closed system: no exchange of particles and energy (in form of heat/work)

i.e. fixed: { internal energy, particle number N, volume V }

Canonical ensamble

- Closed system with no particle exchange, exchange of energy as heat/work
- only mean internal energy is specified
- ensamble parameters are N, T, V
- said to be (1) isothermal (2) isobaric if V = const. and T constant / P constant
- probability distribution of an α-state is given by:

• Boltzmann distribution — distribution of the states

$$f_\alpha = \frac{e^{-\varepsilon_\alpha/kT}}{Z_c} \qquad Z_c = \sum_\alpha e^{-\varepsilon_\alpha/kT}$$

Z_c: canonical partition sum (normalization)

• Helmotz free energy F: $F(T,V,N)$

$$\frac{1}{Z_c} \equiv e^{F/kT} \qquad e^{-F/kT} = Z_c \qquad \boxed{F = -kT\ln(Z_c)}$$

$$\Rightarrow f_\alpha = e^{F/kT} \cdot e^{-\varepsilon_\alpha/kT} = e^{(F-\varepsilon_\alpha)/kT}$$

• $F(T,V,N)$ = const. for the system

• $F = U - TS$ F is Legendre transform of internal energy U
work that a thermodynamical system can do

• $dF(T,V,N) \overset{\circledast}{=} -S\,dT - P\,dV + \mu\,dN \equiv \delta_T F\,dT + \delta_V F\,dV + \delta_N F\,dN$ total differential

$\Rightarrow$ ensamble properties! $\left(S = -\frac{\delta F}{\delta T}\right) \left(\langle P\rangle = -\frac{\delta F}{\delta V}\right) \left(\mu = \frac{\delta F}{\delta N}\right)$

μ = chemical potential change in the free energy due to a change in N

• At equilibrium: F is minimized given a T, V, N

T = const. V = const. $\Rightarrow \mu = 0$, in fact: $dF = -p\,dV^{=0} - S\,dT^{=0} + \mu\,dN = \mu\,dN \overset{!}{=} 0$

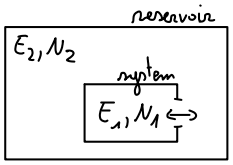
$\circledast$ $dF = dU - T\,dS - S\,dT = \cancel{T\,dS} - P\,dV + \mu\,dN - \cancel{T\,dS} - S\,dT$

$dU = T\,dS - P\,dV + \mu\,dN$

Grand canoncal ensamble

generalization of canonical ensamble N is not fixed, no restriction on # of particles

- Open system, exchange particles and energy with its surroundings
- accessible number of quantum states N_α is unknown
- only mean number is specified
- Ensamble variables are: μ, T, V μ = chemical potential of a specie, describs reactions



$E_2 \gg E_1$
$N_2 \gg N_1$
$N_1 + N_2 = N$ = const.
$E_1 + E_2 = E$ = const.

- $N = \sum_\alpha N_\alpha$ $\quad E(N_\alpha) = \sum_\alpha \varepsilon_\alpha N_\alpha$ Tot. number of states and energy of system

- $f_\alpha = \dfrac{e^{-(E(N_\alpha) - \mu N_\alpha)/kT}}{Z_{gc}}$

$$Z_{gc} = \sum_{N_\alpha} e^{-(E(N_\alpha) - \mu N_\alpha)/kT}$$
$$= \sum_{N_\alpha} e^{-(\sum_\alpha \varepsilon_\alpha N_\alpha - \mu N_\alpha)/kT}$$
$$= \prod_\alpha e^{-\varepsilon_\alpha N_\alpha/kT} \cdot \sum_{\{N_\alpha\}} e^{+\mu N_\alpha/kT}$$

grand canonical partion sum

combination of all possible states →
$$= \prod_\alpha \underbrace{\sum_{\{N_\alpha\}} e^{-(\varepsilon_\alpha - \mu) N_\alpha/kT}}_{Z_\alpha}$$

Partition sum of the α-state

- grand canonical phase-space distribution minimizes Z_{gc}

- As before: Grand canonical potential ϕ

- $\phi(T, V, \mu) = -kT \ln(Z_{gc})$
- $\phi(T, V, \mu) = F(T, V, N) - \mu N$ ϕ is the Legendre transform of F
- $d\phi = dF - N d\mu - \mu dN = -S dT - P dV + \mu dN - N d\mu - \mu dN = -S dT - P dV - N d\mu \equiv \delta_T \phi\, dT + \delta_V \phi\, dV + \delta_\mu \phi\, d\mu$

$$S \underset{(1)}{=} -\frac{\delta\phi}{\delta T} \qquad \langle P \rangle \underset{(2)}{=} -\frac{\delta\phi}{\delta V} \qquad \langle N \rangle \underset{(3)}{=} -\frac{\delta\phi}{\delta\mu} \qquad \mu \underset{(4)}{=} \frac{\delta F}{\delta N} \qquad \langle E \rangle \underset{(5)}{=} \phi + ST + \mu N$$

Ensamble properties!

μ: • describs reactions => change in free energy by a change of particles number
• the total chemical potential $\mu = \sum_i \mu_i$ is conserved in every reaction for constant entropy, volume and number of other species

- $\phi(T, V, \mu) = \overbrace{U - TS}^{F} - \mu N = TS - PV + \mu N - TS - \mu N = -PV$

$U = TS - PV + \mu N$ Euler relation

Case: Fermions and Bosons

$\langle E \rangle = \phi + sT + \sum_i \mu_i N_i$ with more species

- Partition sum of the α-state

Fermions : $N_\alpha = 0, 1$ (Pauli's exclusion principle) $\quad z_\alpha = 1 + e^{-(\varepsilon_\alpha - \mu)/kT}$ Fermi-Dirac (1)

Spin ½: e^+, e^-, μ, p, ν

Bosons : $N_\alpha = 0, 1, 2, \ldots, \infty$ $\quad z_\alpha = (1 - e^{-(\varepsilon_\alpha - \mu)/kT})^{-1}$ Bose-Einstein (2)

Spin 1 : γ

(From geometric series : $\sum_{m=0}^{\infty} e^{-mx} = \sum (e^{-x})^m = (1 - e^x)^{-1}$)

- Number distribution of particles in each state

$$\langle N_\alpha \rangle = -\frac{\delta \phi}{\delta \mu} = +kT \frac{\delta \ln(z_\alpha)}{\delta \mu} = kT \frac{1}{z_\alpha} \frac{\delta z_\alpha}{\delta \mu} \overset{(1)}{=} \frac{\cancel{kT}}{1 + e^{-(\varepsilon_\alpha - \mu)/kT}} \cdot \frac{e^{-(\varepsilon_\alpha - \mu)/kT}}{\cancel{+kT}} = \left(e^{+(\varepsilon_\alpha - \mu)/kT} + 1 \right)^{-1}$$

$$\overset{(2)}{=} \cancel{kT} \left(1 - e^{-(\varepsilon_\alpha - \mu)/\cancel{kT}}\right) (-1) \left(1 - e^{-(\varepsilon_\alpha - \mu)/kT}\right)^{\cancel{-2}} \cdot \frac{e^{-(\varepsilon_\alpha - \mu)/kT}}{\cancel{+kT}}$$

$$= \frac{1}{\left(e^{(\varepsilon_\alpha - \mu)/kT} \pm 1 \right)}$$

(+) fermions
(−) bosons

- Example: photons (bosons) in equilibrium

$\varepsilon = h\nu$, $\mu = 0$, $g = 2$ helicity $(+1, -1)$ = Polarization internal degree of freedom to be multiplied to the number of states

$$N_\nu = \frac{1}{e^{h\nu/kT} - 1}$$ Number distribution of states

$g_s = 4\pi g \nu^2 / c^3$ density of states, $h\nu$ energy

$$U(\nu) = \frac{4\pi g \nu^2/c^3 \, h\nu}{e^{h\nu/kT} - 1}$$ Equilibrium energy density ($g = 2$, "energy")

$U = E_\nu g_\nu N_\nu \quad E_\nu = h\nu$

per steradiant $(1/4\pi)$, per cm^2, per Hz

$$B_T(\nu) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$ Planck spectrum $[W\ sr^{-1}\ m^{-2}\ Hz^{-1}]$

Moreover we can also derive : $P = -\frac{\delta \phi}{\delta V}$ Pressure

$S = -\frac{1}{V} \frac{\delta \phi}{\delta T}$ Entropy density

$\Rightarrow$ you know everything !

Ideal quantum gas

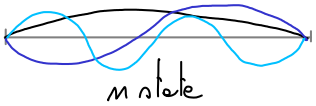
many states → continuous: $\sum \to \int$

- Particle $\quad \bar{p} = \hbar \bar{k} \quad \varepsilon(p) = \sqrt{c^2 p^2 + m^2 c^4}$

- Confined:

mode = state

$k = \frac{2\pi}{L} n = \frac{2\pi n}{V^{1/3}} \Rightarrow n = \frac{V^{1/3}}{2\pi} k \Rightarrow$ 3D box $\; dn^3 = \frac{V}{(2\pi)^3} dk^3$ ← include g: internal degrees of freedom e.g. spin



$d^3p = \hbar^3 d\bar{k}^3 \to$

$dN = \frac{gV}{(2\pi)^3} d^3k$ # of states in interval of k

$= \frac{gV}{(2\pi\hbar)^3} d\bar{p}^3$ # " " " " " p

- Continuous approximation $(\sum \to \int)$

$N = \sum_\alpha N_\alpha \quad N_\alpha = \frac{1}{e^{(\varepsilon-\mu)/kT} \pm 1}$ Fermi-Dirac / Bose-Einstein

! $N = \int N_\alpha \, dN = \int \underbrace{N(p)}_{\text{polar coord.}} d^3p = \int_{-\infty}^{\infty} \frac{1}{e^{(\varepsilon-\mu)/kT} \pm 1} \underbrace{\frac{gV}{(2\pi\hbar)^3} d^3p}_{dN} \longrightarrow$ $\left(d^3p = 4\pi p^2 dp\right)$ (homogeneous) $\quad \left(\cdot \frac{1}{V}\right)$ to get density

$$n = \frac{g}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{e^{(\varepsilon-\mu)/kT} \pm 1} \qquad \text{number density}$$

$$u = \frac{g}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi \varepsilon(p) p^2 dp}{e^{(\varepsilon-\mu)/kT} \pm 1} \qquad \text{energy density} \quad \Big) \cdot \varepsilon(p)$$

Free energy

$\phi = -kT \ln(z_{gc}) = -kT \ln(\prod_\alpha z_\alpha) = -kT \sum_\alpha \ln(z_\alpha)$

$z_\alpha = 1 + e^{-(\varepsilon_\alpha - \mu)/kT}$ Fermi-Dirac (1)

$z_\alpha = (1 - e^{-(\varepsilon_\alpha - \mu)/kT})^{-1}$ Bose-Einstein (2)

continuous limit: $\phi(T, V, \mu) = -kT \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 \ln(z_\alpha) \, dp$

$$= \underset{BE}{\overset{FD}{\mp}} kT \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 \ln\left(1 \pm e^{-(\varepsilon(p)-\mu)/kT}\right) dp$$

Explicit relations of the ensamble properties

E.g. relativistic bosons in equilibrium :

- relativistic $\varepsilon = cp$
- equilibrium $\mu = 0$
- bosons $z_\alpha = (1 - e^{(\varepsilon_p - \mu)/kT})^{-1} = (1 - e^{-cp/kT})^{-1}$

$$\phi_B(T,V,\mu) = kT \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 \ln\left(1 - e^{-(cp/kT)}\right) dp$$

$x \equiv cp/kT$

$p = \frac{xkT}{c} \quad dp = \frac{kT}{c} dx$

$p^2 dp = \left(\frac{kT}{c}\right)^3 x^2 dx$

$$= \frac{(kT)^4}{c^3} \frac{gV4\pi}{(2\pi\hbar)^3} \int_0^\infty (1 - e^{-x}) x^2 dx$$

$= -m!\, \zeta(m+2)$ with $m=2$ Riemann function

$$= -gV \frac{\pi^2}{90} \frac{(kT)^4}{(\hbar c)^3}$$ free energy

$$P_B = -\frac{\delta\phi}{\delta V} = g \frac{\pi^2}{90} \frac{(kT)^4}{(\hbar c)^3}$$ pressure

$$s_B = -\frac{1}{V} \frac{\delta\phi}{\delta T} = gk \frac{2\pi^2}{45} \left(\frac{kT}{\hbar c}\right)^3$$ entropy density

Same for the other cases

	relativistic distributions: Bose-Einstein	Fermi-Dirac	non-relativistic Maxwell-Boltzmann
number density	$n_B = g_B \frac{\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3$	$n_F = \frac{3}{4} \frac{g_F}{g_B} n_B$	$g \left(\frac{kT}{2\pi\hbar}\right)^{3/2} e^{-kT/mc^2}$
energy density	$u_B = g_B \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3}$	$u_F = \frac{7}{8} \frac{g_F}{g_B} u_B$	$\frac{3}{2} nkT$
pressure	$P_B = \frac{u_B}{3} = g_B \frac{\pi^2}{90} \frac{(kT)^4}{(\hbar c)^3}$	$P_F = \frac{u_F}{3} = \frac{7}{8} \frac{g_F}{g_B} P_B$	nkT
entropy	$s_B = g_B k \frac{2\pi^2}{45} \left(\frac{kT}{\hbar c}\right)^3$	$s_F = \frac{7}{8} \frac{g_F}{g_B} s_B$	

Immediate results

Baryons-Photons ratio

- Number density of baryons today, assume they are all in H => m_p protons

$$n_B = \frac{\rho_B}{m_p} = \frac{\Omega_B}{m_p}\rho_B^{-1} \simeq \frac{\Omega_B}{m_p}\frac{3H^2}{8\pi G} \simeq 1{,}1\cdot 10^{-5}\,\Omega_m h^2\,\mathrm{cm}^{-3} \qquad \Omega_B = 0{,}04 \text{ measured}$$

(ρ_B^{-1}: Friedmann)

- Number of photons (bosons)

$$n_\gamma = g_r \frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 = 407\,\mathrm{cm}^{-3} \qquad T_{CMB} = 2{,}752 \text{ measured}$$

(g_r: $\zeta(3)$)

- Baryons - Photons fraction

$$n_F = \frac{3}{4}\frac{g_F}{g_B} n_B \quad \Rightarrow \quad \eta \equiv \frac{n_B}{n_\gamma} = \text{const.} \,!$$

$$\eta = 2{,}7\cdot 10^{-8}\,\Omega_B h^2$$

for each baryon you have $> 10^8$ photons

("temperature of the universe" = $T_{photons}$)

=> entropy of universe is dominated by photons

=> irreversible processes are negligible => adiabatic expansion!

η is important because it tunes nucleosynthesis and recombination!

- η is finite, why? i.e. why are there baryons at all?!

during their creation they should have annihilated with anti-baryons: $B^+ + B^- \to \gamma + \gamma$

i.e. $\eta = 0$

=> must have been asymmetry, possible under Sakharov conditions, eg. CP violation ....

Recombination processes: Saha's equation

- At early time photons are energetic enough (T is large) to keep everything photo-dissociated we have a soup of γ, n, p, e^-
- As $a \uparrow$, temperature $T \downarrow$ recombination processes can take place

  e.g. $D + D \to {}^3He + n$ "cosmological nucleosynthesis" = formation of D, He, Be, Li

  e.g. $e^- + p^+ \longleftrightarrow H + \gamma$ "recombination"

- Assumptions: ideal gases composed by indistinguishable particles (low enough densities such to ignore degeneracies)

We now consider recombination which is simpler

- From Helmoltz free energy we derive all ensamble properties

$$F(T,V,N) = -kT\ln(Z_c) \quad \text{with} \quad Z_c = \prod_\alpha \sum_{\{N_\alpha\}} e^{-(\varepsilon_\alpha-\mu)N_\alpha/kT} \qquad i = e,p,H$$

here: $Z_c = \dfrac{Z_e^{N_e} Z_p^{N_p} Z_H^{N_H}}{N_e!\,N_p!\,N_H!}$ ← • because, all possible combination of states

$\Phi = -kT\ln\left(\dfrac{Z_e^{N_e} Z_p^{N_p} Z_H^{N_H}}{N_e!\,N_p!\,N_H!}\right)$ • $N_B = N_p + N_H$ $\quad N_e = N_p$ (neutral universe) $\quad N_H = N_B - N_e$

$= -kT\left[\ln\left(Z_e^{N_e} Z_p^{N_p} Z_H^{N_H}\right) - \ln(N_e!\,N_p!\,N_H!)\right]$ ⇒ N_e determines everything (for fixed N_B)

• γ serves as the heat reservoirs controlling T

$N_H(\ln N_H - 1) = (N_B - N_e)(\ln N_H - 1)$

$= -kT\left[N_e \ln Z_e + N_p \ln Z_p + N_H \ln Z_H - \ln(N_e!) - \ln(N_p!) - \ln(N_H!)\right]$

$N_p = N_e$; $N_H = N_B - N_e$; $-2\ln(N_e!) \simeq -2N_e(\ln N_e - 1)$

- Equilibrium at the minimum of F ⇒ $\dfrac{\delta F}{\delta N_e} = 0$ stirling formula $\lim_{x\to\infty} \ln(x) = x(\ln(x)-1)$

$$\frac{\delta F}{\delta N_e} = -kT\left[\ln Z_e + \ln Z_p - \ln Z_H - 2\left(\ln N_e - \cancel{1} + \frac{\cancel{N_e}}{\cancel{N_e}}\right) + \ln N_H\right] \overset{!}{=} 0$$

$$\ln\left(\frac{Z_e Z_p}{Z_H}\right) \overset{N_e = N_p}{=} \ln\frac{N_e^2}{N_H} = \ln\frac{N_e^2}{N_B - N_e} = \ln\frac{x^2 N_B}{1-x} \quad \Rightarrow \quad \frac{x^2}{1-x} = \frac{Z_e Z_p}{Z_H N_B} \qquad x \equiv \frac{n_e}{n_B}$$ degree of ionization

- $n_B = \eta n_\gamma$ η ! ratio of baryons and photons and # of photons
- ionization potential of H: $\chi = (m_e + m_p - m_H)c^2 = 13{,}6\,eV$ =
- non relativistic particles: $\varepsilon = m_i c^2 + \dfrac{p^2}{2m_i}$ $\quad i = e,p,H$ *
- plug Z_e, Z_p, Z_H use continuous distribution (M-B) for the i-th specie

$$Z_i = \frac{4\pi g_i V}{(2\pi\hbar)^3}\int_0^\infty p^2 e^{-(\varepsilon - \mu_i)/kT}\,dp = \frac{g_i V (2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} e^{-(m_i c^2 - \mu_i)/kT}$$

(* : $\varepsilon - \mu_i = 0$) $\qquad \mu_e + \mu_p = 0 = \mu_H$ at equilibrium

$$\frac{Z_e Z_p}{Z_H} = \frac{g_e g_p}{g_H} V \frac{(2\pi \frac{m_e \cancel{m_p}}{\cancel{m_H}} kT)^{3/2}}{(2\pi\hbar)^3} e^{-\frac{(m_e + m_p - m_H)c^2}{kT}} \qquad \frac{g_H}{g_e g_p} = \frac{4}{2\cdot 2} \qquad g_H = 1 + 3$$

(singlet, triplets)

$$\Rightarrow \quad \frac{x^2}{1-x} = \frac{(2\pi m_e kT)^{3/2}}{(2\pi\hbar)^3 n_B} e^{-\chi/kT} \qquad \frac{N_B}{V} \quad n_B = \eta n_\gamma = 2\eta \frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}\,\zeta(3)\,\eta}\left(\frac{m_e c^2}{kT}\right)^{3/2} e^{-\chi/kT}$$ Saha's equation $\quad \eta$ tunes the process!

End of recombination

- When medium is no more ionized: most of e, p are bound in H

  degree of ionization $X = \frac{n_e}{n_B} \ll 1 \Rightarrow \frac{X^2}{1-X} \simeq X^2 \propto e^{-\chi/kT} \ll 1 \Rightarrow kT \ll \chi$

  i.e. photons are "cold" enough: for $x=0,1$ $T_{rec} \approx 3500\,K$ $kT_{rec} = 0,3\,eV$

  scattering, $e^- + \gamma \to e^- + \gamma$, drops $\Rightarrow$ matter and radiation decouple

- This is later than the naive (wrong) consideration $T = \frac{\chi}{k_B} = 10^5\,K$ with $\chi = 13.6\,eV$

  - The "delay" is due to the very large number of photons: $\frac{N_B}{N_\gamma} \equiv \eta = 2,7 \cdot 10^{-8} \Omega_B h^2$ is very small
  - many photons have ν high enough to partially ionize back the medium
    (remember - photons energy distribution is not a delta function)
  - This is a "retarded reaction"

- In reality the process is even slower than what predicted by Saha's eq.

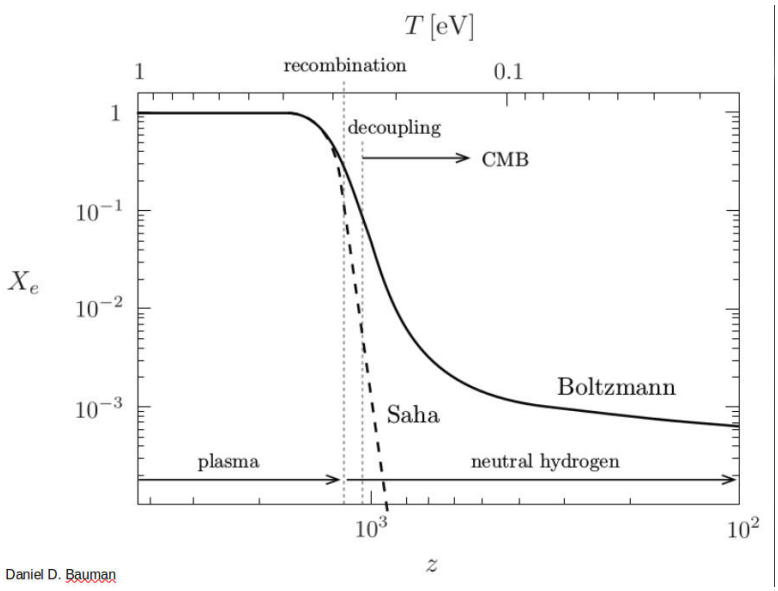
  - the reaction is $e^- + p^+ \longleftrightarrow H + \gamma$ (double arrow) captured e^- transition to ground state

    Lyman-α: $2P \to 1S \Rightarrow h\nu \geq E_{Ly\alpha} = \frac{3\chi}{4} = 10,2\,eV$

  - Reaction proceeds only if lower ν photons are emitted:
    forbidden transition $2S \to 1S$ (2 photons are emitted)
    low probability of transition $\Rightarrow$ slow $\Rightarrow$ source of a further delay

- Time of recombination:
  - photons are the termal reservoir $\Rightarrow$ they set T $T \propto a^{-1}$ (relativistic specie)
  - radiation dominated era $H^2 = -\frac{8\pi G}{3}\rho_\gamma \Rightarrow a(t) \Rightarrow T(t)$ $t_{rec} \simeq 380\,000\,yr$
    $z \simeq 1100$
    last scattering surface

- This process is not istantaneous, it lasts for some time
  $\delta T \approx 200\,K$



- At recombination, the free e^- are gone
  - => no more Thomson scattering
  - → photons are no more coupled to matter
  - => photons background: Cosmic Microwave Background radiation (CMB)

this is the wavelength they have today

Cosmological nucleosynthesis

- Binding energy of nuclei $\sim MeV$ => for $K_B T > 1 MeV$ p and n can not form bond states immediate photon-dissociation
  $e^{-\Delta mc^2/T_D}$
  Afterwards they can combine

- At $T_D \sim 0{,}8 MeV$ p and n not more equilibrium thanks to reactions such as $n + \nu_e \rightleftarrows p + e^-$
  => freez-out: their number density freezes

  $m_n > m_p$ by a little => for equilibrium Boltzmann distribution $n_p > n_n$

  $\boxed{\frac{n_n}{n_p} = e^{-(m_n - m_p)c^2/kT} \approx 1/6}$ $\Delta mc^2 = 1{,}4 MeV$

- When did this happened?

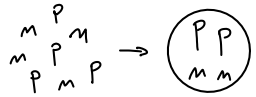
  relativistic particles: $T \propto a^{-1}$, today at a_0 $T_0 = 2{,}7K \rightarrow KT_0 = 2{,}4 \cdot 10^{-4} eV$ $\frac{a_D}{a_0} = \frac{T_0}{T_D} = 3{,}4 \cdot 10^{-10}$ $(=1)$

  radiation domination: $\left(\frac{\dot a}{a}\right)^2 \equiv H^2 = H_0^2 \Omega_R a^{-4}$ $\boxed{t_D = \frac{1}{H_0 \sqrt{\Omega_R}} \int_{z_D}^{\infty} \frac{dz}{(1+z)^3}} \approx \boxed{10 s}$

- Spontaneous decay of n, decay time: lifetime $\sim 900 s$ => little drop in n_n: $\boxed{\frac{n_n}{n_p} = 1/7}$ at $KT \sim 0{,}1 MeV$

- Helium abundance, rough estimate:

  all n_n bond in $\frac{n_n}{2}$ ^4He nuclei

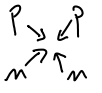
  each ^4He nuclei has 4 nucleons, # of ^4He nuclei

  $\Rightarrow$ $\boxed{Y = \frac{4\left(\frac{n_n}{2}\right)}{(n_n + n_p)}} \approx \boxed{0{,}25}$

  tot # of n and p

- Helium abundance, how to do it:

  - Need to form ^4He nuclei 

     4-body interaction not likely to occour

  - The number of neutron is crucial

  - Only 2 body interactions are likely to occur => need to go through intermediate steps
    Need Deuterium, $D = {}^2H$, to build up ^4He

$$p + n \rightarrow D + \gamma \quad \begin{cases} D + p \rightarrow {}^3He + \gamma \\ D + D \rightarrow {}^3He + n \\ D + D \rightarrow T + p \end{cases} : \quad \begin{cases} {}^3He + n \rightarrow T + p \\ {}^3He + D \rightarrow {}^4He + p \end{cases} \quad \begin{matrix} {}^3He + {}^4He \rightarrow Be + \gamma \\ {}^4He + T \rightarrow {}^7Li + \gamma \end{matrix} \quad : \quad T + D \rightarrow {}^4He + n$$

Saha's equation

- The entire universe is like the interior of a star: p, n, ν, γ, $T \sim 10^7 K$, nuclear fusion
- n_n is crucial
- D is crucial (Gamow criteria):
  Too much D => to few free n to proceed with the reaction
  Too few D => the chain can not proceed because of its lack
- Only light elements form, not enough time to proceed

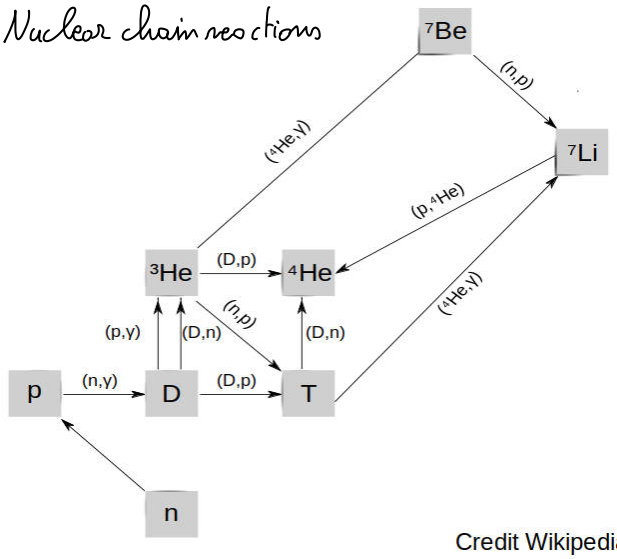
- Expansion tunes the process (i.e. the faster, the less time to process the material):
  time for reactions is short (stars keep the plasma contained because of gravity => heavier elements)
  => There is time to create only light elements (already very little amount of Li, Be)

- Radiation era => γ rules! photo dissociation: $\eta = \frac{N_B}{N_\gamma}$ leading parameter (recall Saha's eq.)

- Also this is a delayed reaction (Saha's eq.) => Nucleosynthesis occurs well after $KT \sim 1\,MeV$
  i.e at $KT \sim 0{,}1\,MeV$ => $z_{nucl} \sim 10^8$, $t_{nucl} = 100\,s$

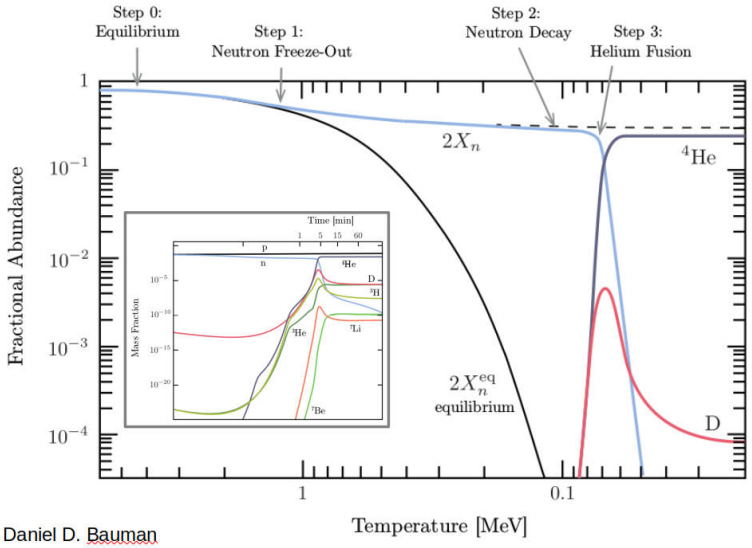
- These reactions are all interconnected:

Nuclear chain reactions



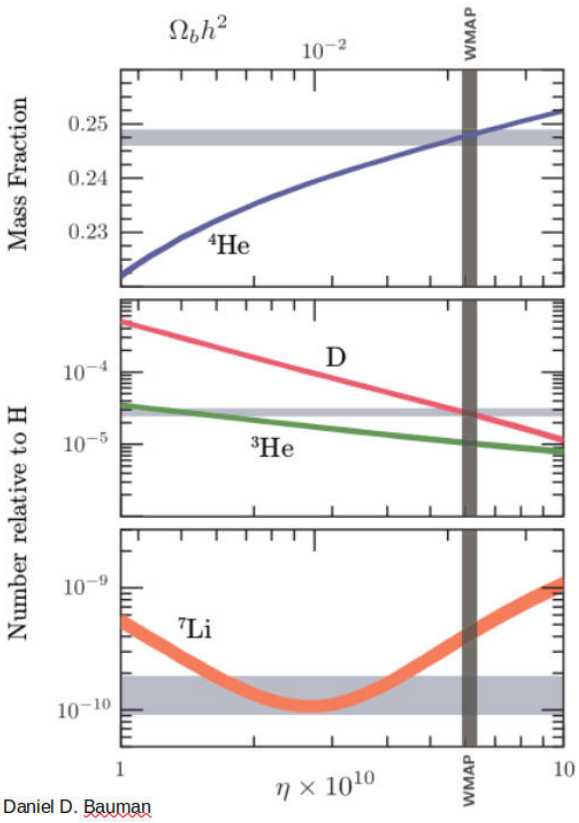
Credit Wikipedia

Evolution of abundances

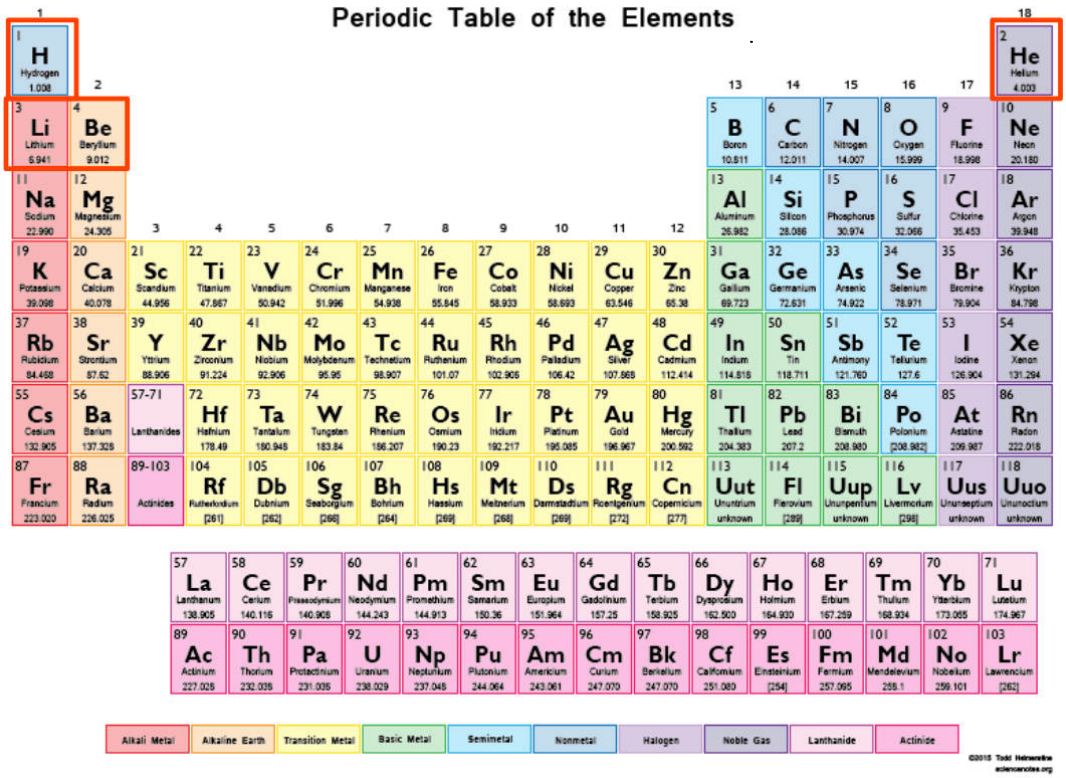


Daniel D. Bauman

Amount of the products



All other elements come from the stars and humans



Neutrino decoupling

- At $T \gg 1\,MeV$ $T_\nu = T_\gamma$ because equilibrium is achieved through weak interaction
- No interaction between γ and ν

  => They should maintain the same T forever ... No! Photon fluid heated up by e^+, e^- annihilation

---

weak interaction

$\nu + \bar{\nu} \longleftrightarrow e^+ + e^-$

- at $T_\nu \approx 10^{10,5}\,K \approx 2,7\,MeV$ => neutrino ν thermal freez-out $\frac{\Gamma}{H} \sim \left(\frac{T}{2,7\,MeV}\right)^3$
- ν are ultrarelativistic => $N_\nu = const$

=> Neutrino Background ($C\nu B$)

$e^+ + e^- \longleftrightarrow \gamma + \gamma$

- bi-directional until $\langle h\nu \rangle < (2 m_e c^2) \approx 1\,MeV$

  the process become irreversible

  γ can no more create e^- or e^+ because their energy is too low (redshift)

$e^+ + e^- \longrightarrow \gamma + \gamma$

- this is a different kind of freez out
- all energy (entropy) goes into photons with $\langle h\nu \rangle \sim m_e c_2 > "\langle h\nu_{old} \rangle"$

=> Reheating of radiation

$S_B = g_{BK} \frac{2\pi^2}{45} \left(\frac{KT}{\hbar c}\right)^3 \quad S_F = \frac{7}{8} \frac{g_F}{g_B} S_B$

full annihilation

$S_{e^+} + S_{e^-} + S_\gamma = S'_\gamma$

before after

Relativistic distri.: $S_F = \frac{7}{8} \frac{g_F}{g_B} S_B$ (e^-, e^+) (γ) with $g_{e^+} = g_{e^-} = g_\gamma = 2$ => internal degrees of freedom $\frac{S_F}{S_B} = \frac{7}{8}$

$\left(\frac{7}{8} + \frac{7}{8} + 1\right) S_\gamma = S'_\gamma$ (11/4)

$S_B = g_{BK} \frac{2\pi^2}{45} \left(\frac{KT}{\hbar c}\right)^3$ => $(11/4)^{1/3}$ $1,5\,T_\gamma \simeq T'_\gamma$

Today: $T_\gamma = 2,752\,K$ (measured)

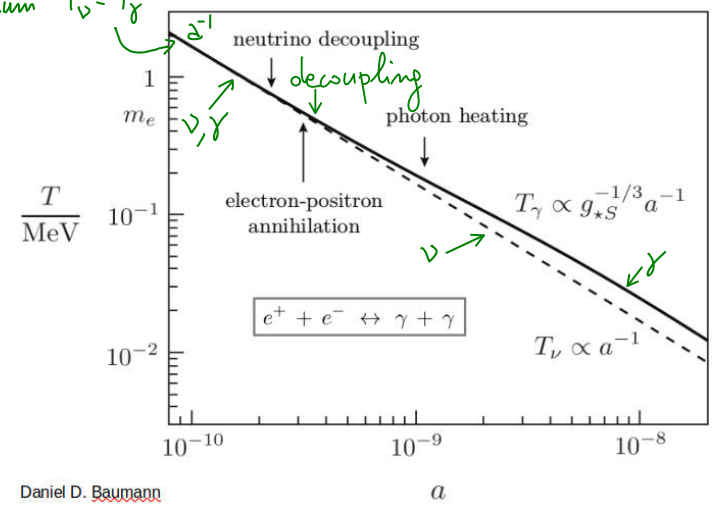
=> $T_\nu = 1,95\,K$ (expected)

photons are hotter

"Reheating"

Thermal equilibrium $T_\nu = T_\gamma$

relativistic



Could neutrinos be the famous dark matter?

- Density of neutrinos $\rho_\gamma, \rho_\nu \sim T^4 \Rightarrow \rho_{\nu_{tot}} = \rho_{\nu+\bar{\nu}} = \frac{7}{8} N_\nu g_\nu \left(\frac{4}{11}\right)^{4/3}$
- $N_\nu g_\nu \approx 3 \pm 0{,}3$ (CMB measures)
  3 families in standard model of particles and $g_\nu = 1 \Rightarrow \rho_{\nu_{tot}} \approx 0{,}68\, \rho_\gamma$
- There are $2N_\nu$ species of neutrinos, ν, and anti-neutrinos, $\bar{\nu}$, and $n_\gamma = \alpha T^3$ α = const.

$$\Rightarrow n_{\nu^*} = \frac{3}{4}(2N_{\nu^*}) n_\gamma(T_\nu) = \frac{3}{2} N_{\nu^*} \alpha T_\nu^3 = \frac{3}{2} N_{\nu^*} \alpha T_\gamma \left(\frac{T_\nu}{T_\gamma}\right)^{1/3} = \frac{6}{11} N_\nu n_\gamma(T_\gamma) \qquad n_{\nu^*} = 340 \frac{1}{cm^3}$$

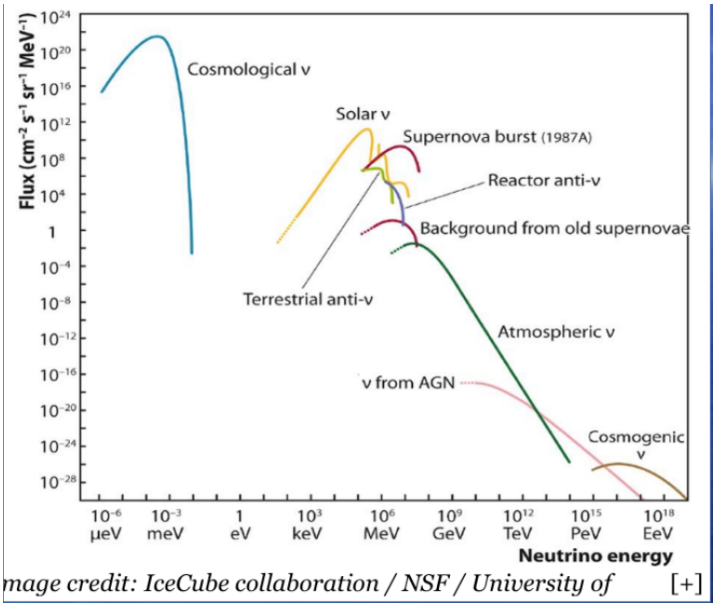
$n_{\nu^*} = n_{\nu_{tot}} + n_{\bar{\nu}}$

- Neutrino mass $m_\nu < 1\,eV$ (measured from solar neutrino oscillations + cosmic structure formations)

$$\Rightarrow \text{today energy density} \quad \rho_{\nu^*,0} = \frac{6}{11} n_\gamma \Sigma m_\nu$$

$$\Omega_{\nu^*,0} = \frac{8\pi G}{3H_0^2} \frac{6}{11} n_\gamma \Sigma m_\nu \overset{!}{=} 0{,}3 \quad \text{with } H_0 \approx 70 \Rightarrow m_\nu = 5\,eV$$

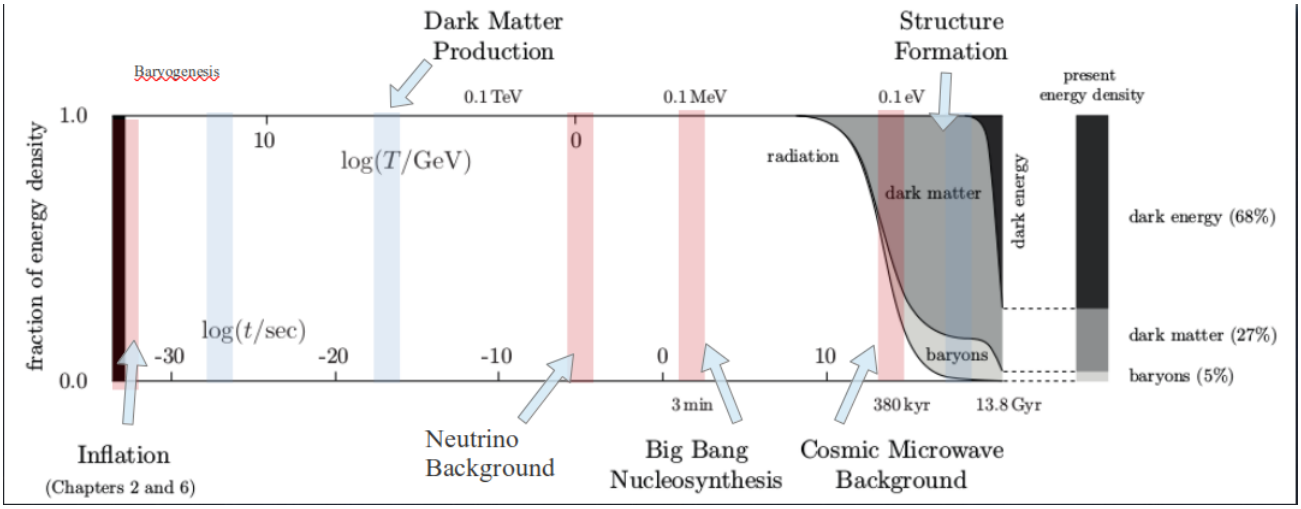
N_ν is too large $\Rightarrow$ Neutrinos alone can not compose all dark matter!



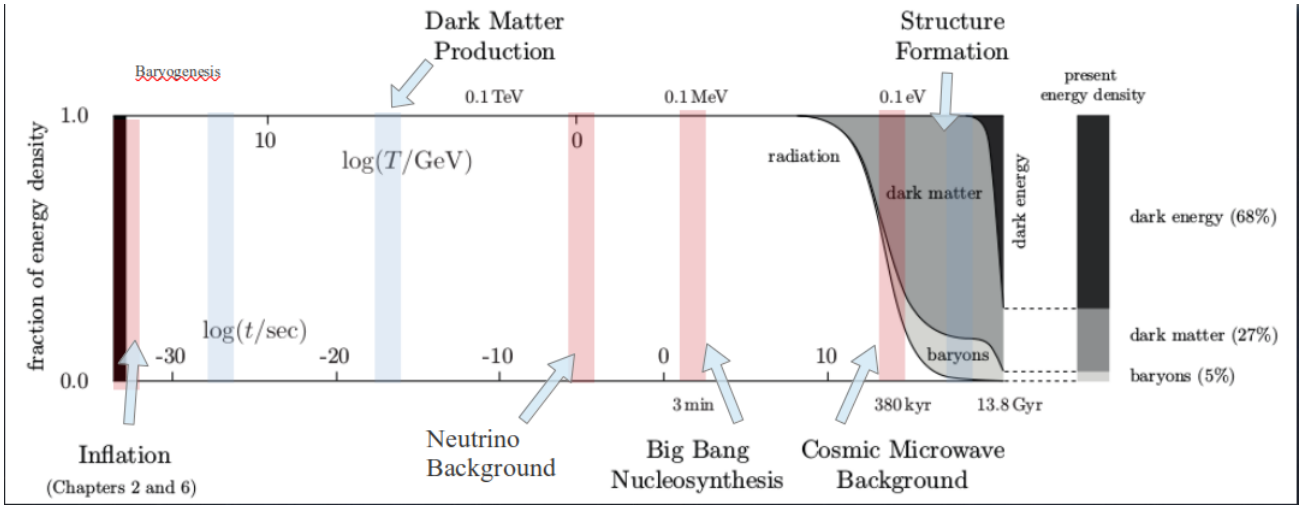
mage credit: IceCube collaboration / NSF / University of [+]

Last remark

- Be carefull with the chronology...
  I order them backward in time to better group physical concepts
- The chronological order is:
  1) - ν decoupling / photons reheating
  2) Big-Bang nucleosynthesis
  3) recombination and photons-matter decoupling



Thermal hystory of the universe



Event	time t	redshift z	temperature T	
Inflation	10^{-34} s (?)	–	–	> Inflaton field
Baryogenesis	?	?	?	> Particle anti-particle annihilation, small asymmetry
EW phase transition	20 ps	10^{15}	100 GeV	> Particles get mass (Higgs mechanism)
QCD phase transition	20 μs	10^{12}	150 MeV	> Quarks and gluons combine → Baryons , Quarks and anti-quarks combine → Mesons
Dark matter freeze-out	?	?	?	> Quite early to explain its weak interaction (t_coll >>t_exp)
Neutrino decoupling	1 s	6×10^9	1 MeV	> Weak inteaction $e^+ + e^- \longleftrightarrow$ nu + nu-
Electron-positron annihilation	6 s	2×10^9	500 keV	> $e^+ + e^- \leftrightarrow \gamma + \gamma$
Big Bang nucleosynthesis	3 min	4×10^8	100 keV	> p, n → D, H, He, Li, (Be)
Matter-radiation equality	60 kyr	3400	0.75 eV	
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV	> $e^- + p^+ \longleftrightarrow$ H + gamma
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV	> CMB
Reionization	100–400 Myr	11–30	2.6–7.0 meV	> H + gamma $\longleftrightarrow e^- + p^+$
Dark energy-matter equality	9 Gyr	0.4	0.33 meV	> Acceleration "start"
Present	13.8 Gyr	0	0.24 meV	> Life



These phases occours when certain processes are no more reversible due to the temperature drop

- e.g.. neutrinos freez-out : $e^+ + e^- \leftrightarrow \gamma + \gamma$ (reversible process) Adiabatic process
  $e^+ + e^- \rightarrow \gamma + \gamma$ become irreversible when $h\nu < m_e c^2$

Key concepts

- $P_1 + P_2 \overset{\text{interaction}}{\longleftrightarrow} P_3 + P_4$ Reaction
  - e.g. $e^+ + e^- \to \gamma + \gamma$ annihilation
    $e^- + \mathcal{Z}^+ \to e^- + \mathcal{Z}^+$ scattering process
  - 3 or 4 body interactions are rare

- Reaction is governed by Γ = rate of collisions $t_{coll} \equiv 1/\Gamma$
  H = " " expansion $t_{exp} \equiv 1/H$

- if $\begin{cases} \Gamma \gg H \Rightarrow \text{particles interact} \Rightarrow \text{Thermal equilibrium} \\ \Gamma \ll H \Rightarrow \text{no interactions} \Rightarrow \text{Particles decouple / Freeze-out} \end{cases}$

- The decoupling time depends on the interaction "$\longleftrightarrow$"

- $\Gamma = n \langle \sigma v \rangle$ $\left[\frac{1}{m^3} m^2 \frac{m}{s}\right] = [s^{-1}]$
  n = number density
  σ = interaction cross section
  v = relative velocity $\Rightarrow v(T)$
  $dN = n \langle \sigma v \rangle dt$ $\Gamma \equiv \frac{dN}{dt}$

$\Rightarrow$ We need to study n, T in a cosmological context

A "geometric" approach: $P = h/\lambda$ de-Broglie
$E = k_B T$ Thermal energy $\quad \lambda \propto a$

Non relativistic: $E = k_B T = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} \Rightarrow \boxed{T \propto a^{-2}}$

Relativistic: $E = k_B T = cp = \frac{h}{\lambda} \Rightarrow \boxed{T \propto a^{-1}}$

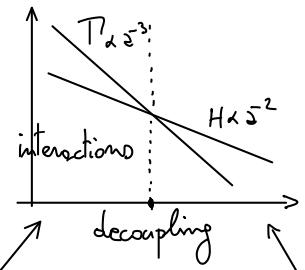
Condition for equilibrium at early times

- Early times: radiation domination, $K=0$, Λ or DE negligible $\Rightarrow H = \sqrt{\frac{8\pi G}{3}\rho}$ Friedmann eq.

$$\begin{cases} H = \sqrt{\frac{8\pi G}{3}\rho} \approx (G\rho)^{1/2} \propto a^{-2} & \uparrow a^{-4} \text{ radiation} \\ \Gamma = (n \langle \sigma v \rangle) \propto a^{-3} & \uparrow a^{-3} \text{ Bosons} \end{cases}$$

$n_B \propto T^3$ $T \propto a^{-1} \Rightarrow n_B \propto a^{-3}$

$\frac{\Gamma}{H} \propto \frac{a^{-3}}{a^{-2}} \propto a^{-1} \xrightarrow{\text{for } a \to 0} \infty$



$a(t) \uparrow \quad T(t) \downarrow$

Equilibrium is guaranteed at early times! freeze-out

Conditions for freez-out

- With no collisions : continuity eq. ⇔ number conservation n_s = const.

$$\dot{n}_s + \vec{\nabla}(n_s \vec{v}) = 0 \qquad \vec{v} = H\vec{r} \qquad \vec{\nabla}(n_s H \vec{r}) = n_s H \vec{\nabla}\vec{r} = 3Hn_s$$
$$\dot{n}_s + 3Hn_s = 0 \quad \text{(expansion)} \qquad \frac{\delta x}{\delta x}+\frac{\delta y}{\delta y}+\frac{\delta z}{\delta z} = 3$$

- With collisions : source term

$$\dot{n}_s + 3Hn_s = j - \alpha n_s$$

α: destruction rate of particles
j: creation " " "

$P_1 + P_2 \underset{j}{\overset{\alpha}{\rightleftharpoons}} P_3 + P_4$ (destruction α, creation j)

$j - \alpha n_{sT} = 0 \qquad n_{sT} = \frac{j}{\alpha}$ at equilibrium

$$= \alpha n_s \left(\frac{j}{\alpha n_s} - 1\right) = \alpha n_s \left(\frac{n_{sT}}{n_s} - 1\right)$$

$j = \langle\sigma v\rangle n_T^2$
$\alpha = -\Gamma$

- Go comoving for convenience

$$a^3(\dot{n}_s + 3Hn_s) = \alpha n_s a^3\left(\frac{n_{sT}a^3}{n_s a^3} - 1\right)$$

comooving: $N_s \equiv n_s a^3$

$$\dot{N}_s = \dot{n}_s a^3 + n_s 3a^2\dot{a} = a^3(\dot{n}_s + 3Hn_s)$$

$$\dot{N}_s = \alpha N_s\left(\frac{N_{sT}}{N_s} - 1\right)$$

$$\frac{d}{dt} = \frac{da}{dt}\frac{d}{da} = aH\frac{d}{da} = H\frac{d}{d\ln a} \qquad \dot{N}_s = H\frac{dN_s}{d\ln a}$$

$$\boxed{\frac{d\ln N_s}{d\ln a} = \frac{\alpha}{H}\left(\frac{N_{sT}}{N_s} - 1\right)}$$

Freez-out equation / Boltzmann eq. it tells you how the reaction proceeds

- Consequences :

1) if equilibrium : $\frac{d\ln N_s}{d\ln a} = 0 \Rightarrow N_s = N_{sT}$ N_s does not change

2) if off equilibrium $\Rightarrow$ N_s changes by moving toward equilibrium
if $N_s > N_{st} \Rightarrow \dot{N}_s < 0$ destruction →
if $N_s < N_{st} \Rightarrow \dot{N}_s > 0$ creation ←

3) if $\alpha \ll H$ ($\alpha = -\Gamma$) $\Rightarrow \dot{N}_s \to 0$ Freez-out (their number is conserved) reaction stops ⇏
particles do not participate any more to the reaction, they leave a background (relics)
e.g. CMB photons, neutrino background

- $\Gamma \ll H$ equivalent to : mean free path of interaction $\lambda \ll cH^{-1}$ cosmic horizon

- Relativistic particles (Fermions and Bosons) $\begin{cases} n_s \propto T^3 \\ T \propto a^{-1} \end{cases} \Rightarrow n_s \propto a^{-3} \Rightarrow N_s \equiv a^3 n = \text{const.}$

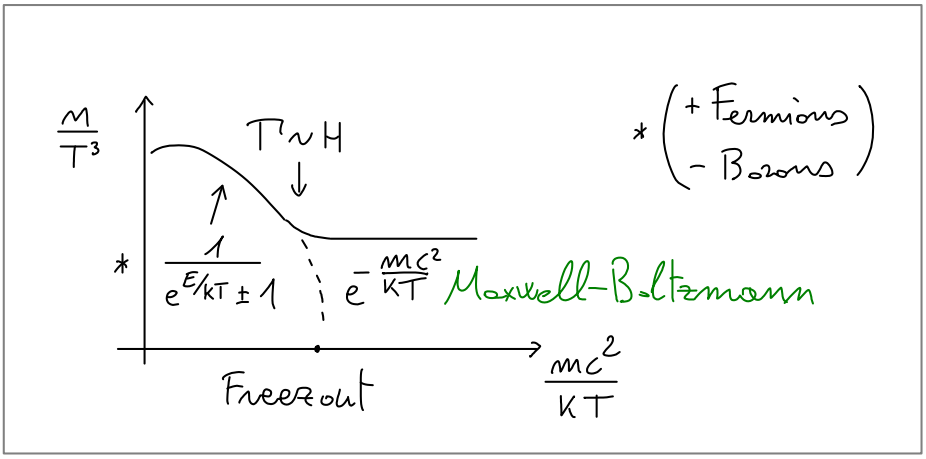
$\frac{d\ln N_s}{d\ln a} = \frac{\alpha}{H}\left(\frac{N_{sT}}{N_s} - 1\right) = 0 \Rightarrow$ i.e. $N_s = N_{sT}$ remains in T. equilibrium regardless $\frac{\alpha}{H}$!

- Non relativistic particles $\begin{cases} n_s \propto T^{3/2} e^{-mc^2/kT} \\ T \propto a^{-2} \end{cases} \Rightarrow N_s = \cancel{a^3}\,\cancel{a^{-3}}\, e^{-\frac{mc^2}{kT}}$

$\Rightarrow$ for $kT \ll mc^2$ "cold", exponential drop $\Rightarrow N_s \ll N_{sT}$: $\frac{d\ln N_s}{d\ln a} = -\frac{\alpha}{H} \to 0$

- to find out $n_s(T)$ we need to study quantum gases .... Next topic!

(Another kind of freez-out
- e.g. neutrinos freez-out : $e^+ + e^- \leftrightarrow \gamma + \gamma$ (reversible process) Adiabatic process
$e^+ + e^- \to \gamma + \gamma$ become irreversible when $h\nu < m_e c^2$
see later....)

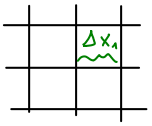


Quantum statistics

Basic concepts

- Particle : · $(\bar{x}, \bar{p})$ point in a 6-dimensional space = phase-space
  · it occupies a quantum energy state with energy ε_α α: label of the state
  · phase-space is quantized (also for free particles because of indeterminination principle)

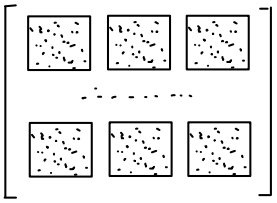
  $\Delta x \Delta p > h$ => phase-space "cells"



- Statistical approach :
  · we do not care about each single particle, we care about ensamble properties
  · Phase space distribution $f(\bar{x}, \bar{p}, t)$: probability of having a particle in a certain $(\bar{x}, \bar{p})$ cell
    it can evolve as a function of time (dynamical system)
  · We are interested probability of having a certain number of particles in a given state

- System : collection of N particles 

- Ensamble : very large collection of identical systems
  (different realizations with the same macro properties)
  We consider 3 types of ensambles : Microcanonical
  Canonical
  Grand canonical



- Surrounding : what the system is in contact with, the thermal bath

- We consider 2-body interactions : $P_1 + P_2 \longrightarrow P_3 + P_4$ e.g. $e^+ + e^- \longleftrightarrow \gamma + \gamma$

Our universe: Assumptions

adiathermal $\Rightarrow$ adiabatic $\Leftarrow$ reversible $\Leftarrow$ no S generation
$\Uparrow$ $\Uparrow$

- Adiabatic expansion : no heat flux because of isotropy + entropy dominated by photons (CMB)
- Thermal equilibrium : despite expansion, the system has the time to thermalize $\frac{\alpha}{H} \ll 1$
- Ideal fluids : no long scale interactions, direct collisions

Micro-canonical ensamble

Thermodynamical isolated/closed system: no exchange of particles and energy (in form of heat/work)

i.e. fixed: { internal energy, particle number N, volume V }

Canonical ensamble

- Closed system with no particle exchange, exchange of energy as heat/work
- only mean internal energy is specified
- ensamble parameters are N, T, V
- said to be (1) isothermal (2) isobaric if V = const. and T constant / P constant
- probability distribution of an α-state is given by:

• Boltzmann distribution — distribution of the states

$$f_\alpha = \frac{e^{-\varepsilon_\alpha/kT}}{Z_c} \qquad Z_c = \sum_\alpha e^{-\varepsilon_\alpha/kT}$$

Z_c: canonical partition sum (normalization)

• Helmotz free energy F: $F(T,V,N)$

$$\frac{1}{Z_c} \equiv e^{F/kT} \qquad e^{-F/kT} = Z_c \qquad \boxed{F = -kT\ln(Z_c)}$$

$$\Rightarrow f_\alpha = e^{F/kT} \cdot e^{-\varepsilon_\alpha/kT} = e^{(F-\varepsilon_\alpha)/kT}$$

• $F(T,V,N)$ = const. for the system

• $F = U - TS$ F is Legendre transform of internal energy U — work that a thermodynamical system can do

• $dF(T,V,N) \overset{\circledast}{=} -S\,dT - P\,dV + \mu\,dN \equiv \delta_T F\,dT + \delta_V F\,dV + \delta_N F\,dN$ total differential

$\Rightarrow$ ensamble properties! $\left(S = -\frac{\delta F}{\delta T}\right)$ $\left(\langle P\rangle = -\frac{\delta F}{\delta V}\right)$ $\left(\mu = \frac{\delta F}{\delta N}\right)$

μ = chemical potential change in the free energy due to a change in N

• At equilibrium: F is minimized given a T, V, N

T = const. V = const. $\Rightarrow \mu = 0$, in fact: $dF = -P\,dV^{=0} - S\,dT^{=0} + \mu\,dN = \mu\,dN \overset{!}{=} 0$

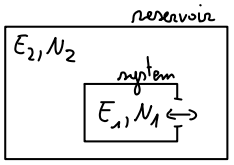
$\circledast$ $dF = dU - T\,dS - S\,dT = \cancel{T\,dS} - P\,dV + \mu\,dN - \cancel{T\,dS} - S\,dT$

$dU = T\,dS - P\,dV + \mu\,dN$

Grand canoncal ensamble

generalization of canonical ensamble N is not fixed, no restriction on # of particles

- Open system, exchange particles and energy with its surroundings
- accessible number of quantum states N_α is unknown
- only mean number is specified
- Ensamble variables are: μ, T, V μ = chemical potential of a specie, describs reactions



$E_2 \gg E_1$
$N_2 \gg N_1$
$N_1 + N_2 = N = \text{const.}$
$E_1 + E_2 = E = \text{const.}$

- $N = \sum_\alpha N_\alpha$ $\quad E(N_\alpha) = \sum_\alpha \varepsilon_\alpha N_\alpha$ Tot. number of states and energy of system

- $f_\alpha = \dfrac{e^{-(E(N_\alpha) - \mu N_\alpha)/kT}}{Z_{gc}}$

$$Z_{gc} = \sum_{N_\alpha} e^{-(E(N_\alpha) - \mu N_\alpha)/kT} \quad \text{grand canonical partion sum}$$
$$= \sum_{N_\alpha} e^{-(\sum_\alpha \varepsilon_\alpha N_\alpha - \mu N_\alpha)/kT}$$
$$= \prod_\alpha e^{-\varepsilon_\alpha N_\alpha/kT} \cdot \sum_{\{N_\alpha\}} e^{+\mu N_\alpha/kT}$$

combination of all possible states →
$$= \prod_\alpha \underbrace{\sum_{\{N_\alpha\}} e^{-(\varepsilon_\alpha - \mu) N_\alpha/kT}}_{Z_\alpha} \quad \text{Partition sum of the } \alpha\text{-state}$$

- grand canonical phase-space distribution minimizes Z_{gc}

- As before: Grand canonical potential Φ

- $\Phi(T, V, \mu) = -kT \ln(Z_{gc})$
- $\Phi(T, V, \mu) = F(T, V, N) - \mu N$ Φ is the Legendre transform of F
- $d\Phi = dF - N d\mu - \mu dN = -S dT - P dV + \mu dN - N d\mu - \mu dN = -S dT - P dV - N d\mu \equiv \delta_T \Phi dT + \delta_V \Phi dV + \delta_\mu \Phi d\mu$

$$S \underset{(1)}{=} -\frac{\delta\Phi}{\delta T} \quad \langle P \rangle \underset{(2)}{=} -\frac{\delta\Phi}{\delta V} \quad \langle N \rangle \underset{(3)}{=} -\frac{\delta\Phi}{\delta\mu} \quad \mu \underset{(4)}{=} \frac{\delta F}{\delta N} \quad \langle E \rangle = \Phi + ST + \mu N \underset{(5)}{} \quad \text{Ensamble properties!}$$

μ: • describs reactions => change in free energy by a change of particles number
• the total chemical potential $\mu = \sum_i \mu_i$ is conserved in every reaction for constant entropy volume and number of other species
• $\mu_1 + \mu_2 = \mu_3 + \mu_4$ Chemical equilibrium => chemical potentials are related
• $\mu_\gamma = 0$ because the number of photons is not constrained (emission/absorbtion)
e.g. Compton scattering $e^- + \gamma \leftrightarrow e^- + \gamma + \gamma$ $\quad \mu_{e^-} + \mu_\gamma = \mu_{e^-} + 2\mu_\gamma \Rightarrow \mu_\gamma = 0$
• $\mu_X = -\mu_{\bar{X}}$ particles/antiparticle, in fact: $X + \bar{X} \leftrightarrow \gamma + \gamma$ $\quad \mu_X + \mu_{\bar{X}} = 0$

- $\Phi(T, V, \mu) = \overbrace{U - TS}^{F} - \mu N = TS - PV + \mu N - TS - \mu N = -PV$
$U = TS - PV + \mu N$ Euler relation

Case: Fermions and Bosons

$\langle E \rangle = \phi + sT + \sum_i \mu_i N_i$ with more species

- Partition sum of the α-state

Fermions : $N_\alpha = 0, 1$ (Pauli's exclusion principle) $z_\alpha = 1 + e^{-(\varepsilon_\alpha - \mu)/kT}$ Fermi-Dirak (1)

Spin ½: e^+, e^-, μ, p, ν

Bosons : $N_\alpha = 0, 1, 2, \dots, \infty$ $z_\alpha = (1 - e^{-(\varepsilon_\alpha - \mu)/kT})^{-1}$ Bose-Einstein (2)

Spin 1 : γ

(From geometric series : $\sum_{m=0}^{\infty} e^{-mx} = \sum (e^{-x})^m = (1-e^x)^{-1}$)

- Number distribution of particles in each state

$$\langle N_\alpha \rangle = -\frac{\delta \phi}{\delta \mu} = +kT \frac{\delta \ln(z_\alpha)}{\delta \mu} = kT \frac{1}{z_\alpha} \frac{\delta z_\alpha}{\delta \mu} \overset{(1)}{=} \frac{kT}{1 + e^{-(\varepsilon_\alpha - \mu)/kT}} \cdot \frac{e^{-(\varepsilon_\alpha - \mu)/kT}}{+kT} = \left(e^{+(\varepsilon_\alpha - \mu)/kT} + 1 \right)^{-1}$$

$$\overset{(2)}{=} kT (1 - e^{-(\varepsilon_\alpha - \mu)/kT})(-1)(1 - e^{-(\varepsilon_\alpha - \mu)/kT})^{-2} \cdot \frac{e^{-(\varepsilon_\alpha - \mu)/kT}}{+kT}$$

$$= \frac{1}{\left(e^{(\varepsilon_\alpha - \mu)/kT} \pm 1 \right)}$$

(+) fermions

(−) bosons

- Example : photons (bosons) in equilibrium

$\varepsilon = h\nu$, $\mu = 0$, $g = 2$ helicity $(+1, -1)$ = Polarization internal degree of freedom to be multiplied to the number of states

$N_\nu = \frac{1}{e^{h\nu/kT} - 1}$ Number distribution of states

$g_s = 4\pi g \nu^2 / c^3$ density of states, $h\nu$ energy

$U(\nu) = \frac{4\pi g \nu^2/c^3 \, h\nu}{e^{h\nu/kT} - 1}$ Equilibrium energy density ($g = 2$, "energy")

$U = E_\nu g_\nu N_\nu \quad E_\nu = h\nu$

per steradiant $(1/4\pi)$, per cm^2, per Hz

$B_T(\nu) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$ Planck spectrum $[W\, sr^{-1}\, m^{-2}\, Hz^{-1}]$

Moreover we can also derive : $P = -\frac{\delta \phi}{\delta V}$ Pressure

$S = -\frac{1}{V} \frac{\delta \phi}{\delta T}$ Entropy density

$\Rightarrow$ you know everything !

Ideal quantum gas

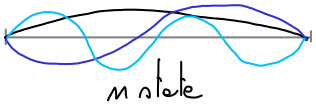
many states → continuous: $\sum \to \int$

- Particle $\quad \bar{p} = \hbar \bar{k} \quad \varepsilon(p) = \sqrt{c^2 p^2 + m^2 c^4}$

- Confined:

mode = state

$k = \frac{2\pi}{L} n = \frac{2\pi n}{V^{1/3}} \Rightarrow n = \frac{V^{1/3}}{2\pi} k \Rightarrow$ 3D box $\; dn^3 = \frac{V}{(2\pi)^3} dk^3$ ← include g: internal degrees of freedom e.g. spin



$d^3p = \hbar^3 d\bar{k}^3 \to$

$dN = \frac{gV}{(2\pi)^3} d^3k$ # of states in interval of k

$= \frac{gV}{(2\pi\hbar)^3} d\bar{p}^3$ # " " " " " p

- Continuous approximation $(\sum \to \int)$

$N = \sum_\alpha N_\alpha \quad N_\alpha = \frac{1}{e^{(\varepsilon-\mu)/kT} \pm 1}$ Fermi-Dirac / Bose-Einstein

! $N = \int N_\alpha \, dN = \int \mathcal{N}(p) \, d^3p = \int_{-\infty}^{\infty} \frac{1}{e^{(\varepsilon-\mu)/kT} \pm 1} \underbrace{\frac{gV}{(2\pi\hbar)^3} d^3p}_{dN}$ → polar coord. $(d^3p = 4\pi p^2 dp)$ to get density $(\cdot \frac{1}{V})$ (homogeneous)

$$n = \frac{g}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{e^{(\varepsilon-\mu)/kT} \pm 1} \quad \text{number density}$$

$$u = \frac{g}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi \varepsilon(p) p^2 dp}{e^{(\varepsilon-\mu)/kT} \pm 1} \quad \text{energy density} \quad \Big) \cdot \varepsilon(p)$$

Free energy

$\phi = -kT \ln(z_{gc}) = -kT \ln(\prod_\alpha z_\alpha) = -kT \sum_\alpha \ln(z_\alpha)$

$z_\alpha = 1 + e^{-(\varepsilon_\alpha - \mu)/kT}$ Fermi-Dirac (1)

$z_\alpha = (1 - e^{-(\varepsilon_\alpha - \mu)/kT})^{-1}$ Bose-Einstein (2)

continuous limit: $\phi(T, V, \mu) = -kT \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 \ln(z_\alpha) \, dp$

$$= \underset{BE}{\overset{FD}{\mp}} kT \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 \ln\left(1 \pm e^{-(\varepsilon(p)-\mu)/kT}\right) dp$$

Explicit relations of the ensamble properties

E.g. relativistic bosons in equilibrium :

- relativistic $\varepsilon = cp$
- equilibrium $\mu = 0$
- bosons $z_\alpha = (1 - e^{(\varepsilon_p - \mu)/kT})^{-1} = (1 - e^{-cp/kT})^{-1}$

$$\phi_B(T,V,\mu) = kT \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 \ln\left(1 - e^{-(cp/kT)}\right) dp$$

$x \equiv cp/kT$

$p = \frac{xkT}{c} \quad dp = \frac{kT}{c} dx$

$p^2 dp = \left(\frac{kT}{c}\right)^3 x^2 dx$

$$= \frac{(kT)^4}{c^3} \frac{gV4\pi}{(2\pi\hbar)^3} \int_0^\infty (1 - e^{-x}) x^2 dx$$

$= -m!\, \zeta(m+2)$ with $m=2$ Riemann function

$$= -gV \frac{\pi^2}{90} \frac{(kT)^4}{(\hbar c)^3}$$ free energy

$$P_B = -\frac{\delta\phi}{\delta V} = g \frac{\pi^2}{90} \frac{(kT)^4}{(\hbar c)^3}$$ pressure

$$s_B = -\frac{1}{V}\frac{\delta\phi}{\delta T} = gk \frac{2\pi^2}{45}\left(\frac{kT}{\hbar c}\right)^3$$ entropy density

Same for the other cases

- continuous case
- $\varepsilon_\alpha = mc^2 + p^2/2m$ 1° order relat. limit
- $z_\alpha = \frac{gV(2\pi m kT)^{3/2}}{h^3} e^{-(mc^2-\mu)/kT}$

↓

	relativistic distributions: Bose-Einstein	Fermi-Dirac	non-relativistic Maxwell-Boltzmann
number density	$n_B = g_B \frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$	$n_F = \frac{3}{4}\frac{g_F}{g_B} n_B$	$g\left(\frac{kT}{2\pi\hbar}\right)^{3/2} e^{-kT/mc^2}$
energy density	$u_B = g_B \frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3}$	$u_F = \frac{7}{8}\frac{g_F}{g_B} u_B$	$\frac{3}{2} n kT$
pressure	$P_B = \frac{u_B}{3} = g_B \frac{\pi^2}{90}\frac{(kT)^4}{(\hbar c)^3}$	$P_F = \frac{u_F}{3} = \frac{7}{8}\frac{g_F}{g_B} P_B$	nkT
entropy	$s_B = g_B k \frac{2\pi^2}{45}\left(\frac{kT}{\hbar c}\right)^3$	$s_F = \frac{7}{8}\frac{g_F}{g_B} s_B$	

Table 3.2: *Particle content of the Standard Model.*

type		mass	spin	g
quarks	$t, \bar{t}$	173 GeV	$\frac{1}{2}$	$2 \cdot 2 \cdot 3 = 12$
	$b, \bar{b}$	4 GeV		
	$c, \bar{c}$	1 GeV		
	$s, \bar{s}$	100 MeV		
	$d, \bar{s}$	5 MeV		
	$u, \bar{u}$	2 MeV		
gluons	g_i	0	1	$8 \cdot 2 = 16$
leptons	$\tau^{\pm}$	1777 MeV	$\frac{1}{2}$	$2 \cdot 2 = 4$
	$\mu^{\pm}$	106 MeV		
	$e^{\pm}$	511 keV		
	$\nu_\tau, \bar{\nu}_\tau$	< 0.6 eV	$\frac{1}{2}$	$2 \cdot 1 = 2$
	$\nu_\mu, \bar{\nu}_\mu$	< 0.6 eV		
	$\nu_e, \bar{\nu}_e$	< 0.6 eV		
gauge bosons	W^+	80 GeV	1	3
	W^-	80 GeV		
	Z^0	91 GeV		
	γ	0		2
Higgs boson	H^0	125 GeV	0	1

From D. D. Baumann

Baryons-Photons ratio

- Number density of baryons today, assume they are all in Hidrogen H => m_p protons

$$n_B = \frac{\rho_B}{m_p} = \frac{\Omega_B}{m_p}\rho_c \simeq \frac{\Omega_B}{m_p}\frac{3H^2}{8\pi G} \simeq 1{,}1\cdot 10^{-5}\,\Omega_m h^2\,\mathrm{cm}^{-3} \qquad \Omega_B = 0{,}04 \text{ measured}$$

- Number of photons (bosons)

$$n_\gamma = g_r \frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 = 407\,\mathrm{cm}^{-3} \qquad T_{CMB} = 2{,}752 \text{ measured}$$

- Baryons - Photons fraction

$$n_F = \frac{3}{4}\frac{g_F}{g_B} n_B \quad \Rightarrow \quad \eta \equiv \frac{n_B}{n_\gamma} = \text{const.}\,! \qquad \eta = 2{,}7\cdot 10^{-8}\,\Omega_B h^2$$

for each baryon you have $>10^8$ photons

("temperature of the universe" = $T_{photons}$)

=> entropy of universe is dominated by photons

=> irreversible processes are negligible => adiabatic expansion!

η is important because it tunes nucleosynthesis and recombination!

- why are there baryons at all?! i.e. $\eta > 0$?

during their creation they should have annihilated with anti-baryons: $B^+ + B^- \to \gamma + \gamma$

=> must have been asymmetry in baryon / anti-baryon number,

possible under Sakharov conditions, eg. CP violation ....

Reactions and Saha's equation

- At early time photons are energetic enough (T is large) to keep everything photo-dissociated we have a soup of γ, n, p, e^-
- As a ↑, temperature T ↓ certain recombination processes take place

e.g. $D + D \to {}^3He + n$ "cosmological nucleosynthesys" = formation of D, He, Be, Li

e.g. $e^- + p^+ \longrightarrow H + \gamma$ "recombination"

- Assumptions: ideal gases composed by indistinguishible particles

low enough densities such to ignore degeneracies

Recombination

We now consider recombination which is simpler $\boxed{e^- + p^+ \longleftrightarrow H + \gamma}$

- From Helmoltz free energy we derive all ensamble properties

$$F(T,V,N) = -kT\ln(Z_c) \quad \text{with} \quad Z_c = \prod_\alpha \sum_{\{N_\alpha\}} e^{-(\varepsilon_\alpha - \mu)N_\alpha/kT} \qquad i = e, p, H$$

here: $Z_c = \dfrac{Z_e^{N_e} Z_p^{N_p} Z_H^{N_H}}{N_e!\, N_p!\, N_H!}$ ←

- because, all possible combination of states
- $N_B = N_p + N_H$ $\quad N_e = N_p$ (neutral universe) $\quad N_H = N_B - N_e$
  ⇒ N_e determines everything (for fixed N_B)
- γ serves as the heat reservoirs controlling T

$$\Phi = -kT\ln\left(\frac{Z_e^{N_e} Z_p^{N_p} Z_H^{N_H}}{N_e!\, N_p!\, N_H!}\right)$$

$$= -kT\left[\ln\left(Z_e^{N_e} Z_p^{N_p} Z_H^{N_H}\right) - \ln\left(N_e!\, N_p!\, N_H!\right)\right]$$

$N_H(\ln N_H - 1) = (N_B - N_e)(\ln N_H - 1)$

$$= -kT\left[N_e \ln Z_e + \underbrace{N_p}_{N_e} \ln Z_p + \underbrace{N_H}_{N_B - N_e} \ln Z_H - \ln(N_e!) - \underbrace{\ln(N_p!)}_{=N_e} - \ln(N_H!)\right]$$

$-2\ln(N_e!) \simeq -2N_e(\ln N_e - 1)$

- Equilibrium at the minimum of F ⇒ $\dfrac{\delta F}{\delta N_e} = 0$

stirling formula $\lim_{x\to\infty} \ln(x) = x(\ln(x) - 1)$

$$\frac{\delta F}{\delta N_e} = -kT\left[\ln Z_e + \ln Z_p - \ln Z_H - 2\left(\ln N_e - \cancel{1} + \frac{\cancel{N_e}}{\cancel{N_e}}\right) + \ln N_H\right] \stackrel{!}{=} 0$$

$$\ln\left(\frac{Z_e Z_p}{Z_H}\right) \stackrel{\uparrow}{=} \ln\frac{N_e^2}{N_H} = \ln\frac{N_e^2}{N_B - N_e} = \ln\frac{x^2 N_B}{1-x} \quad \Rightarrow \quad \frac{x^2}{1-x} = \frac{Z_e Z_p}{Z_H N_B} \qquad x \equiv \frac{n_e}{n_B}$$ degree of ionization

- $n_B = \eta n_\gamma$ η ! ratio of baryons and photons and # of photons
- ionization potential of H: $\chi = (m_e + m_p - m_H)c^2 = 13{,}6\,eV$
- non relativistic particles: $\varepsilon = m_i c^2 + \dfrac{p^2}{2m_i}$ $\quad i = e, p, H$ *
- plug Z_e, Z_p, Z_H use continuous distribution (M-B) for the i-th specie

$$Z_i = \frac{4\pi g_i V}{(2\pi\hbar)^3}\int_0^\infty p^2 e^{-(\varepsilon - \mu_i)/kT}\,dp = \frac{g_i V (2\pi m_i kT)^{3/2}}{(2\pi\hbar)^3} e^{-(m_i c^2 - \mu_i)/kT}$$

(* : $\varepsilon - \mu_i = 0$) $\qquad \mu_e + \mu_p = 0 = \mu_H$ at equilibrium

$$\frac{Z_e Z_p}{Z_H} = \frac{g_e g_p}{g_H} V \frac{(2\pi \frac{m_e \cancel{m_p}}{\cancel{m_H}} kT)^{3/2}}{(2\pi\hbar)^3} e^{-\frac{(m_e + m_p - m_H)c^2}{kT}} \qquad \frac{g_H}{g_e g_p} = \frac{4}{2\cdot 2} \qquad g_H = \underset{\text{singlet}}{1} + \underset{\text{triplets}}{3}$$

$$\Rightarrow \quad \frac{x^2}{1-x} = \frac{(2\pi m_e kT)^{3/2}}{(2\pi\hbar)^3 n_B} e^{-\chi/kT} \qquad \frac{N_B}{V} \quad n_B = \eta n_\gamma = 2\eta \frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}\,\zeta(3)\,\eta}\left(\frac{m_e c^2}{kT}\right)^{3/2} e^{-\chi/kT}$$

Saha's equation $\qquad \eta$ tunes the process!

End of recombination

- When medium is no more ionized: most of e, p are bound in H

  degree of ionization $X = \frac{n_e}{n_B} \ll 1 \Rightarrow \frac{X^2}{1-X} \simeq X^2 \propto e^{-\chi/kT} \ll 1 \Rightarrow kT \ll \chi$

  i.e. photons are "cold" enough: for $x=0,1$ $T_{rec} \approx 3500K$ $kT_{rec} = 0,3\ eV$

  scattering, $e^- + \gamma \to e^- + \gamma$, drops => matter and radiation decouple

- This is later than the naive (wrong) consideration $T = \frac{\chi}{k_B} = 10^5\ K$ with $\chi = 13.6\ eV$

  - The "delay" is due to the very large number of photons: $\frac{N_B}{N_\gamma} \equiv \eta = 2,7 \cdot 10^{-8} \Omega_B h^2$ is very small
  - many photons have ν high enough to partially ionize back the medium
    (remember - photons energy distribution is not a delta function)
  - This is a "retarded reaction"

- In reality the process is even slower than what predicted by Saha's eq.

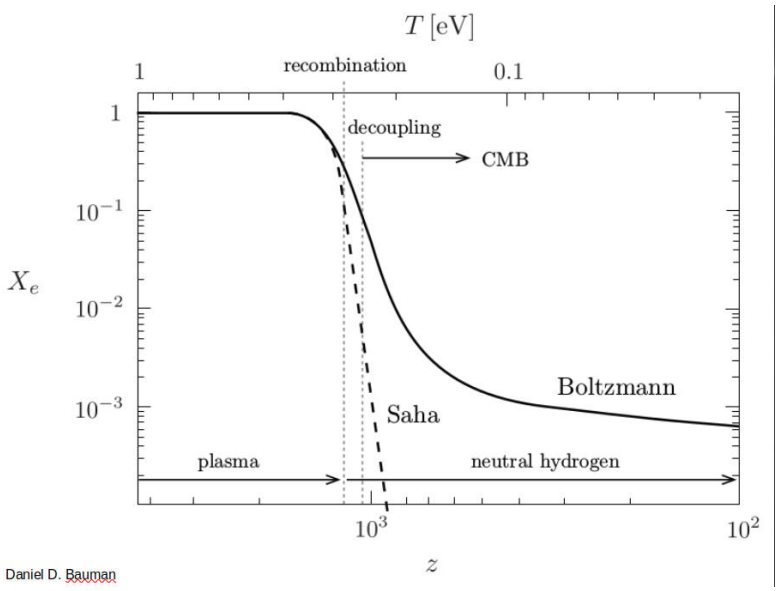
  - the reaction is $e^- + p^+ \longleftrightarrow H + \gamma$ (double arrow) captured e^- transition to ground state

    Lyman-α: $2P \to 1S$ => $h\nu \geq E_{Ly\alpha} = \frac{3\chi}{4} = 10,2\ eV$

  - Reaction proceeds only if lower ν photons are emitted:
    forbidden transition $2S \to 1S$ (2 photons are emitted)
    low probability of transition => slow => source of a further delay

- Time of recombination:
  - photons are the termal reservoir => they set T $T \propto a^{-1}$ (relativistic specie)
  - radiation dominated era $H^2 = -\frac{8\pi G}{3}\rho_\gamma$ => $a(t)$ => $T(t)$ $t_{rec} \simeq 380\cdot 000\ yr$
    $z \simeq 1100$
    last scattering surface

- This process is not istantaneous, it lasts for some time
  $\delta T \approx 200K$



Photons decoupling

pg 32 MdM

- Coupling between photons and plasma was guaranteed by Thompson scattering

  At recombination, the free e^- are "gone"

  $\Rightarrow$ no more Thomson scattering

  $\rightarrow$ photons are no more coupled to matter

  $\Rightarrow$ photons relic background: Cosmic Microwave Background radiation (CMB)

  ↑ this is the wavelength they have today

  $e^- + \gamma \longleftrightarrow e^- + \gamma$ with $\Gamma_\gamma \approx n_e \sigma_T$ σ_T = Thompson cross section

  as $n_e \to 0 \Rightarrow \Gamma_\gamma \to 0$ as well

- using Saha eq. for this "reaction":

  $k T_{dec} \sim 0{,}27\,eV$

  $z_{dec} \sim 1100$ $t_{dec} \sim 380\,000\,yrs$

  $X_e(T_{rec}) \simeq 0{,}1$ $X_e(T_{dec}) \simeq 0{,}01$ i.e. universe becomes transparent only when most e^- are gone

Cosmological nucleosynthesis

- Binding energy of nuclei $\sim MeV$ => for $K_B T > 1 MeV$ p and n can not form bond states immediate photon-dissociation
  $e^{-\Delta mc^2/T_D}$
  Afterwards they can combine

- At $T_D \sim 0{,}8\,MeV$ p and n not more equilibrium thanks to reactions such as $n + \nu_e \rightleftarrows p + e^-$
  => freez-out: their number density freezes

  $m_n > m_p$ by a little => for equilibrium Boltzmann distribution $n_p > n_n$

  $\frac{n_n}{n_p} = e^{-(m_n - m_p)c^2/kT} \approx \boxed{1/6}$ $\quad \Delta mc^2 = 1{,}4\,MeV$

- When did this happened?

  relativistic particles: $T \propto a^{-1}$, today at a_0 $T_0 = 2{,}7K \rightarrow kT_0 = 2{,}4 \cdot 10^{-4} eV$ $\quad \frac{a_D}{a_0} = \frac{T_0}{T_D} = 3{,}4 \cdot 10^{-10}$ $\quad (=1)$

  radiation domination: $\left(\frac{\dot a}{a}\right)^2 \equiv H^2 = H_0^2 \Omega_R a^{-4}$ $\quad \boxed{t_D} = \frac{1}{H_0 \sqrt{\Omega_R}} \int_{z_D}^{\infty} \frac{dz}{(1+z)^3} \approx \boxed{10\,s}$

- Spontaneous decay of n, decay time: lifetime $\sim 900s$ => little drop in n_n: $\boxed{\frac{n_n}{n_p} = 1/7}$ at $kT \sim 0{,}1\,MeV$

- Helium abundance, rough estimate:

  all n_n bond in $\frac{n_n}{2}$ ^4He nuclei $\quad n\,p\,n\,p \rightarrow (p\,p\,n\,n)$ $\quad \Rightarrow$ $\quad \boxed{Y} = \frac{4\left(\frac{n_n}{2}\right)}{(n_n + n_p)} \approx \boxed{0{,}25}$

  each ^4He nuclei has 4 nucleons; # of ^4He nuclei; tot # of n and p

- Helium abundance, how to do it:

  - Need to form ^4He nuclei $(p\,p\,n\,n)$ $\quad$  4-body interaction not likely to occour
  - The number of neutron is crucial
  - Only 2 body interactions are likely to occur => need to go through intermediate steps
    Need Deuterium, $D = {}^2H$, to build up ^4He

$$p + n \rightarrow D + \gamma \quad \begin{cases} D + p \rightarrow {}^3He + \gamma \\ D + D \rightarrow {}^3He + n \\ D + D \rightarrow T + p \end{cases} : \quad \begin{cases} {}^3He + n \rightarrow T + p \\ {}^3He + D \rightarrow {}^4He + p \end{cases} \quad \begin{matrix} {}^3He + {}^4He \rightarrow Be + \gamma \\ {}^4He + T \rightarrow {}^7Li + \gamma \end{matrix} \quad : \quad T + D \rightarrow {}^4He + n$$

Saha's equation

- The entire universe is like the interior of a star: p, n, ν, γ, $T \sim 10^7 K$, nuclear fusion
- n_n is crucial
- D is crucial (Gamow criteria):
  Too much D => to few free n to proceed with the reaction
  Too few D => the chain can not proceed because of its lack
- Only light elements form, not enough time to proceed

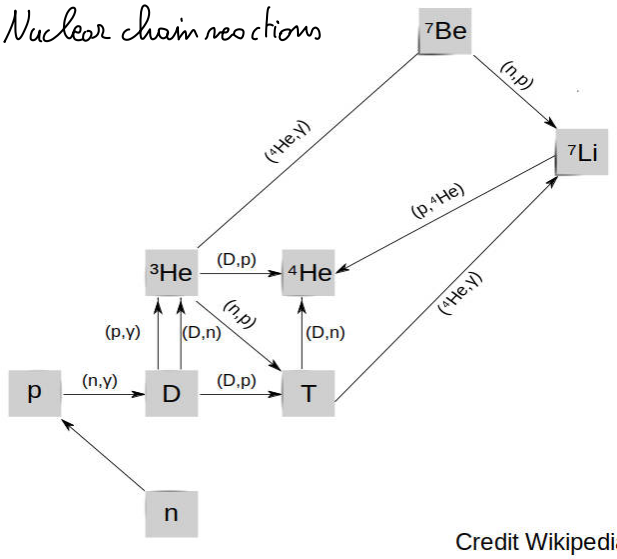
- Expansion tunes the process (i.e. the faster, the less time to process the material):
  time for reactions is short (stars keep the plasma contained because of gravity => heavier elements)
  => There is time to create only light elements (already very little amount of Li, Be)

- Radiation era => γ rules! photo dissociation: $\eta = \frac{N_B}{N_\gamma}$ leading parameter (recall Saha's eq.)

- Also this is a delayed reaction (Saha's eq.) => Nucleosynthesis occurs well after $KT \sim 1\,MeV$
  i.e at $KT \sim 0{,}1\,MeV$ => $z_{nucl} \sim 10^8$, $t_{nucl} = 100\,s$

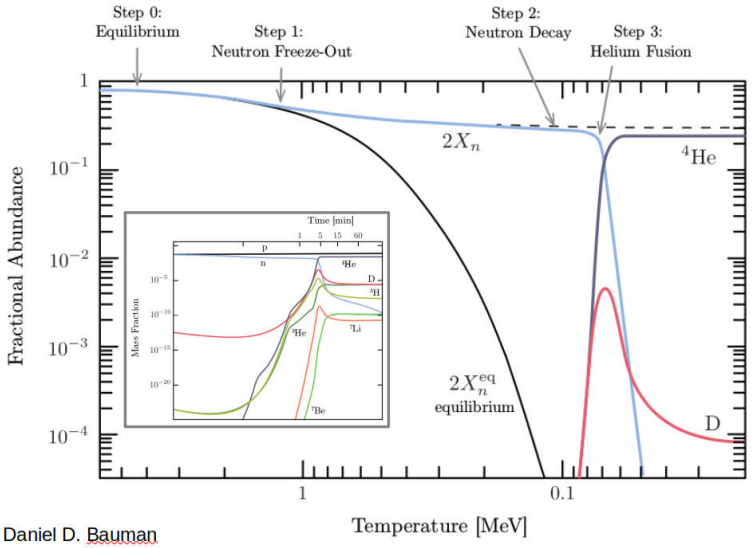
- These reactions are all interconnected:

Nuclear chain reactions



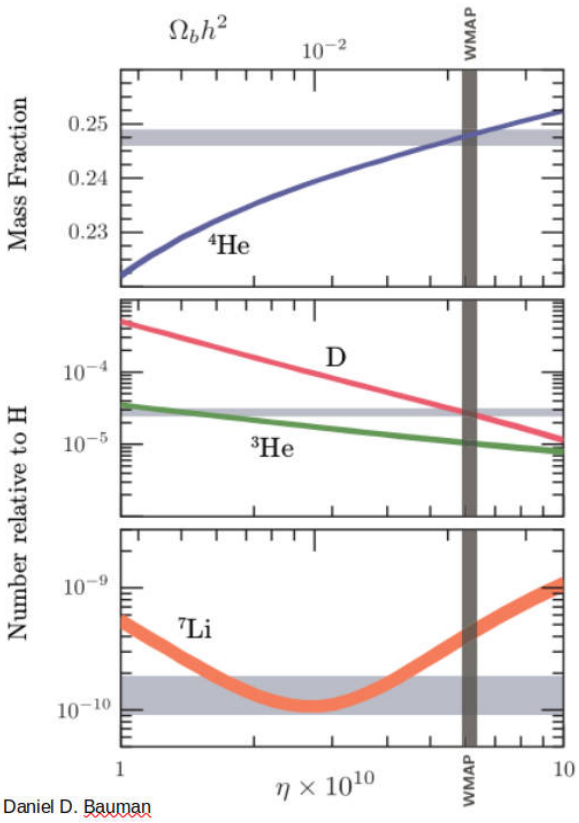
Credit Wikipedia

Evolution of abundances

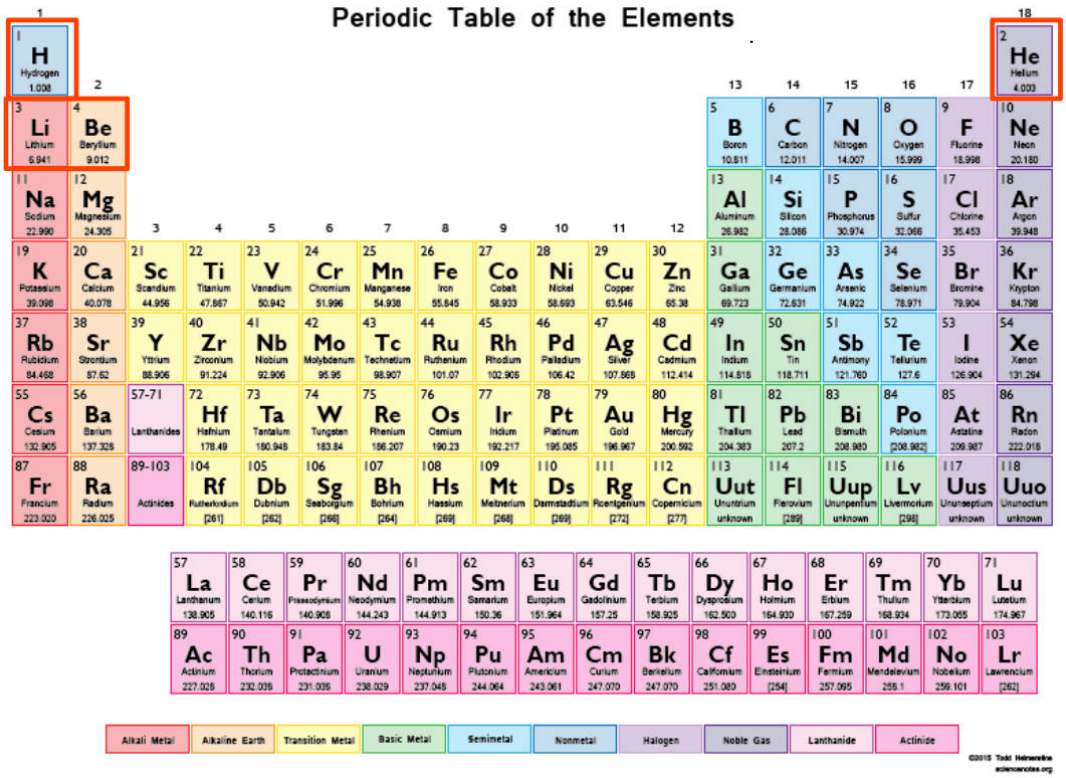


Daniel D. Bauman

Amount of the products



All other elements come from the stars and humans



Neutrino decoupling

- At $T \gg 1\,MeV$ $T_\nu = T_\gamma$ because equilibrium is achieved through weak interaction
- No interaction between γ and ν

  => They should maintain the same T forever ... No! Photon fluid heated up by e^+, e^- annihilation

---

weak interaction

$\nu + \bar{\nu} \longleftrightarrow e^+ + e^-$

- at $T_\nu \approx 10^{10,5}\,K \approx 2,7\,MeV$ => neutrinos ν thermal freez-out $\frac{\Gamma}{H} \sim \left(\frac{T}{2,7\,MeV}\right)^3$
- ν are ultrarelativistic => $N_\nu = const$

=> Neutrino Background ($C\nu B$)

$e^+ + e^- \longleftrightarrow \gamma + \gamma$

- bi-directional untill $\langle h\nu \rangle < (2 m_e c^2) \approx 1\,MeV$

  the process become irreversible

  γ can no more create e^- or e^+ because their energy is too low (redshift)

$e^+ + e^- \longrightarrow \gamma + \gamma$

- this is a different kind of freez out
- all energy (entropy) goes into photons with $\langle h\nu \rangle \sim m_e c_2 > "\langle h\nu_{old} \rangle"$

  => Reheating of radiation

- Entropy is conserved:

full annihilation

$S_{e^+} + S_{e^-} + S_\gamma = S'_\gamma$

before after

Relativistic distri.: $S_F = \frac{7}{8} \frac{g_F}{g_B} S_B$ with $g_{e^+} = g_{e^-} = g_\gamma = 2$ => $\frac{S_F}{S_B} = \frac{7}{8}$

$\left(\frac{7}{8} + \frac{7}{8} + 1\right) S_\gamma = S'_\gamma$ ($\frac{7}{8} + \frac{7}{8} + 1 = 11/4$)

$S_B = g_B K \frac{2\pi^2}{45} \left(\frac{KT}{\hbar c}\right)^3$ => $(11/4)^{1/3}$ $1,5\,T_\gamma \simeq T'_\gamma$

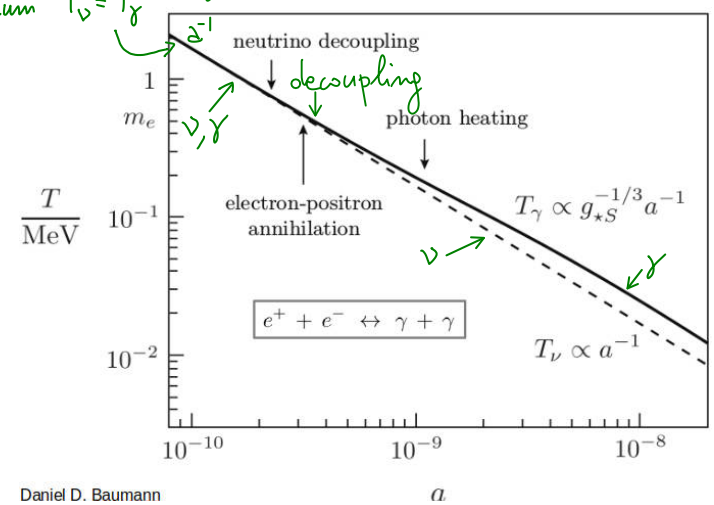
Today: $T_\gamma = 2,752\,K$ (measured)

=> $T_\nu = 1,95\,K$ (expected)

photons are hotter

"Reheating"

Thermal equilibrium $T_\nu = T_\gamma$



Could neutrinos be the famous dark matter?

- Density of neutrinos $\rho_\gamma, \rho_\nu \sim T^4 \Rightarrow \rho_{\nu_{tot}} = \rho_{\nu+\bar{\nu}} = \frac{7}{8} N_\nu g_\nu \left(\frac{4}{11}\right)^{4/3}$
- $N_\nu g_\nu \approx 3 \pm 0{,}3$ (CMB measures)
  3 families in standard model of particles and $g_\nu = 1 \Rightarrow \rho_{\nu_{tot}} \approx 0{,}68\, \rho_\gamma$
- There are $2N_\nu$ species of neutrinos, ν, and anti-neutrinos, $\bar{\nu}$, and $n_\gamma = \alpha T^3$ α = const.

$$\Rightarrow n_{\nu^*} = \frac{3}{4}(2N_{\nu^*}) n_\gamma(T_\nu) = \frac{3}{2} N_{\nu^*} \alpha T_\nu^3 = \frac{3}{2} N_{\nu^*} \alpha T_\gamma \left(\frac{T_\nu}{T_\gamma}\right)^{11/3} = \frac{6}{11} N_\nu n_\gamma(T_\gamma) \qquad n_{\nu^*} = 340 \frac{1}{cm^3}$$

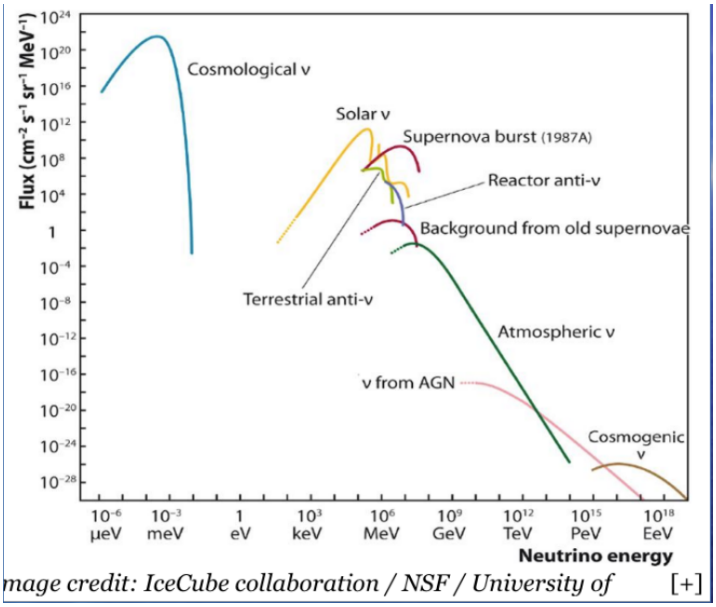
$n_{\nu^*} = n_{\nu_{tot}} + n_{\bar{\nu}}$

$$\Rightarrow \text{today energy density} \quad \rho_{\nu^*,0} = \frac{6}{11} n_\gamma \Sigma m_\nu$$

$$\Omega_{\nu^*,0} = \frac{8\pi G}{3H_0^2} \frac{6}{11} n_\gamma \Sigma m_\nu \overset{!}{=} 0{,}3 \quad \text{with } H_0 \approx 70 \Rightarrow \underline{m_\nu = 5\,eV}$$

But.... neutrino mass is $\underline{m_\nu < 1\,eV}$ (measured from solar neutrino oscillations + cosmic structure formations

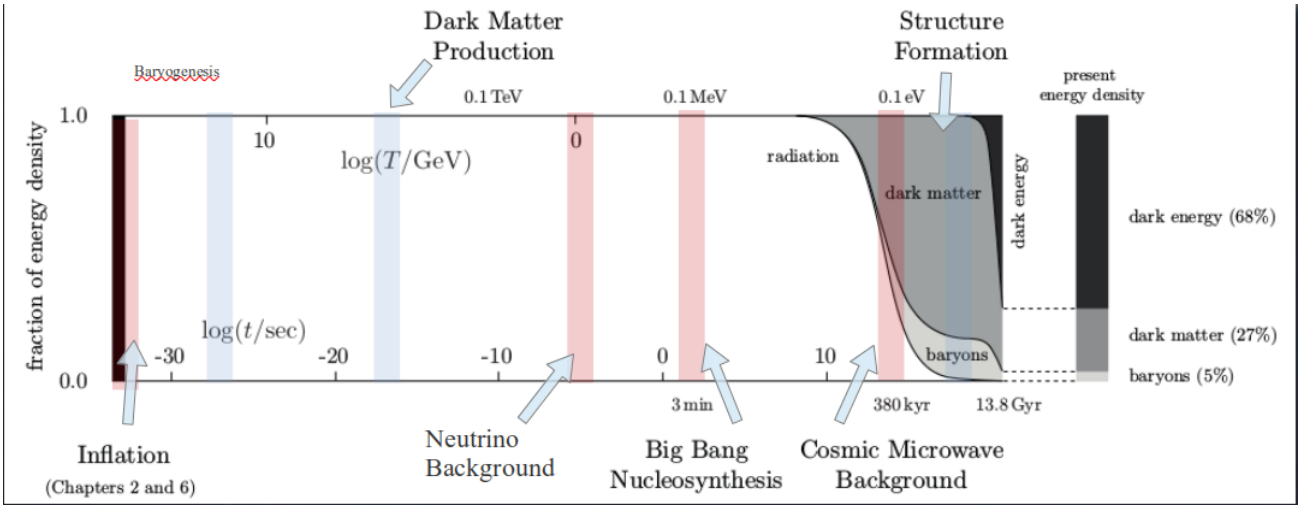
$\underline{N_0 \text{ is large} \Rightarrow \text{Neutrinos alone can not compose all dark matter!}}$



mage credit: IceCube collaboration / NSF / University of [+]

Last remark

- Be carefull with the chronology...
  I order them backward in time to better group physical concepts
- The chronological order is:
  1) - ν decoupling / photons reheating
  2) Big-Bang nucleosynthesis
  3) recombination and photons-matter decoupling



Appendix

Another way to look at the Boltzmann eq.

- $P_1 + P_2 \leftrightarrow P_3 + P_4$
- $\underbrace{\frac{1}{a^3}}_{(4)} \underbrace{\frac{d(n_1 a^3)}{dt}}_{(3)} = \underbrace{-\alpha n_1 n_2}_{(1)} + \underbrace{\beta n_3 n_4}_{(2)}$

1) creation of P_1 particles out of reaction between P_3 and P_4 $\quad \alpha = \langle \sigma v \rangle$
2) destruction of P_1 particles due to their reaction with P_2
3) Total change in the comoving number density of P_1 particles $N_i = n_i a^3$
4) Factor to convert to actual number densities, not comooving

At equilibrium $\dot{N}_1 = 0 \Rightarrow \alpha n_1 n_2 = \beta n_3 n_4 \qquad \beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} \alpha$

$$= -\langle \sigma v \rangle n_1 n_2 \left[1 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} \frac{n_3 n_4}{n_1 n_2}\right]$$

$$\frac{1}{n_1 a^3} \frac{dN_1}{d\ln a} = -\frac{\langle \sigma v \rangle n_2}{H} \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq} \frac{N_3 N_4}{N_1 N_2}\right]$$

$$\frac{d\ln N_1}{d\ln a} = -\frac{\Gamma}{H} \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq} \frac{N_3 N_4}{N_1 N_2}\right]$$

$\frac{d}{dt} = \frac{d\dot{a}}{dt}\frac{d}{da} = aH\frac{d}{da} = H\frac{d}{d\ln a} \qquad \dot{N}_1 = H\frac{dN_1}{d\ln a} = (3)$

$\Gamma \equiv n_2 \langle \sigma v \rangle$ Reaction rate

Dark Matter Relics

- DM particles "nowadays" are a relic
  - ⇒ they broke off equilibrium
  - ⇒ no more interacting with other particles

- Speculations about some unknown particle X, Assume:
  - e.g. Weakly Interacting Massive Particle (WIMPs) with mass M_x
  - No direct interactions with photons ~~$X + \gamma$~~
  - they were in equilibrium with photons at early times via $X + \bar{X} \leftrightarrow \ell + \bar{\ell}$ (annihilation)
  - $\ell, \bar{\ell}$ some light (basically massless) particle/antip. tightly coupled with the other particles
    - ⇒ ℓ are in equilibrium: $n_\ell = n_\ell^{eq}$
  - No initial asymmetry between X and $\bar{X}$ ⇒ $n_X = n_{\bar{X}}$

- Boltzmann eq. for $X + \bar{X} \leftrightarrow \ell + \ell_x$:

$$\frac{dN_x}{dt} = -a^3 \langle\sigma v\rangle N_x^2 \left[1 - \left(\frac{N_x^2}{N_e^2}\right)_{eq} \frac{N_e^2}{N_x^2}\right] = -\frac{\langle\sigma v\rangle}{a^3}\left(N_x^2 - N_{x\,eq}^2\right) \qquad n_x = a^3 N_x$$

Time is interesting when $kT \sim M_x c^2$

- define $x \equiv \frac{M_x}{T}$ $(c = 1 = k)$
- use it as a time variable: $\dot{x} = \frac{d}{dt}\left(\frac{M_x}{T}\right) = -\frac{M_x}{T^2}\dot{T} = -x\frac{\dot{T}}{T} \simeq xH$ (relativistic $T \propto a^{-1}$) $\qquad \frac{dN_x}{dt} = \frac{dN_x}{dx}\frac{dx}{dt} = \frac{dN_x}{dx} xH$
- radiation domination: $H^2 = H_0^2 \Omega_R a^{-4} = H_0^2 \Omega_R g_* T^4 = H_0^2 \Omega_R g_* \frac{M_x^4}{x^4} \equiv H(M_x) x^{-4}$

$$\frac{dN_x}{dx} = -\frac{\langle\sigma v\rangle}{a^3 xH} N_x^2 \left(1 - N_{x\,eq}^2\right) \qquad a \propto T^{-1} \propto \frac{x}{M_x} \qquad a^3 xH \propto \frac{x^3}{M_x^3} x H(M_x) x^{-4}$$

$$= -\beta \cdot \frac{M_x^3 \langle\sigma v\rangle}{H(M_x) x^2}\left(N_x^2 - N_{x\,eq}^2\right)$$

$$= -\frac{\lambda}{x^2}\left(N_x^2 - N_{x\,eq}^2\right) \qquad \lambda = \frac{2\pi^2}{45} g_* \frac{M_x^3 \langle\sigma v\rangle}{H(M_x) x^2}$$

Riccati equation (no analytical solution)

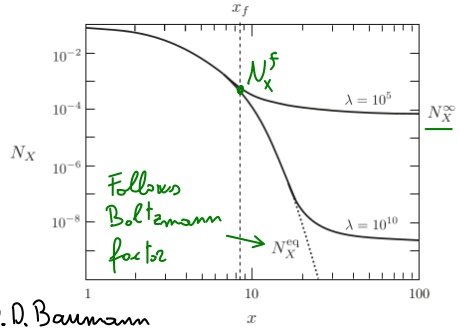


Figure 3.7: *Abundance of dark matter particles as the temperature drops below the mass.*

- in most WIMPs theories λ = const.
- Freeze-out at $x_f \sim 10$ (little dependence on M_x)
- Final abundance $N_x^\infty \equiv N_x(x = \infty)$ of dark matter
- The longer the equilibrium is kept (efficient coupling) the smaller final N_x^∞ will be

Final number density N_x^∞

Much after freeze-out : $N_x^{eq} \ll N_x^\infty \Rightarrow \frac{dN_x}{dx} \simeq \frac{\lambda}{x^2} N_x^2$

integrate $\int_{N_x^f}^{N_x^\infty} \frac{dN_x}{N_x^2} = \int_{x_f}^{\infty} \frac{\lambda}{x^2} dx$ $\quad \frac{1}{N_x^\infty} - \frac{1}{N_x^f} = \frac{\lambda}{x_f}$ $\quad N_x^f \gg N_x^\infty$ $\quad \boxed{N_x^\infty \simeq \frac{x_f}{\lambda}}$ with $x_f \sim 10$

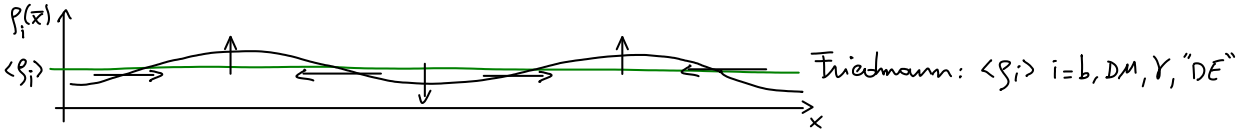
WIMPs Miracle

$\Omega_x = \Omega_{DM} = \frac{\rho_{x,0}}{\rho_{c,0}}$ $\quad \Omega_x h^2 \sim 0{,}1 \left(\frac{x_f}{10}\right)\left(\frac{10}{g_*(M_x)}\right)^{1/2} \frac{10^{-8}\,GeV^{-2}}{\langle \sigma v \rangle} \simeq 0{,}3 \Rightarrow \sqrt{\langle \sigma v \rangle} \sim 10^{-4} GeV^{-1} \sim 0{,}1 \sqrt{G_F}$ (observations)

i.e. such thermal relic has a cross section characteristic of weak interaction!

Cosmic structure formation

- The universe is not perfectly homogeneous
- Initial energy-density fluctuations produced by inflation (seeds)
- Sound waves in the medium for which $p \neq 0$ (harmonic oscillations)
- Gravity and cosmic expansion cause them to evolve



- Formation of filaments and voids on scales of ~ 50 Mpc (very large but still $\ll R_H$)
  " " galaxy clusters " " " ~ 1 Mpc
  " " galaxies $\sim 0{,}1$ Mpc
- Treat the components as fluids => use hydrodynamics
- small overdensities: Weack field limit
- scales $\ll R_H$: ignore retardation and curvature

  } => Newtonian gravity suffice

- Use perturbative approach when in linear regime: $\rho(\vec{x}) = \langle \rho \rangle + \delta\rho \quad \delta\rho \ll \langle \rho \rangle$
- Spherical collapse and N-body simulation when in non linear regime

- Cold Dark Matter (CDM)

  Its existence and characteristics are necessary to explain observations
  - ↳ cold : non relativistic before decoupling
  - ↳ non baryonic : "exotic" particle
  - ↳ non collisional : zero pressure $P = 0$
  - ↳ dark : only gravitational interaction (no emissions / absorbtions)
    => does not radiate away!

- Baryons → radiates => further collapse
  → galaxies / stars form later
  → complex physics: feedback, Super-Novae, AGNs, ...

- Neutrinos → remains relativistic for longer time (because of small mass)
  → their diffusion affects structure formation

Length scales and structure formation

- Relevant scales: 1) Cosmological horizon $R_H(t) = a(t)\int_0^t c\,dt$ causal connection
  2) Jeans scale $\lambda_J = R_J$ collapse yes / no
  3) Diffusion scale λ_D diffusion

2) Jeans scale: Criteria for gravitationalle collapse

- R_J sets a criteria for having a gravitational collapse for $R > R_J$ (Jeans criteria)
- it is related to the balance of the system

- over-dense region: R, ρ, $M = \rho\frac{4\pi}{3}R^3$, v = typical values from Maxwell-Boltzmann distrib.

- Energy balance: $\begin{cases} E_k \simeq \frac{1}{2}Mv^2 \\ E_G \simeq -\frac{GM^2}{R} \end{cases}$ $\Rightarrow$ (virial theorem) $E_k + \frac{1}{2}V = 0$ $\frac{1}{2}Mv^2 = \frac{1}{2}\frac{GM^2}{R} \simeq \frac{G}{R}\rho R^5$

$$R_J = v\sqrt{\frac{1}{G\rho}}$$

- Force balance (per unit of mass): $\begin{cases} F_p \simeq \frac{P\cdot A}{M} = \frac{P\cdot R^2}{M} = \frac{P R^2}{\rho R^3} = \frac{c_s^2}{R} \\ F_G \simeq -\frac{GM^2}{MR^2} \end{cases}$ $\Rightarrow$ $\frac{c_s^2}{R} = \frac{GM}{R^2} = \frac{G\rho R^3}{R}$

$$R_J = c_s\sqrt{\frac{1}{G\rho}}$$

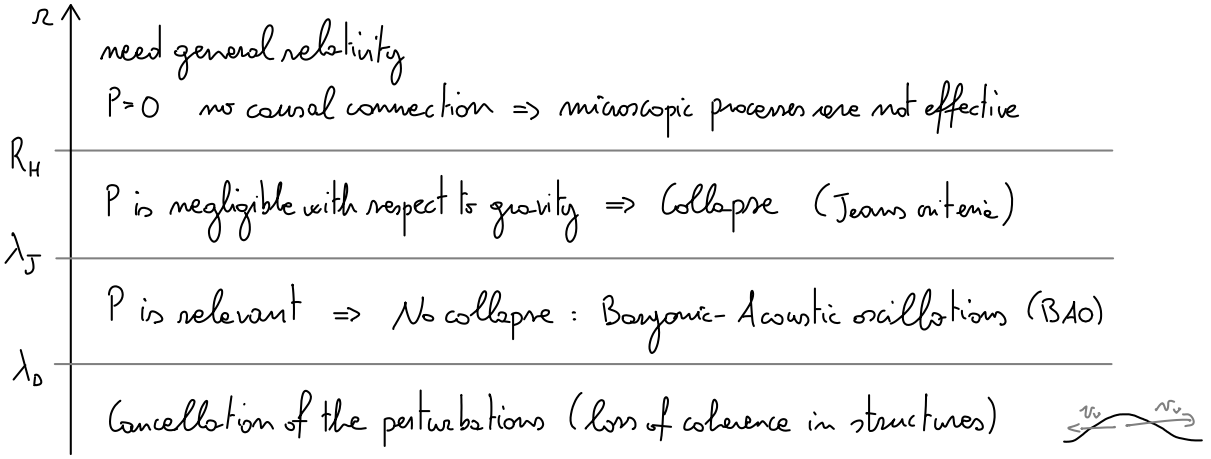
- Time scales: $\begin{cases} \tau_{ff} \sim (G\rho)^{-1/2} & \text{free fall time} \\ \tau_{cross} \sim \frac{2R}{c_s} & \text{crossing time} \end{cases}$ $\Rightarrow$ $2(G\rho)^{-1/2} = \frac{2R}{c_s}$

$$R_J = c_s\sqrt{\frac{1}{G\rho}}$$

- Jeans mass: $M_J \equiv \frac{4\pi}{3}R_J^3\rho$ mass within R_J

3) Diffusion scale λ_D it sets a scale "smoothing" the overdensities

depends on the velocity of particles



The density contrast and a model for gravitational collapse

- For all relativistic species and/or $\lambda > R_H$ you need GR
- We have a simple general GR solution valid for $\lambda > \lambda_J$ and $\lambda > R_H$:
  - treat overdensity as a "closed universe" immersed in a flat one (background) $\Rightarrow$ use Friedmann eq.

$$H^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} \qquad \begin{cases} H_B = \frac{8\pi G}{3}\rho_B & (K=0) \\ H_1^2 = \frac{8\pi G}{3}\rho_1 - \frac{c^2}{a^2} & (K=1) \end{cases}$$

$\delta(a)$ = density contrast

$$\boxed{H_B = H_1} \qquad \frac{8\pi G}{3}\rho_B\left(\frac{\rho_1 - \rho_B}{\rho_B}\right) = \frac{c^2}{a^2}$$

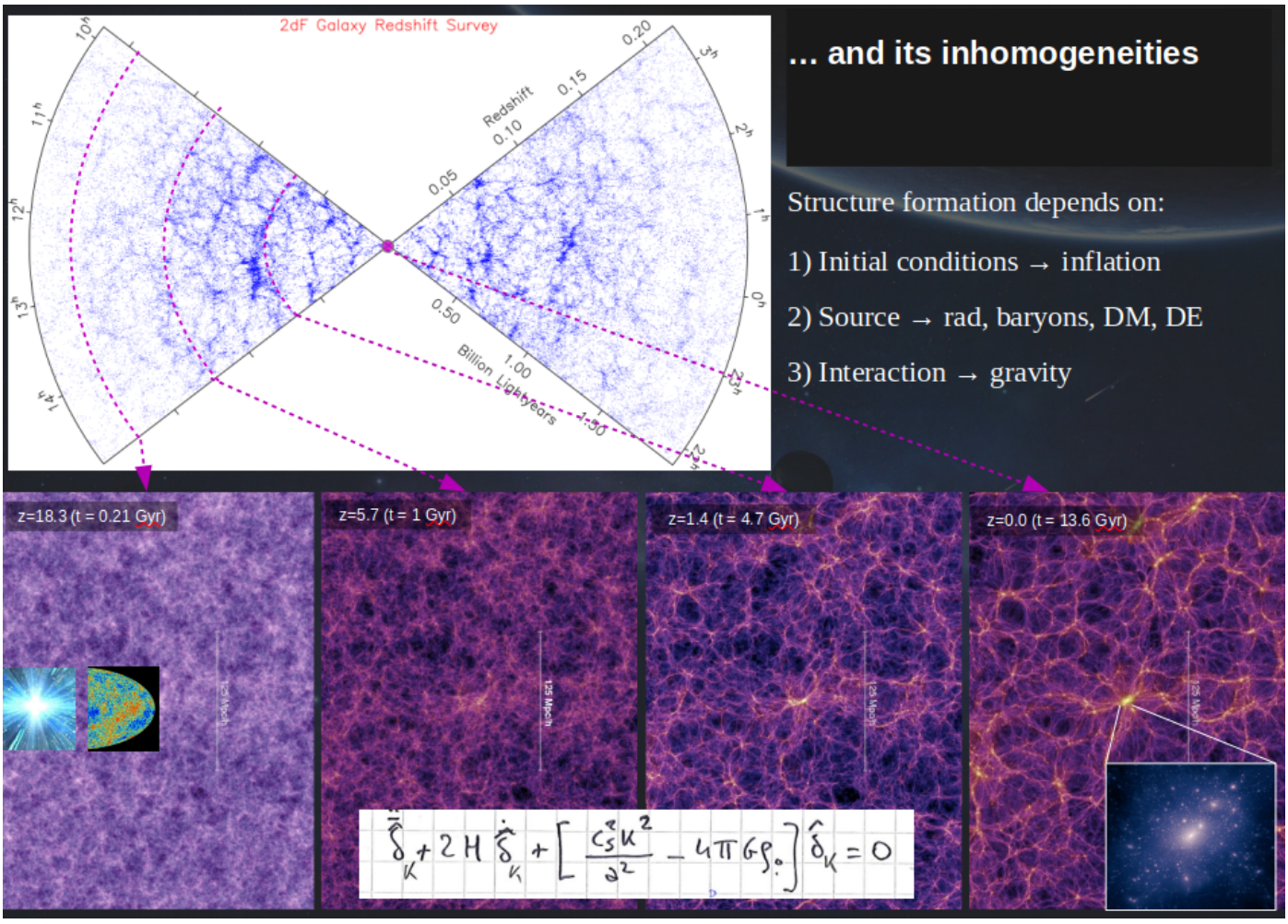
$$\Rightarrow \boxed{\delta(a)} = \frac{3c^2}{8\pi G \rho_B(a) a^2} \propto a^{3+3w-2} \propto \boxed{a^{1+3w}} \propto \begin{cases} a^2 \propto t & \text{for } w = \frac{1}{3} \quad \text{Radiation domination era} \\ a \propto t^{1/3} & \text{for } w = 0 \quad \text{Matter domination era} \end{cases}$$

$\rho = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$ politropic eq. of state

↑ the non dominant components follow the dominant one

- $(\delta) \longleftrightarrow (a) \longleftrightarrow (t)$ $\Rightarrow$ δ can be used as a time variable

growth factor, more later...



Structure formation in the linear regime

▫ Assumptions:

- $\Lambda \leftrightarrow$ DE enters in the dynamic of the univers, they do not participate to gravit. collapse
- Focus on scales $\lambda < R_H$: neglect { curvature; finite speed of light (retardation); we deal with non relativistic velocities; use: Newtonian gravity / dynamic
- Many neutral particles → treat as fluid → macroscopic quantities

  Properties: density, velocity, pressure, entropy, self gravity

  $\rho(t,\vec{x})$ (1) $\quad \vec{v}(t,\vec{x})$ (3) $\quad P(t,\vec{x})$ (1) $\quad S$ (1) $\quad \varphi(t,\vec{x})$ (1) $\quad$ 7 variables

▫ Hydrodynamic

Ideal Fluid: $T_{\mu\nu} = \left(\rho + \frac{P}{c}\right) u_\mu u_\nu + P\,\eta_{\mu\nu}$

Energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ eq. of motion of fluids } Non relativistic limit

$\vec{\nabla} \equiv (\partial_x, \partial_y, \partial_z)^T$

(1) energy conservation ($\nu = 0$) $\quad \partial_t \mathcal{E} + \vec{\nabla}(\mathcal{E}\vec{v}) + P\vec{\nabla}\vec{v} = 0$ ⊛

(2) momentum conservation ($\nu = 1,2,3$) $\quad \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}P}{\rho} - \vec{\nabla}\varphi$ (Euler eq.)

(3) matter conservation $\quad \frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0$ (Continuity equation)

(4) equation of state $\quad P(\rho, S)$

entropy evolution ⊛ $\frac{\partial S}{\partial t} = 0$ (adiabatic) } $P(\rho) = wc^2\rho \quad 0 \le w < 1$

(5) eq. of motion of gravity $\quad \vec{\nabla}^2\varphi = 4\pi G\rho$ (Poisson eq.)

↭ Closed system of equations => 7 variables and 7 equations

⊛ $\partial_t \mathcal{E} + \vec{\nabla}(\mathcal{E}\vec{v}) + P\vec{\nabla}\vec{v} = \rho T \frac{d\tilde{S}}{dt} = 0 \qquad \tilde{S} \equiv \frac{S}{\rho}$ specific entropy

↑ 1° law of thermodynamics $dE + PdV = TdS$

(1) Linearize equations for small perturbations

$$\begin{cases} \rho = \rho_B + \delta\rho = \rho_B(1+\delta) \\ \bar v = \bar v_B + \delta\bar v \\ P = P_B + \delta P \\ \varphi = \varphi_B + \delta\varphi \end{cases} \qquad \delta(t,\bar r) \equiv \frac{\rho(t,\bar r) - \rho_B(t,\bar r)}{\rho_B(t,\bar r)} = \frac{\delta\rho}{\rho_B} \ll 1 \qquad |\delta\bar v| \ll |\bar v_B| \qquad \delta P \ll P_B \qquad |\delta\varphi| \ll |$$

(2) Meaning of the velocity perturbation

$\bar r$ = physical coordinates
$\bar x \equiv$ comoving "
$\bar u \equiv \frac{\delta\bar v}{a}$ comoving proper velocity

$$\bar r = a\bar x \qquad \bar v = \frac{d\bar r}{dt} = \dot a\bar x + a\dot{\bar x} = \underbrace{H\bar r}_{\bar v_B} + \underbrace{a\dot{\bar x}}_{\delta\bar v}$$

$\bar\nabla \bar v_B \neq 0$ "radial"
$H\bar r = \bar v_B$ Hubble flow
$\delta\bar v = a\dot{\bar x}$ Proper motion

(3) "Transform" time derivatives accounting for comoving coordinates

$$\bar\nabla \equiv \bar\nabla_{\bar r} = \frac{\delta}{a\delta x} = \frac{1}{a}\bar\nabla_x \qquad d(a\bar x)$$

$$df(t,\bar r) = \delta_t f\, dt + \bar\nabla f\, d\bar r = \delta_t f\, dt + a\bar\nabla f\, d\bar x + \bar x\, a\bar\nabla f \frac{\dot a}{a} dt = \underbrace{\left(\delta_t + \bar x H\bar\nabla_x\right)}_{\equiv \delta_t \equiv \text{"}\cdot\text{"}\ \text{redefine}\ (*)} f dt + \bar\nabla_x f\, d\bar x$$

a) Continuity eq.

$$\delta_t\rho + \bar\nabla(\rho\bar v) = \partial_t(\rho_B+\delta\rho) + \bar\nabla\left[(\rho_B+\delta\rho)(\bar v_B+\delta\bar v)\right] = \underbrace{\delta_t\rho_B + \bar\nabla(\rho_B\bar v_B)}_{=0 \text{ continuity for background}} + \delta_t\delta\rho + \bar\nabla(\rho_B\delta\bar v + \delta\rho\bar v_B + \cancel{\delta\rho\delta\bar v}) = 0 \quad (2^{\text{nd}}\text{ order})$$

$$\simeq \delta_t\delta\rho + \rho_B\bar\nabla\delta\bar v + \underbrace{H a\bar x}_{\bar v_B}\bar\nabla\delta\rho + \delta\rho\bar\nabla\bar v_B = 0$$

$\leftarrow \delta\rho = \rho_B\cdot\delta$: $\delta_t\delta\rho = \rho_B\delta_t\delta + \delta\cdot\delta_t\rho_B = \rho_B\delta_t\delta - \delta\cdot\rho_B\bar\nabla\bar v_B$

$\leftarrow \bar\nabla\delta\rho = \rho_B\bar\nabla\delta$ $\quad \uparrow = -\bar\nabla(\rho_B\bar v_B)$ continuity

$$(*) = \cancel{\rho_B\delta_t\delta} - \cancel{\delta\rho\bar\nabla\bar v_B} + \cancel{\rho_B\bar\nabla\delta\bar v} + \cancel{\rho_B H\bar x a\bar\nabla\delta} + \cancel{\delta\rho\bar\nabla\bar v_B} = 0 \implies \boxed{\dot\delta + \bar\nabla_x\bar u = 0}$$

b) Euler eq.

$$\left(\delta_t + \bar v\bar\nabla\right)\bar v = -\frac{\bar\nabla_{\bar r}P}{\rho} - \bar\nabla\varphi \qquad \left[\delta_t + (\bar v_B+\delta\bar v)\bar\nabla\right](\bar v_B+\delta\bar v) = -\frac{\bar\nabla(P_B+\delta P)}{(\rho_B+\delta\rho)_{\text{neglect}}} - \bar\nabla(\varphi_B+\delta\varphi)$$

$$\cancel{\delta_t\bar v_B} + \delta_t\delta\bar v + \cancel{\bar v_B\bar\nabla\bar v_B} + \bar v_B\bar\nabla\delta\bar v + \delta\bar v\bar\nabla\bar v_B = -\frac{\cancel{\bar\nabla\rho_B}}{\rho_B} - \frac{\bar\nabla\delta P}{\rho_B} - \cancel{\bar\nabla\varphi_B} - \bar\nabla\delta\varphi$$

as before, background follows Euler eq.

$$\underbrace{\delta_t\delta\bar v}_{\dot{\delta\bar v}\ (3)} + \underbrace{(\bar v_B\bar\nabla)}_{(2)}\delta\bar v + \underbrace{(\delta\bar v\bar\nabla)}_{(1)}\bar v_B = -\frac{\bar\nabla\delta P}{\rho_B} - \bar\nabla\delta\varphi \qquad \bar\nabla = \frac{\bar\nabla_x}{a}\ \div a$$

$$\boxed{\dot{\bar u} + 2H\bar u = -c_s^2\frac{\rho_B\bar\nabla_x\delta}{\rho_B a^2} - \bar\nabla_x\frac{\delta\varphi}{a^2}}$$

1) $(\delta\bar v\bar\nabla)\bar v_B^i = \delta v^j\nabla_j(Hx^i) = \delta v^j H\frac{\delta x^i}{\delta x^j} = \delta v^j H = a\dot u^i H$

2) $(\bar v_B\bar\nabla)\delta\bar v = (Ha\bar x\bar\nabla)\delta\bar v = (H\bar x\bar\nabla_x)\delta\bar v$

3) $\dot{\delta\bar v} = (a\dot{\bar u}) = \dot a\bar u + a\dot{\bar u} = aH\bar u + a\dot{\bar u}$

4) $\frac{\delta P}{\delta\rho} = c_s^2$ (eq. of state) $\quad \delta\rho = \rho_B\delta$

c) Poisson eq. $\quad \bar\nabla^2\varphi = 4\pi G\rho \qquad \bar\nabla^2(\cancel{\varphi_B}+\delta\varphi) = 4\pi G(\cancel{\rho_B}+\delta\rho)$

$$\boxed{\bar\nabla_x^2\delta\varphi = 4\pi G a^2\rho_B\delta}$$

- From now on, $\bar{\nabla} \equiv \bar{\nabla}_x$

$$\dot{\delta} + \bar{\nabla}\bar{u} = 0 \qquad \text{Continuity eq.}$$
$$\dot{\bar{u}} + 2H\bar{u} = -c_s^2 \frac{\bar{\nabla}\delta}{a^2} - \bar{\nabla}\frac{\delta\varphi}{a^2} \qquad \text{Euler eq.}$$
$$\bar{\nabla}^2 \delta\varphi = 4\pi G a^2 \rho_B \delta \qquad \text{Poisson eq.}$$
$$\frac{\delta P}{\delta \rho} = c_s^2 \qquad \text{eq. of state } P = P(\rho)$$

6 equations for 6 variables: $\delta, \bar{u}, P, \delta\phi$

closed system of eq.

⇓

Solve it

δ_ϵ(continuity): $\ddot{\delta} + \bar{\nabla}\dot{\bar{u}} = 0$

$\bar{\nabla}$(Euler): $\bar{\nabla}\dot{\bar{u}} + 2H\bar{\nabla}\bar{u} = -c_s^2\frac{\bar{\nabla}^2\delta}{a^2} - \frac{\bar{\nabla}^2\delta\varphi}{a^2} = -c_s^2\frac{\bar{\nabla}^2\delta\rho}{\rho_B a^2} - 4\pi G\rho_B\delta$

(Poisson; continuity)

$$\ddot{\delta} + 2H\dot{\delta} = c_s^2\frac{\bar{\nabla}^2\delta}{a^2} + 4\pi G\rho_B\delta$$

Density perturbation eq.

- H acts as a dumping term: expansion affects structure formation!
- Gravity and Pressure are the source term
  they depend on the fluid, e.g. DM is pressureless

- Decompose in plane waves i.e. go to Fourier space

$$\delta(t,\bar{x}) = \int \frac{d^3k}{(2\pi)^3}\hat{\delta}(t,\bar{k})\, e^{-i\bar{k}\bar{x}}$$

Ansatz for the solution

$\hat{\delta}_k$ fourier mode, spatial derivatives become $\bar{\nabla} \to -i\bar{k}$

$$\bar{\nabla}\delta(t,\bar{x}) = \int \frac{d^3k}{(2\pi)^3}(-i\bar{k})\hat{\delta}(t,\bar{k})\, e^{-i\bar{k}\bar{x}}$$

$\equiv -\omega_k^2$

$$\ddot{\hat{\delta}}_k + 2H\dot{\hat{\delta}}_k = \left(-\frac{c_s^2k^2}{a^2} + 4\pi G\rho_B\right)\hat{\delta}_k$$

For matter components

$$\ddot{\hat{\delta}}_k + 2H\dot{\hat{\delta}}_k = \left(-\frac{c^2k^2}{3a^2} + \frac{32\pi}{3}G\rho_B\right)\hat{\delta}_k$$

For relativistic fluid

here, we used: $P = \frac{1}{3}\rho c^2 \Rightarrow c_s^2 \equiv \frac{P}{\rho}$ $\quad c_s = \frac{c}{\sqrt{3}}$ (radiation domination)

$\bar{\nabla}(\rho\bar{v}) \to \bar{\nabla}\left[\left(\rho + \frac{P}{c^2}\right)\bar{v}\right] = \frac{4}{3}\bar{\nabla}(\rho\bar{v})$ in continuity eq.

$\rho \to 2\rho$ in Poisson eq.

- in the linear regime, $\hat{\delta}_k$ modes are decoupled:
  - ⇒ they evolve independently from each other
  - ⇒ No power transfer to $\neq$ modes
    i.e. shape of power spectrum will not change $P(k) \propto k^{n_s}$ $n_s = 0,96$

Simple case: static universe: H=0

- Matter domination

$\ddot{\hat{\delta}}_k + \omega_k \delta_k = 0 \qquad \omega_k \equiv \sqrt{\frac{c_s^2 k^2}{a^2} - 4\pi G \rho_B}$ oscillator -> solution $\hat{\delta}_k \propto e^{i\omega_k}$

$\omega_k \in \mathbb{R}$: if $\frac{c_s^2 k^2}{a^2} \geq 4\pi G \rho_B \Rightarrow k \geq \frac{a}{c_s}\sqrt{4\pi G \rho_B} \equiv k_J \qquad \lambda_J \equiv \frac{2\pi}{k_J} = \frac{c_s}{a}\sqrt{\frac{4\pi^2}{4\pi G \rho_B}}$ Jeans scale

$\lambda \leq \lambda_J$ harmonic oscillations (No collapse!)

$\omega_k \in \mathbb{C}$: $\lambda > \lambda_J$ growth / decay

with $\omega_k = a + ib \quad b \neq 0 \qquad \hat{\delta}_k \propto e^{ia} \cdot e^{b} \qquad b > 0$ damped oscillations

- Radiation domination

$\omega_k = \sqrt{-\frac{c^2 k^2}{3a^2} + \frac{32\pi}{3} G \rho_B} \doteq 0 \Rightarrow k_J = \frac{a}{c}\sqrt{32\pi G \rho_B}$

$\lambda_J = \frac{2\pi}{k_J} = \frac{c}{a}\sqrt{\frac{4\pi^2}{32\pi G \rho_B}} = \frac{c}{2a}\sqrt{\frac{\pi}{2G\rho_B}} = \frac{c}{2a}\sqrt{\frac{\pi 2\pi G}{2G 3H^2}} = \frac{\pi}{\sqrt{3}}\frac{c}{H} \simeq R_H$ ($\pi \approx 1$)

=> growth is forbidden within the Hubble horizon!

-> as baryons are coupled to radiation, their fluctuations will not grow either!

Photons: $w = \frac{1}{3} \Rightarrow c_s^2 = \frac{P}{\rho} = \frac{1}{3}c^2 \qquad \lambda_J \simeq R_H \Rightarrow$ no growth within the horizon

Baryons: $c_s > 0 \Rightarrow (\lambda > \lambda_J)$ growth or $(\lambda < \lambda_J)$ Baryonic Acoustic Oscillations (BAO)

before decoupling with radiations $c_s \sim c \Rightarrow$ baryons perturbations are dumped out

after " " " $c_s \ll c \Rightarrow$ they can fall in DM potential well that could grow because not coupled to photons

Dark Matter: $w = 0 \Rightarrow P = 0 \Rightarrow c_s^2 = \frac{P}{\rho} = 0 \Rightarrow \lambda_J = 0$ growth on all scales

$\Rightarrow c_s$ has no meaning... but: $\lambda_J = \langle v^{-2} \rangle^{-1/2} \sqrt{\frac{\pi}{G\rho_B}}$

You will get virialized structures: Dark Matter Haloes, later ....

- Cold Dark Matter (CDM): $v \sim 0 \Rightarrow \lambda_D = 0 \Rightarrow$ "all" scales can collapse
- Warm " " (WDM) ....
- Hot Dark Matter (HDM) v larger (relativistic) $\lambda_D > 0 \Rightarrow$ the spectrum is truncated ("smaller" structures can not form)

WDM and HDM are ruled out by observations

- The Growth Factor: D

▫ Assume

$\lambda \gg \lambda_J$ or $c_s = 0$ (pressureless fluid) — for λ large, pressure is negligible

$\Omega = 1$ (flat geometry) and $\Omega_\Lambda = 0$ ("early times") $\Rightarrow H^2 = \frac{8\pi G}{3}\rho_B$ Friedman eq.

▫ Apply assumptions

M: $\ddot{\hat\delta}_k + 2H\dot{\hat\delta}_k = \left(-\frac{c_s^2 k^2}{a^2} + 4\pi G \rho_B\right)\hat\delta_k$ $\quad \ddot{\hat\delta}_k + 2H\dot{\hat\delta}_k = 4\pi G\,\hat\delta_k \frac{3H^2}{8\pi G}$ $\quad \ddot{\hat\delta}_k + 2H\dot{\hat\delta}_k = \frac{3}{2}H^2\hat\delta_k$ (1)

R: $\ddot{\hat\delta}_k + 2H\dot{\hat\delta}_k = \frac{32}{3}\pi G\,\hat\delta_k \frac{3H^2}{8\pi G}$ $\quad \ddot{\hat\delta}_k + 2H\dot{\hat\delta}_k = 4H^2\hat\delta_k$

▫ Solve eq.

· Matter domination: $\rho_B = \rho_{Matter}$ $H = \frac{2}{3t}$ ansatz $\hat\delta \propto t^n$

$n(n-1)t^{n-2} + 2\cdot\frac{2}{3t}nt^{n-1} = \frac{3}{2}\frac{4}{9t^2}t^n$ $\quad n^2 + \frac{1}{3}n - \frac{2}{3} = 0$ $\quad n_\pm = -1, \frac{2}{3}$

· Radiation domination: $\rho_B = \rho_R$ $H = \frac{1}{2t}$

$n(n-1)t^{n-2} + 2\frac{1}{2t}nt^{n-1} = 4\frac{1}{4t^2}t^n$ $\quad n^2 - 1 = 0$ $\quad n_\pm = 1, -1$

$\Rightarrow$ $\hat\delta_k \propto \begin{cases} a \\ a^{-3/2} \end{cases}$ Matter domination era

$\hat\delta_k \propto \begin{cases} a^2 \\ a^{-1} \end{cases}$ Radiation " "

decaying modes are irrelevant

▫ For $\Omega \neq 1$ and $\Omega_\Lambda > 0$ slowlier evolution at late time

$\delta(a) = \delta_0 D_+(a)$ $\quad \boxed{D_+(a) = \frac{5a}{2}\Omega_m\left[\Omega_m^{4/7} - \Omega_\Lambda + \left(1+\frac{\Omega_m}{2}\right)\left(1+\frac{\Omega_\Lambda}{70}\right)\right]}$ Excellent approximation

Velocity perturbations

Back to the linearized equations (matter domination)

$\dot{\delta} + \bar{\nabla}\bar{u} = 0$ Continuity eq.

$\dot{\bar{u}} + 2H\bar{u} = -\frac{c_s^2}{a^2}\bar{\nabla}\delta - \bar{\nabla}\frac{\delta\varphi}{a^2}$ Euler eq. ← Neglect pressure, i.e. consider large scales

$\bar{\nabla}^2\delta\varphi = 4\pi G a^2 \rho_B \delta$ Poisson eq.

A numerical solution is needed but we can make 2 simple considerations:

1) velocity fields point in the direction given by $-\bar{\nabla}\varphi$
2) amplitude $u(t)$ is time dependent only (the growth factor is $D(t)$)

$\Rightarrow \boxed{\bar{u}(\bar{x},t) = -u(t)\bar{\nabla}\delta\varphi}$

$\bar{\nabla}\bar{u}(\bar{x},t) = -u(t)\bar{\nabla}^2\delta\varphi = -u(t)\,4\pi G a^2 \delta$) Poisson eq

$H\frac{\delta\delta}{\delta\ln a} = u(t)\,4\pi G a^2 \rho_B \delta = u(t)\frac{3}{2}H^2\Omega_m a^2\delta$

) Continuity eq. $\bar{\nabla}\bar{u} = -\dot{\delta} = -\frac{\delta\delta}{\delta a}\dot{a} = -H\frac{\delta\delta}{\delta\ln a}$

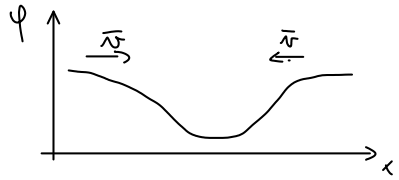
) $4\pi G\rho_0 = \frac{3H^2}{2}\Omega_m = \frac{3}{2}H^2\Omega_m$

) growth factor $\delta = \delta_0 D_+(a)$

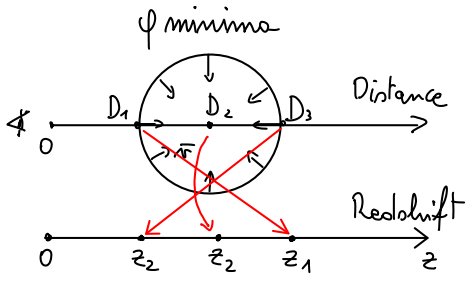
$\frac{\delta\ln D_+}{\delta\ln a} = \frac{3}{2}H\Omega_m a^2 u(t)$

$u(t) = \frac{2f(\Omega_m)}{3H\Omega_m a^2}$ $f(\Omega_m) \equiv \frac{\delta\ln D_+}{\delta\ln a} \simeq \Omega_m^{0.6}$ very good approximation

$\Rightarrow \boxed{\bar{u}(\bar{x},t) = -\frac{2f(\Omega_m)}{3H\Omega_m a^2}\bar{\nabla}\delta\varphi}$ $\delta\bar{v} = a\bar{u}$



in-flow toward potential minima



$D_1 < D_2 < D_3$ but.... $z_3 > z_2 > z_1$!

↑ Hubble flow

Redshift space distortion

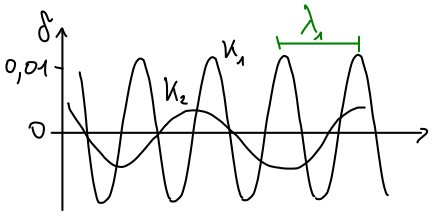
Fingers of God

in the non-linear regime

- Vortical solutions with $\bar{\nabla}\bar{u} = 0$ are $\bar{u} = \bar{\nabla}\times\bar{a}$ occurs only where $\delta = 0$

Quantifying the density fluctuaitons

- Recall:
  - Field $\delta = \frac{\rho - \rho_B}{\rho_B}$ $\quad \langle \delta \rangle = 0$ by construction
    decomposed in k-modes δ_k



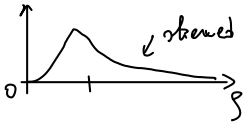
Total δ field as superposition of $\hat{\delta}_k$ modes
In the linear regime, $\hat{\delta}_k$ modes are decoupled

- δ, v, ... are random variables, therefore:
  - we know how they evolve
    eg. $\delta(a, \vec{x}) = \delta_0(\vec{x}) \cdot a$ (matter domination, linear regime) $\Rightarrow$ if $\delta_0 = 0$ at a certain $\vec{x}$ it will remain 0
    $\delta_0 > 0$ " " " " it will grow
    $\delta_0 < 0$ " " " " " " decrease
    valid until $|\delta| \sim 1$ (linear regime)
  - but we do not know the value in a certain position $\vec{x}$

$\Rightarrow$ Need a statistical approach

do not study $\delta(x, y, z)$, study moments: mean, variance, skewness, ...

- mean is not interesting: $\langle \delta \rangle = 0$ by construction
- variance as a function of separation $|\vec{r}|$ $\quad \xi(r) \equiv \langle \delta(\vec{x}) \delta(\vec{x} + \vec{r}) \rangle_{\vec{x}}$ Correlation function
  " " " " " mode $|\vec{k}|$ $\quad P(k)$ Power spectrum
- higher momentums are necessary only if the field is not Gaussian
- As the field evolves it will become non-Gaussian because $\rho \in [0, \infty[= [$"voids", "galaxies", black hole$]$
  ρ can become 0 (void), ρ can grow arbitrarily



$$\left(\hat{\delta}(t, \vec{k}) = \int_{-\infty}^{\infty} d^3x \, \delta(t, \vec{x}) e^{i \vec{k} \vec{x}} \quad \text{Fourier transform} \right)$$

$$\left(\delta(t, \vec{x}) = \int_{-\infty}^{+\infty} \frac{d k^3}{(2\pi)^3} \hat{\delta}(\vec{k}, t) e^{-i \vec{k} \vec{x}} \quad \text{back Fourier transform} \right)$$

Correlation function and Power spectrum

- Correlation function

$$\xi(x,t) \equiv \langle \delta^*(\bar{x},t)\,\delta(\bar{x}+\bar{r},t)\rangle = \left\langle \int \frac{dk^3}{(2\pi)^3}\,\hat{\delta}^*(\bar{k},t)\,e^{i\bar{k}\bar{x}} \cdot \int \frac{dk'^3}{(2\pi)^3}\,\hat{\delta}(\bar{k}',t)\,e^{-i\bar{k}'(\bar{x}+\bar{r})}\right\rangle$$

$\delta = \delta^*$ because $\delta \in \mathbb{R}$

$$= \frac{1}{(2\pi)^6}\int dk^3 \int dk'^3 \,\langle \hat{\delta}^*(\bar{k},t)\,\hat{\delta}(\bar{k}',t)\rangle\, e^{i(\bar{k}\bar{x}-\bar{k}'\bar{x}-\bar{k}'\bar{r})}$$

$$= \frac{1}{(2\pi)^6}\int dk^3 \int dk'^3 \,(2\pi)^3 P(\bar{k},t)\,\delta_D(\bar{k}-\bar{k}')\, e^{i(\bar{k}\bar{x}-\bar{k}'\bar{x}-\bar{k}'\bar{r})}$$

$$= \frac{1}{(2\pi)^3}\int dk^3\, P(\bar{k},t)\, e^{-i\bar{k}'\bar{r}}$$

Wiener-Kitching Theorem ("Parseval")

- Power spectrum

$$\langle \hat{\delta}^*(\bar{k},t)\,\hat{\delta}(\bar{k}',t)\rangle \equiv (2\pi)^3 P(k,t)\,\delta_D(\bar{k}-\bar{k}')$$

- uncorrelated modes
- power of each mode, $k=|\bar{k}|$ not $\bar{k}$ isotropy
- expectation value over all volumes (ensambles)
- ergodic principle : $\neq$ volumes are statistically equivalent

- A convenient expression: $\xi(r) \overset{\text{isotropy}}{=} 2\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \int_0^\pi \sin\theta\, d\theta\, e^{-ikr\cos\theta} = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \frac{\sin(kr)}{kr}$

- "The way around": $P(k) = \int d^3x\, \xi(x)\, e^{i\bar{k}\bar{x}} = 2\pi \int_0^\infty \xi(x)\, x^2 dx \int_0^\pi e^{ikx\cos\theta} \sin\theta\, d\theta = 4\pi \int_0^\infty x^2 \frac{\sin(kx)}{kx}\,\xi(x)\,dx$

isotropy => spherical coord.

The correlation function is the back Fourier transform of the Power spectrum
" Power spectrum " " Fourier transform " " correlation function

- Probability distribution of the modes

$p_k(\hat{\delta})$ = probability to have a fluctuation $\hat{\delta}_k$ in a $d\hat{\delta}$ bin

$$P(\bar{k},t) = \frac{1}{(2\pi)^3}\langle \hat{\delta}^2(\bar{k},t)\rangle = \int_{-\infty}^{\infty} d\hat{\delta}\; p_k(\hat{\delta}_k)\,\hat{\delta}^2(\bar{k},t)$$

if p_k is Gaussian => $P(k)$ stores all information (if Gaussianity)
No need of tri-spectrum $\langle \delta(k)\delta(k')\delta(k'')\rangle = 0$
" " " quadri ....

Normalization of the power spectrum and σ_8

- $\sigma_R^2 \equiv \int_{-\infty}^{\infty} \frac{dk^3}{(2\pi)^3} P(k) \hat{W}_R^2(k)$ $\quad$ window function $\hat{W}$:
  $= 4\pi \int_0^{\infty} \frac{k^2 dk}{(2\pi)^3} P(k) W_R^2(k)$
  - normalized such that: $\int dx^3 W(\vec{x}) \equiv 1$
  - "selects / weights" certain modes
  - used to have a finite variance because $\int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^2} P(k) = \infty$, e.g. $P(k) \propto k^{0.96}$ ($P(k) > 0$)

- variance of a convolved field $\delta_c(\vec{x})$, in fact:

$$\delta_c(\vec{x}, t) = \int \delta(\vec{x}, t)\, W_R(|\vec{x} - \vec{x}'|)\, d^3x' \quad \Longleftrightarrow \quad \hat{\delta}_c(\vec{k}, t) = \hat{\delta}(\vec{k}, t)\, \hat{W}_R(k)$$

convolution of δ in real space — convolution in Fourier space

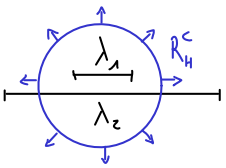
- σ_R measures the "roughness" of the density field

- Cosmological parameter σ_8

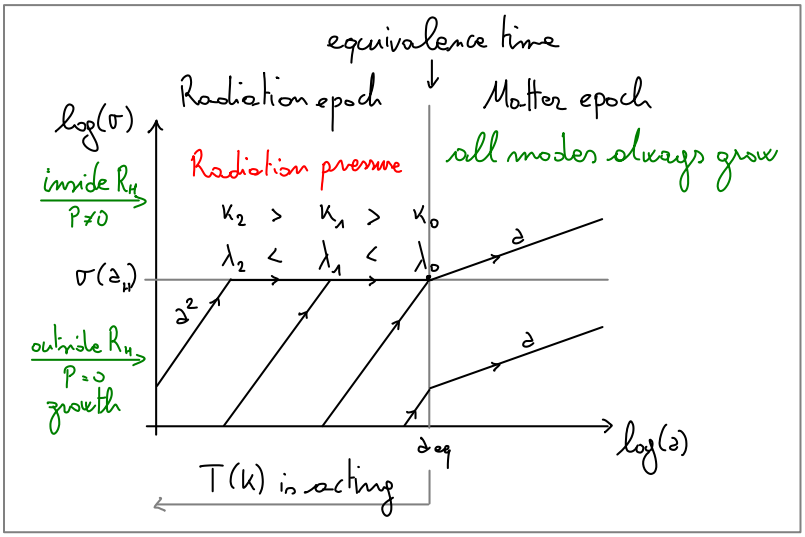
W_R = Top hat function, $\boxed{R = 8\ Mpc/h}$ $\Rightarrow$ $\boxed{\sigma_8}$

CDM power spectrum

- During radiation and matter domination, no accelerated expansion $\Rightarrow$ R_H^c grows
- Modes with $\neq \lambda$ enter the horizon at $\neq$ times
- shorter modes $\hat{\delta}_k$ (i.e. larger k) enter R_H earlier



- when $\lambda > R_H^c$ any time = growth
- when $\lambda < R_H^c$ before equivalence time $\Rightarrow$ growth stops because of radiation Pressure $P \neq 0$
- " " after " always grow (matter domination, DM, $P=0$)



	$a < a_{eq}$	$a > a_{eq}$
$\lambda < R_H$	$\hat{\delta}_k \propto$ const	$\hat{\delta}_k \propto a$
$\lambda > R_H$	$\hat{\delta}_k \propto a^2$	$\hat{\delta}_k \propto a$
$R_H = \frac{c}{H}$	$R_H \propto a^{3/2}$	$R_H \propto a^2$

- λ_0 threshold for continuous growth if $\lambda > \lambda_0$

$$\lambda_0 = R_H(a_{eq}) \equiv \frac{c}{H} = \frac{c}{H_0(\Omega_{R0} a_{eq}^{-4} + \Omega_{M0} a_{eq}^{-3})^{1/2}} = \frac{c}{H_0} \frac{a_{eq}^{3/2}}{\sqrt{2\Omega_{M0}}}$$

$$\Omega_{R,0} a_{eq}^{-4} = \Omega_{M,0} a_{eq}^{-3}$$

$$k_0 \equiv \frac{2\pi}{\lambda} = 2\pi \frac{H_0}{c} \sqrt{\frac{2\Omega_{M0}}{a_{eq}}} = 2\pi \frac{H_0}{c} \Omega_{M,0} \sqrt{\frac{2}{\Omega_{R0}}}$$

k_0 is known from observations: $H_0, \Omega_{M0}, \Omega_{R0}$ (!!)

- Each scale is affected differently according to k

this is quantified by a filter: the transfer function

• Transfer function $T(k)$

- $T(k)$ quantifies the modification of the initial power spectrum due to entrance in R_H
- Initial power spectrum $P_i(k) \propto k^{ns}$ (from inflation) $n_s = 0,96 \approx 1$
- Today " " $P(k) = P_i(k)\, T^2(k)$ T = transfer function (filter)

▫ When does a k mode enter the horizon?

$$\lambda \cdot a_{enter} = \frac{2\pi}{k} a_{enter} = R_H \propto \begin{cases} a^2_{enter} & (\text{for } a_{enter} < a_{eq}) \\ a^{3/2}_{enter} & (\text{for } a_{enter} > a_{eq}) \end{cases} \Rightarrow a_{enter} \propto \begin{cases} k^{-1} & \text{small } (k \text{ large}) \text{ earlier} \\ k^{-2} & \end{cases}$$

Radiation domination

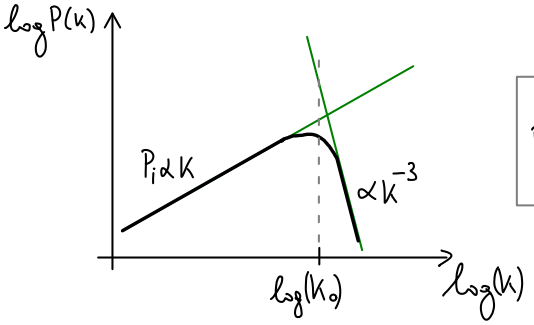
matter domination

▫ Relative suppression of growth

for $a < a_{eq}$ and $\lambda < \lambda_0$ $\quad T(k) = \left(\frac{a_{enter}}{a_{eq}}\right)^2 = \left(\frac{k_0}{k}\right)^2$ $\quad \hat{\delta}_k = \hat{\delta}_{ki}\, a^2_{enter} = \hat{\delta}_{ki} \cdot k^{-2}$ $\Rightarrow$ $\boxed{P(k) = P_i(k)\, k^{-4}}$

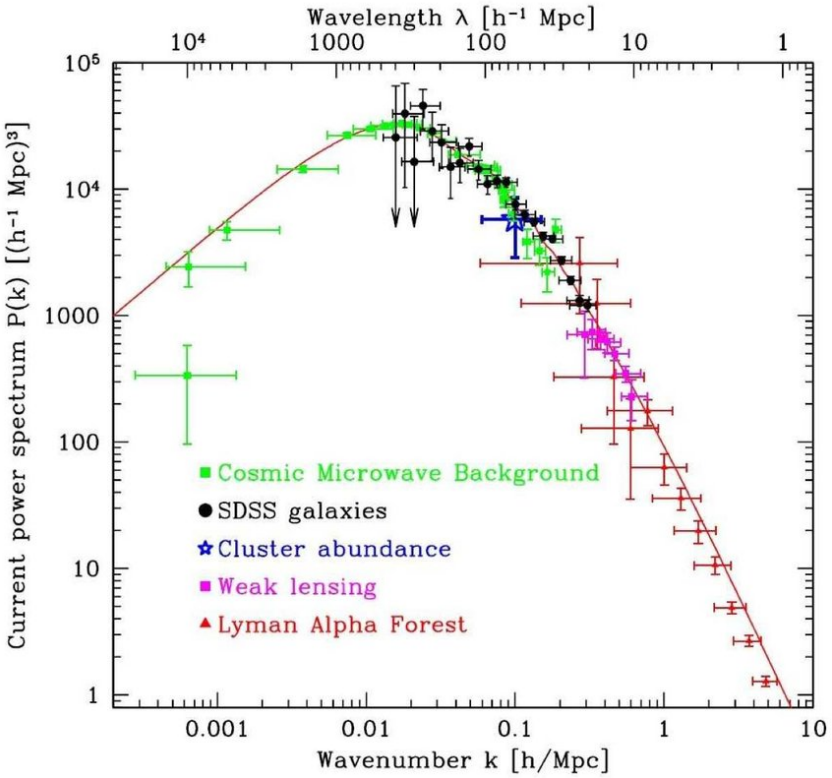
$\hat{\delta}_{ki} a^2_{enter}$ grows up to

k



$$P(k) \propto \begin{cases} k < k_0 \\ k^{-3} \gg k_0 \end{cases}$$

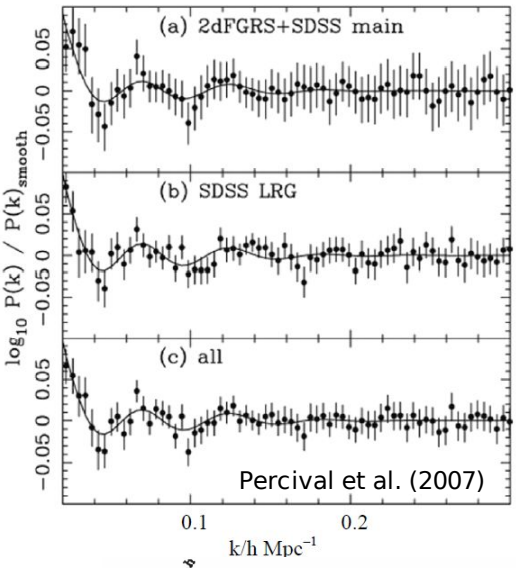
CDM power spectrum

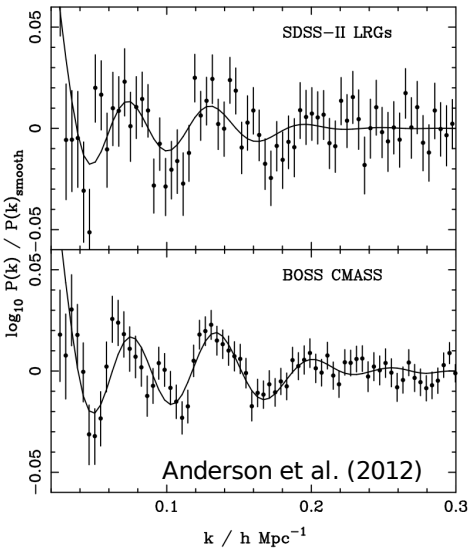


Tegmark et. al (2004)

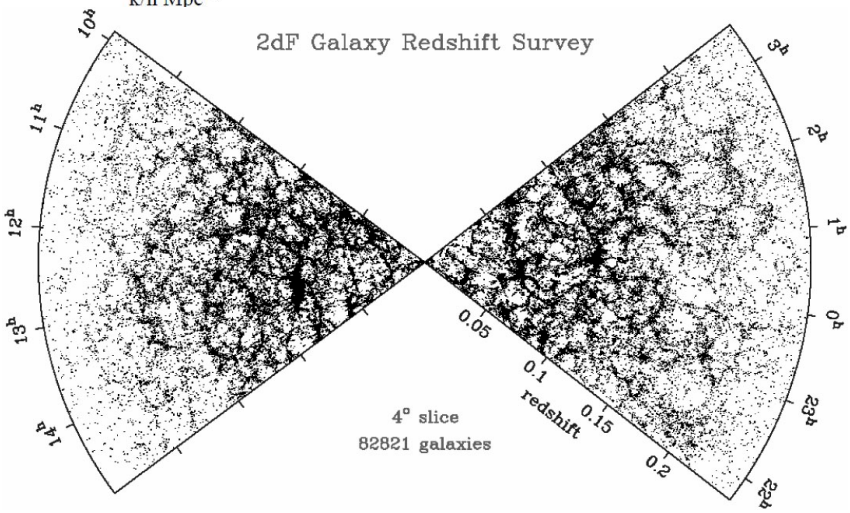
- Actual spectrum Need to account for :

(1) BAOs

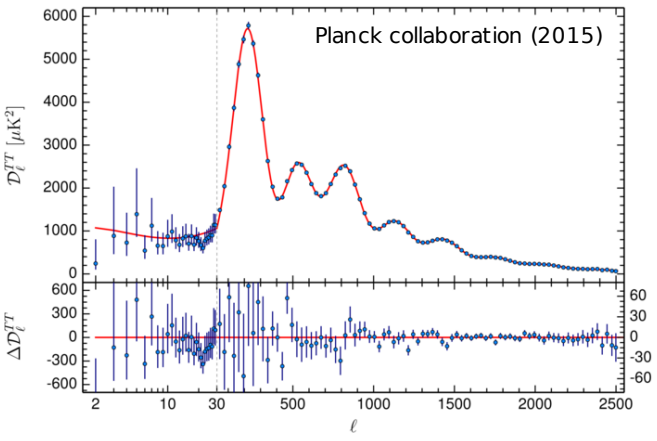


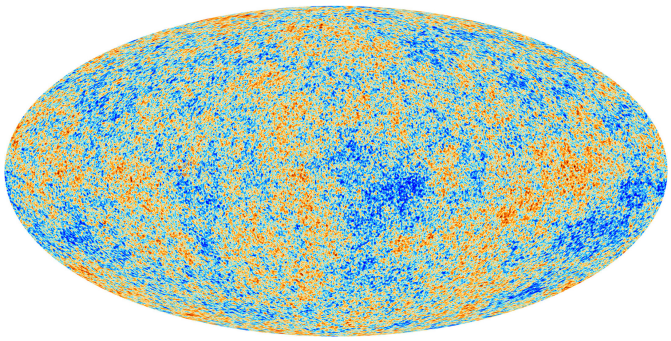


Spectroscopic galaxy surveys →



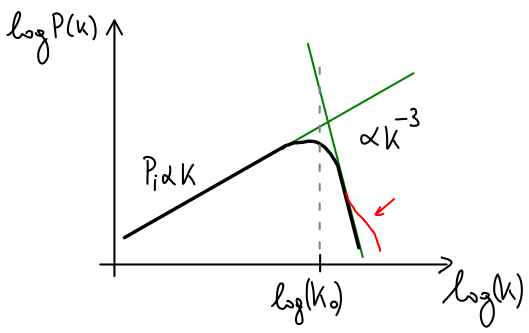
CMB temperature fluctuations ↓





Planck collaboration (2015)

(2) Non-linear growth at large k



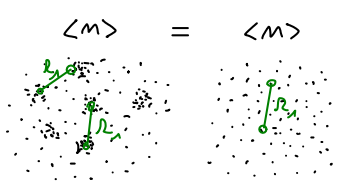
Correlation function from galaxy surveys

- Observationally: Use galaxies as density tracers $\rho(\vec{x},t) \to n(\vec{x},t)$
- Consider 2 volumes separated by r: n_1, dV_1 — r — n_2, dV_2
- If random distribution (uncorrelated field):



  - given a galaxy, the probability to find another galaxy separated by r is the same $\forall r$
  - probability to find one galaxy in dV_1 and one in dV_2 is: $dP = n dV_1 \cdot n dV_2 = n^2 dV_1 dV$

- For non random distributions: (correlated field)
  there is an excess of probability on certain scales
  Pairs separated by r_1 are more likely than random



$$dP = \langle n_1 dV_1 \; n_2 dV_2 \rangle = \langle n(1+\delta_1) dV_1 \; n(1+\delta_2) dV_2 \rangle \qquad n(\vec{x}_i) = n(1+\delta(\vec{x}_i))$$

$$= n^2 \langle (1+\delta_1)(1+\delta_2) \rangle dV_1 dV_2$$

$$= n^2 (1 + \cancel{\langle \delta_2 \rangle}^{=0} + \cancel{\langle \delta_1 \rangle}^{=0} + \langle \delta_1 \delta_2 \rangle) dV_1 dV_2$$

$$= n^2 [1 + \xi(r)] dV_1 dV_2$$

↦ Correlation function = excess of probability for scale r with respect to random !!

- Volume limited survey
  - Count number of pairs of galaxies to estimate $\xi(r)$ in data: $\langle DD \rangle$
  - compare it with expected number for a randomize sample: $\langle RR \rangle$

$$\Rightarrow \quad \boxed{\xi = \frac{\langle DD \rangle}{\langle RR \rangle} - 1}$$

The "random" is generated with brute force to account for masked areas
(you can not observe where there are bright stars)

The bias factor

- Caveat: δ (real density contrast) but we use δ_g (visible galaxies!)... they might be different!
  ↳ assume: $\frac{\delta n}{n}(\vec{x}) \equiv \delta_g(\vec{x}) = b \cdot \delta(\vec{x}) = \delta(\vec{x}) + (b-1)\delta(\vec{x})$ b = bias factor (a rescaling)
  b can be measured via velocity measurements

- bias might be z dependend on z and even the color of galaxies....
  red galaxies are more clustered than the blue ones
- You can do the same with the clustering of galaxy clusters

Spherical collapse and dark matter haloes

- Matter dominated spherical overdensity : It behaves as a universe on its own
- A closed "universe" collapses : matter in universe "organized" in Dark-Matter Haloes

Condition for the collapse of a perturbation

$$\rho_p(t_i) = \rho(t_i)(1+\delta_i) \quad \div \rho_{cr}(t_i) \qquad \Omega_p(t_i) = \Omega(t_i)(1+\delta_i) > 1 \quad \Rightarrow$$

collapse !

linear criteria

$$\boxed{\delta_i > \frac{1-\Omega(t_i)}{\Omega(t_i)}}$$

t_i = time of collapse — perturbation — universe

if $\begin{cases} 1-\Omega(t_i) < 0 & (\Omega > 1) \\ 1-\Omega(t_i) > 0 & (\Omega < 1) \end{cases}$ but $\delta_i \geq 0$ by construction $\Rightarrow$ collapse is always possible

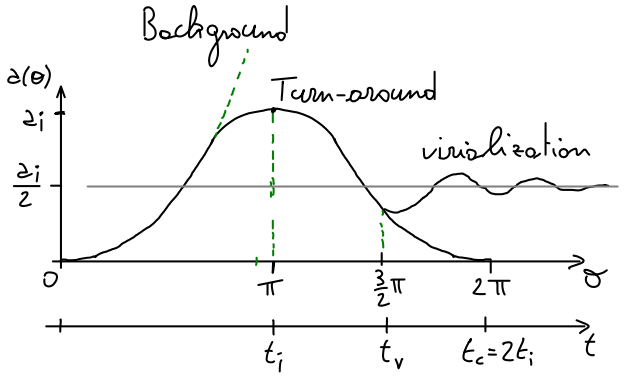
$\Rightarrow$ there is a minimum δ_i for collapse to happen

Time and radious of region at collapse

Solve Friedmann eq. for matter domination

$$a(\theta) = \frac{a_0}{2}\frac{\Omega_{p0}}{(\Omega_{p_0}-1)}(1+\cos\theta)$$

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_{p0}}{(\Omega_{p0}-1)^{3/2}}(\theta - \sin\theta)$$



Collapse starts at $a = max$, i.e $\theta = \pi$

$$a_i = a_0\frac{\Omega_{p0}}{(\Omega_{p_0}-1)} \rightarrow (1) \quad \left(\frac{a_i}{a_0}\right) = \frac{\Omega_{p0}}{(\Omega_{p_0}-1)}$$

$$t_i = \frac{\pi}{2H_0}\frac{\Omega_{p0}}{(\Omega_{p0}-1)^{3/2}} \rightarrow (2) \quad \Omega_p = \frac{2H_0}{\pi}(\Omega_{p0}-1)^{3/2}\, t_i$$

Overdensity at collapse

$$\rho_p(t_i) = \rho_{p0}\left(\frac{a_i}{a_0}\right)^{-3} \overset{(1)}{=} \cancel{\Omega_{p0}}\,\rho_{cr0}\cdot\frac{\Omega_{p0}^{\cancel{-3}\,-2}}{(\Omega_{p_0}-1)^{-3}} = \rho_{cr0}\frac{\pi^2(\Omega_{p_0}-1)^3}{4H_0^2(\Omega_{p0}-1)^3 t_i^2} = \frac{3\cancel{H_0^2}}{8\pi G}\frac{\pi^{\cancel{2}}}{4\cancel{H_0^2}t_i^2} = \frac{3\pi}{32Gt_i^2}$$

Density at virialization

$$E_i = E_{pot\,i} = \frac{GM^2}{R_i}$$

$$E_v = E_{pot\,v} + E_{kv} = E_{pot\,v} - \frac{1}{2}E_{pot\,v} = \frac{1}{2}\frac{GM^2}{R_v}$$

virial theorem $E_{pot\,v} + \frac{1}{2}E_{kv} = 0$

at collapse, virialization

$$\Rightarrow E_i = E_v \qquad R_v = \frac{1}{2}R_i$$

(!!)

$$\boxed{\rho_p(t_v) = 8\,\rho_p(t_i)}$$

R^3

• Evolution of the overdensity

▫ Background

at turn-around t_i $\quad \rho_{Bi} = \rho_{m,0}\, a_i^{-3}$ $\qquad a_i = \left(\frac{3}{2} H_0 t_i\right)^{2/3} = \left(\frac{3}{2} H_0 \frac{\pi}{2 H_0} \frac{\Omega_{p0}}{(\Omega_{p0}-1)^{3/2}}\right)^{\frac{2}{3}}$ (1)

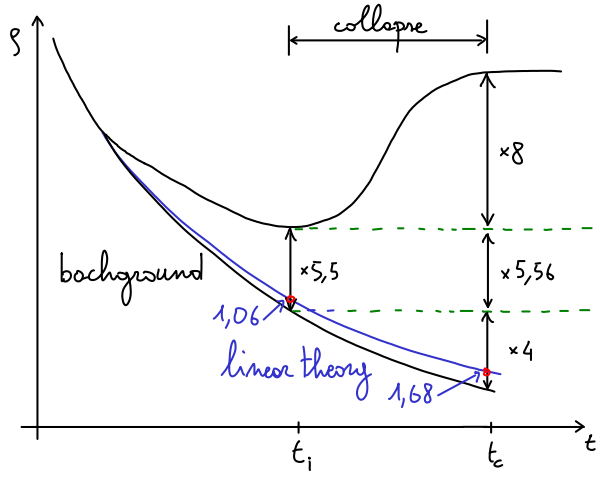
▫ Overdensity $\delta = \frac{\rho_i}{\rho_B} - 1 \equiv \chi - 1$

- at turnaround: t_i $\qquad \chi(t_i) = \frac{\rho_{pi}}{\rho_{Bi}} = \frac{\rho_{m0}(\Omega_{p0}-1)^3}{\Omega_{p0}^2} \frac{1}{\rho_{m0}} \cdot \left(\frac{3\pi}{4} \frac{\Omega_{p0}}{(\Omega_{p0}-1)^{3/2}}\right)^2 = \frac{9\pi^2}{16} \simeq 5{,}56$

- at collapse: t_c $\qquad \boxed{\chi_c = 8\, \chi_i \cdot 2^2 \simeq 178}$

from density at virialization ↑ ↳ because of further drop in $\rho_B \downarrow$ $t_c = 2t_i$ in eq. (1)

- at virialization: t_v $\qquad \chi \simeq 145$ $\qquad$ "this value is never used



Linear theory gives very wrong values at t_i, t_c
but it can still be used to set "land marks"

• Linear criteria for collapse

- Trick: map the timing of the spherical collapse on the one given by the linear theory

=> at turn around t_i: $\boxed{\delta_{i\,lin} \simeq 1{,}06}$ when structure begins to collapse (wrong by a factor of 4,5)

=> at collapse t_c: $\boxed{\delta_{c\,lin} \simeq 1{,}69}$ " " is collapsed (against the non linear 180)

↳ very small dependency on cosmological parameters

Including $\Omega_{\Lambda 0} = 1 - \Omega_{m0}$ (flat): $\delta_c = 1{,}69 \cdot (1 + 0{,}0123 \log_{10} \Omega_m)$

=> $\delta_{c\,lin} \simeq 1{,}69$ can use this number for all cosmologies

- Advantage: link to statistic of random fields

=> Power spectrum quantifies how many regions reach $\delta_{c\,lin}(\vec{x}, \epsilon)$

=> derive Mass function of dark matter haloes

Press-Schechter mass function

▫ Assumptions:

(1) Direct collapse (no merging between haloes)

(2) Use linear perturbation theory:

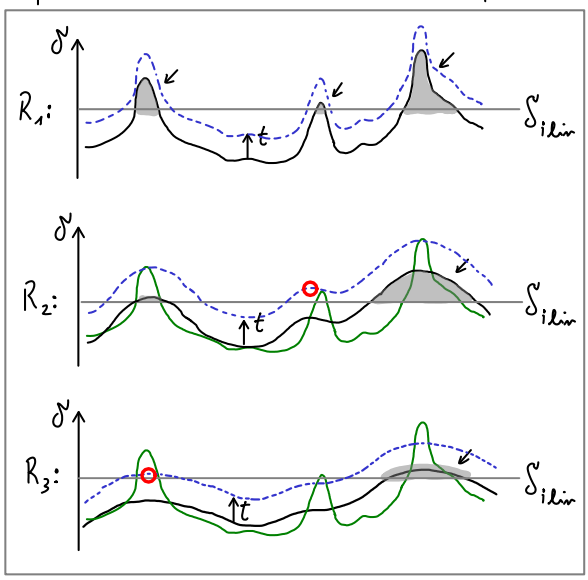
↳ Random density field $\delta(\vec{x}, t)$

↳ Linear criteria for collapse $\delta(\vec{x}, t) > \delta_{i\,lin}$

↳ Relate mass to scales $M = \frac{4\pi}{3}\rho_B R^3$ $\quad \sigma_R^2 = 4\pi \int \frac{k^2 dk}{(2\pi)^3} P(k) \hat{W}_R^2(k)$

Top hat W_R to link k (i.e. scale λ) to M within R

▫ Different scales cross $\delta_{i\,lin}$ at different times



- $\delta \to \delta_R$ convolved
- e.g. 3 scales: $R_1 < R_2 < R_3$, $k_1 > k_2 > k_3$, $\sigma_1 < \sigma_2 < \sigma_3$
- $t \uparrow$ time evolution: $\delta_R = \delta_{oR} D_+(t)$ (linear growth), $\sigma_R(z) = \sigma_R D_+(z)$
- smaller structures form first
  - ■ cross $\delta_{i\,lin}$ first, ○ cross at a later time

↳ Hierarchical structure formation (!)

• Probability distribution to have a certain δ_R

$$p_R(\delta, z) = \frac{1}{\sqrt{2\pi}\,\sigma_R(z)} \exp\left(\frac{-\delta^2}{2\sigma_R^2(z)}\right) \quad \text{(Gaussianity)}$$

• Probability to exceed $\delta_{i\,lin}$

$$F(R, z) = \int_{\delta_{i\,lin}}^{\infty} p_R(\delta, z)\, d\delta = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\delta_{i\,lin}}{\sqrt{2}\,\sigma_R(z)}\right) \qquad \mathrm{erfc}: \text{complementary error function}$$

• Distribution of haloes over mass

use: $\frac{d}{dM} = \frac{d}{d\sigma_R}\frac{d\sigma_R}{dM}$ and $\frac{d}{dx}\mathrm{erfc}(x) = -\frac{2}{\sqrt{\pi}} e^{-x^2}$

$$\Rightarrow \frac{\delta F}{\delta M} = \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \exp\left(\frac{-\delta_{i\,lin}^2}{2\sigma_R^2(z)}\right) \frac{\delta_{i\,lin}}{\sqrt{2}\,\sigma_R^2(z)} \cdot \frac{d\sigma_R}{dM} = \frac{\delta_{i\,lin}}{\sqrt{2\pi}\,\sigma_R(z)} \exp\left(\frac{-\delta_{i\,lin}^2}{2\sigma_R^2(z)}\right) \frac{d\ln\sigma_R}{dM}$$

- Fix normalization :

$$\int_0^\infty \frac{\delta F(M,z)}{\delta M} dM \stackrel{!}{=} \frac{1}{2}$$

because prescription do not account for decreasing modes that have $\delta > \delta_{ilin}$ already

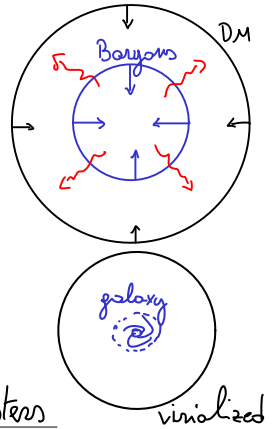
- Press-Schecter mass function (comoving number density)

$$N(M,z)\,dM \equiv \frac{\delta F}{\delta M} \cdot 2 \cdot \frac{1}{V} = \sqrt{\frac{2}{\pi}} \frac{\delta_{ilin}}{\sigma_R(z)} \exp\left(\frac{-\delta_{ilin}^2}{2\sigma_R^2(z)}\right) \frac{d\ln\sigma_R}{d\ln M} M \cdot \frac{1}{V} \qquad M = \rho_B V$$

$$= \sqrt{\frac{2}{\pi}} \frac{\rho_B \delta_{ilin}}{\sigma_R(z)} \exp\left(\frac{-\delta_{ilin}^2}{2\sigma_R^2(z)}\right) \frac{d\ln\sigma_R}{d\ln M} \qquad (!!)$$

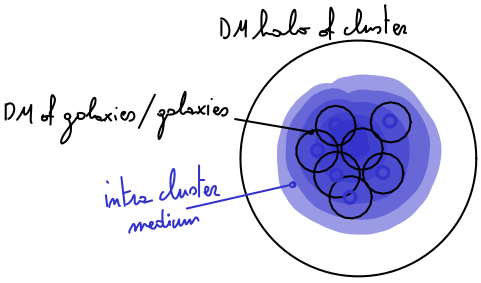
$\sigma_R D_+(z)$

- Galaxies and galaxy clusters



- Baryons will participate to the gravitational collapse
- They will collapse more because they can radiate away energy
- Star will form from the further collapse of local substructures
- Galaxies are formed
- Smaller dark matter haloes (galaxies) will merge to form galaxy clusters

- Constraining the cosmological parameters



1) Clusters counts

- detect dark matter haloes (galaxy clusters) and
  - optical : identify the galaxies they contain
  - X-rays : Bremsstrahlung emission from intra cluster medium (ICM)
  - Millimetric : Sunjaev-Zel'dovic effect (compton scattering of CMB photons with ICM free e^-)
  - Gravitational lensing : shear peaks
- count them as a func of M and z
  => Constraints on Ω_m, σ_8 and geometry (Ω_Λ) through V

2) Clustering

- of galaxie
- of galaxy clusters

Properties of DM haloes

- M_{200}, R_{200} : mass, M_{200}, within a radius, R_{200}, for which $\rho = 200\,\rho_{cr}$

$$M_{200} \equiv 200\,\rho_{cr}\frac{4\pi}{3}R_{200}^3 = 200\,\frac{3H^2}{8\pi G}\frac{4\pi}{3}R_{200}^3 = \frac{100\,H^2R_{200}^3}{G} \qquad R_{200} \equiv \left(\frac{GM_{200}}{100\,H^2}\right)^{1/3}$$

200 comes from $x_c = 178$ of spherical collapse

- M_{200m} : alternative definition based on the mean density : $M_{200m} \equiv 200\,\rho_B\frac{4\pi}{3}R_{200}^3$ (!)

this definition is closer to a dynamical definition such as M_{vir} but requires Ω_m

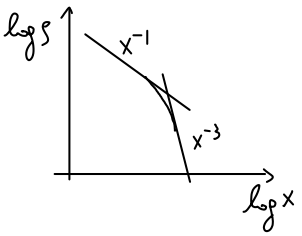
- M_{vir}, R_{vir} : dynamical motivation from virial theorem, $M_{200m} \simeq M_{vir}$

(these definitions do not require to specify the radial density profile)

- Radial profile of haloes

- Fit to N-body numerical simulations

Navarro-Frenk-White (NFW) profile $\quad \rho(r) = \frac{\rho_s}{x(1+x)^2} \quad x \equiv \frac{r}{R_s}$



well supported by observations: gravitational lensing, radial velocities, ...

- Concentration

$c \equiv \frac{R_{200}}{R_s}$ $\quad c(M_{200}, z, \Omega)$ because of the hierarchical growth:

smaller haloes have higher c because they formed earlier when ρ_c was higher

$\Rightarrow$ (the same mass was contained in a smaller radius

- Mass within R:

$$M(R) = 4\pi\int_0^R \rho(r')\,dr' = 4\pi R_s^3\int_0^x x'^2\rho(x')\,dx' = 4\pi\rho_s R_s^3\int_0^x \frac{x'\,dx'}{(1+x')^2} = 4\pi\rho_s R_s^3\left[\ln(1+x) - \frac{x}{1+x}\right]$$

set R_s based on $M(R_{200})$ $\Rightarrow$ Profile fully determined by 1 parameter (mass)

- Careful ! $\lim_{R\to\infty} M(R) = \infty$ logarithmically divergent

$\lim_{R\to 0}\rho(r) = \infty$ density diverges at zero, never mind... in any case on must account for baryons that in the core are relevant !

- Other profiles : generalized NFW, truncate NFW, Einasto profile

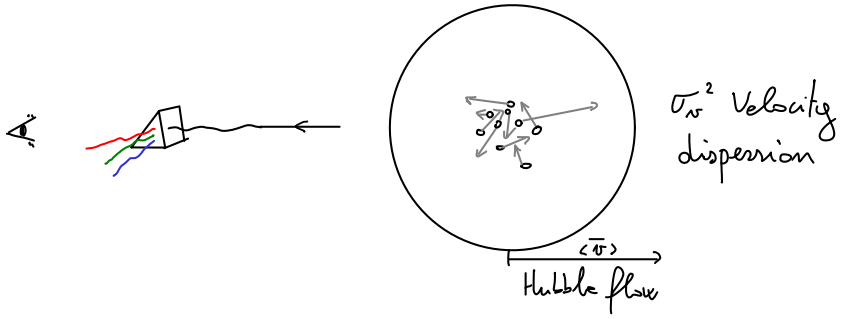
Dark matter in galaxy clusters

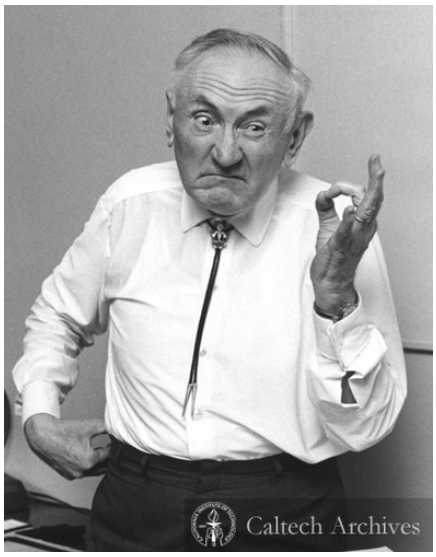
The first to speak about "dunkle Materie" Fritz Zwicky (1933)

Velocity dispersion measure of Coma cluster

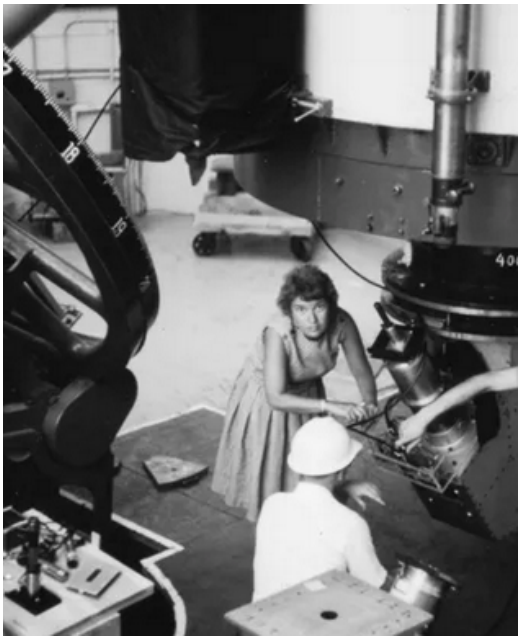
Virial theorem

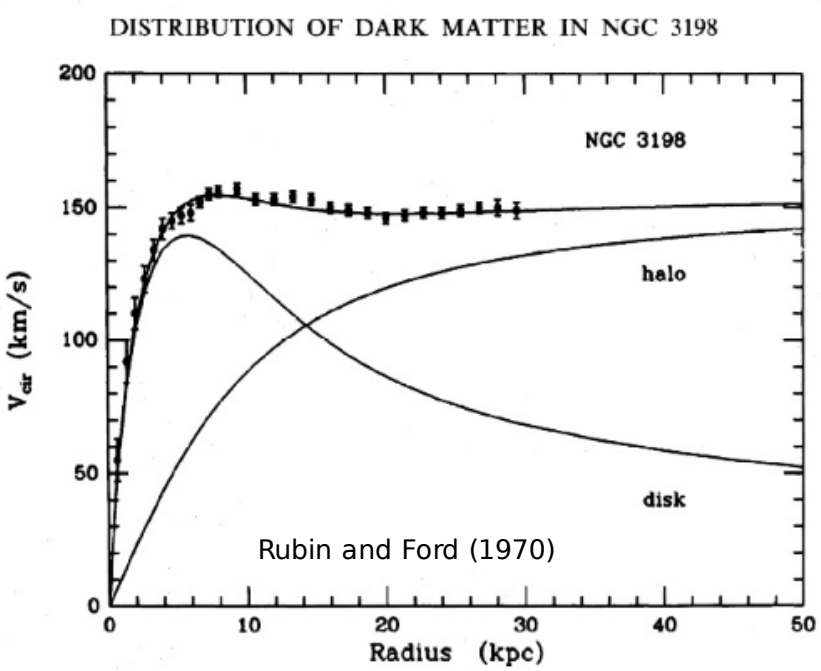
He measured 400 times more mass than just the visible one



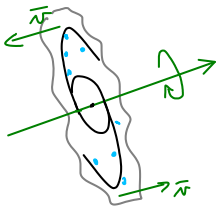


Dark matter in galaxies











Galaxy Clusters

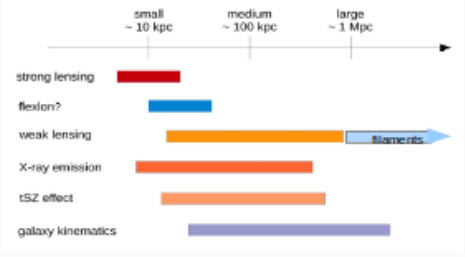
What they are, how to use them

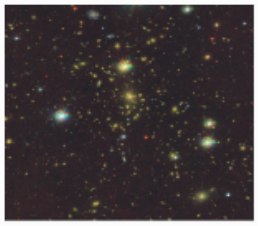
Properties:

Mass $M_v \sim 10^{14}–10^{15}\ M_\odot$
Virial radius $r_v \sim 1–3$ Mpc

+ Dark-matter halo
+ gas $T \sim 10^7–10^8$ K
+ stars & galaxies

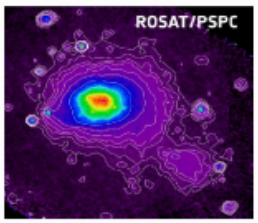
Different regions
Different response to distance

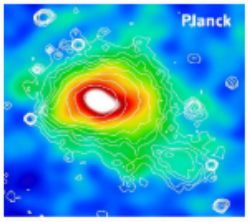




(1) Stars / galaxies

Optical emission from galaxies

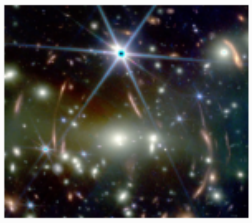




(2) ICM Temperature $T \sim 10^7–10^8$ K

Bremsstrahlung emission
- *X-ray luminosity* $L_x \simeq 10^{45}$ *erg/s*

Inverse Compton scattering of CMB photons
- *tSZ effect (CMB)* $\Delta T\ tSZ \sim 100\ \mu K$
- *kSZ effect (CMB)* $\Delta T\ kSZ \sim 10 - 30\ \mu K$



(3) Dark matter + baryons

Gravitational Lensing (background galaxies & CMB)
- *strong* $R_E \approx 5$”
- *weak*

An example of cosmology based on galaxy clusters: AMICO-KiDS-DR3

Number count of galaxy lcusters: Lesci et al. (2022a)

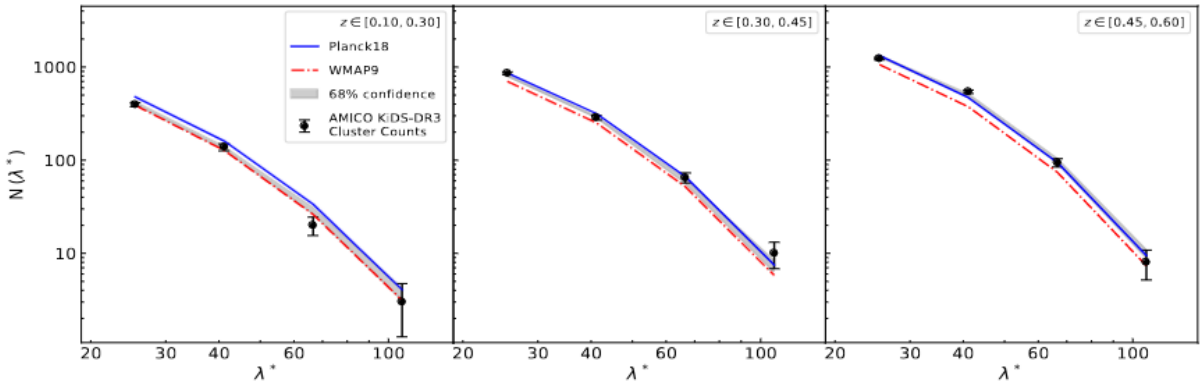


Fig. 4. Number counts from the AMICO KiDS-DR3 cluster catalogue as a function of the intrinsic richness λ^*, in the redshift bins $z \in [0.10, 0.30]$, $z \in [0.30, 0.45]$, and $z \in [0.45, 0.60]$, *from left to right*. The black dots represent the counts directly retrieved from the catalogue, where the error bars are given by the Poissonian noise. The solid blue lines represent the model computed by assuming the cosmological parameters obtained by Planck Collaboration VI (2020) (Table 2, TT, TE, and EE+lowE), while the red dashed lines show the results based on the WMAP cosmological parameters (Hinshaw et al. 2013) (Table 3, WMAP-only Nine-year). Both in the Planck and WMAP cases, the scaling relation parameters and the intrinsic scatter have been fixed to the median values listed in Table 2, retrieved from the modelling. The grey bands represent the 68% confidence level derived from the multivariate posterior of all the free parameters considered in the cosmological analysis.

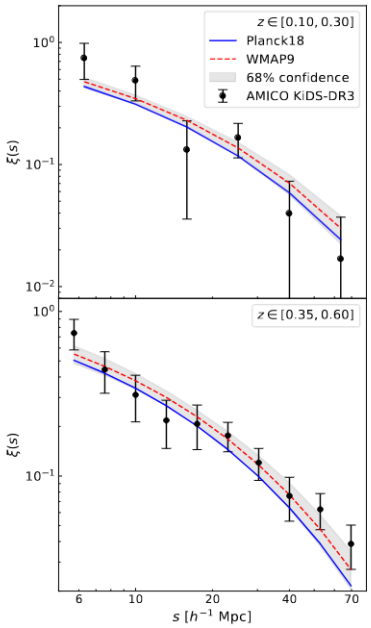


Fig. 2. Redshift-space 2PCF (black dots) of the AMICO KiDS-DR3 clusters in the spatial range $s \in [5, 80]\,h^{-1}$ Mpc, and redshift ranges $z \in [0.10, 0.30]$ (*top panel*) and $z \in [0.35, 0.60]$ (*bottom panel*). In both panels, the grey band represents the 68% confidence level derived from the multivariate posterior of the free parameters considered in the cosmological analysis, described in Sect. 4.1. The model computed by assuming the cosmological parameters derived by *Planck* (Planck Collaboration VI 2020, Table 2, TT, TE, and EE+lowE; blue lines) and WMAP (Hinshaw et al. 2013, Table 3, WMAP-only Nine-year; red lines) is represented by the solid blue lines and by the dashed red lines, respectively. In both cases, the median values of the scaling relation parameters derived by Lesci et al. (2022) are assumed.

Clustering of galaxy clusters: Lesci et al. (2022b)

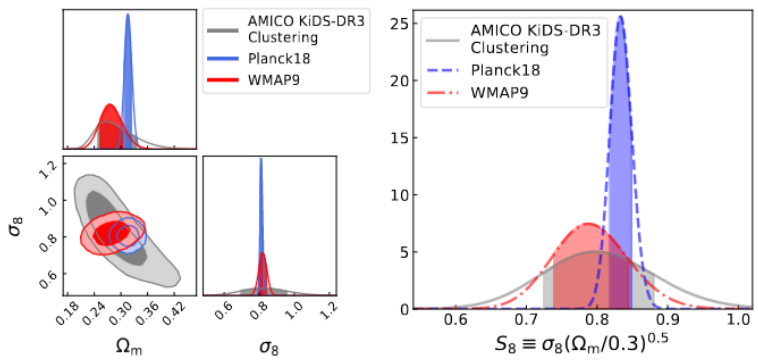
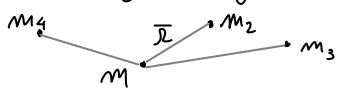


Fig. 3. Constraints obtained from the cosmological analysis, compared to WMAP and *Planck* results. In the *left panel*, we show the 68% and 95% confidence levels in the $\Omega_m - \sigma_8$ parameter space, along with the 1D marginalised posteriors with the relative intervals between the 16th and 84th percentiles, in the case of the cluster clustering analysis of the AMICO KiDS-DR3 catalogue (grey lines). In the same panel, we also display the results from WMAP (Hinshaw et al. 2013, Table 3, WMAP-only Nine-year; red lines) and *Planck* (Planck Collaboration VI 2020, Table 2, TT, TE, and EE+lowE; blue lines). In the *right panel*, we show the posteriors for the parameter S_8, where the bands show the intervals between 16th and 84th percentiles. The colours are the same as in the *left panel*.

N-Body numerical simulations

- Gravitational interaction only => DM (not collisional)
- Matter field as composed by "macro-particles" eg. $m \simeq 10^{10} M_{sun}/h$
- Solve for: $\begin{cases} \dot{\bar{v}} = \bar{F}/m \\ \dot{\bar{x}} = \bar{v} \\ \bar{\nabla}^2 \varphi = 4\pi G \rho \end{cases}$ $\bar{F}$ = force given by all particles



- Discretize time steps $\begin{cases} \bar{v}(t+\Delta t) \overset{!}{=} \bar{v}_{new} = \bar{v}(t) + \frac{\bar{F}}{m}\Delta t \\ \bar{x}(t+\Delta t) \overset{!}{=} \bar{x}_{new} = \bar{x}(t) + \bar{v}_{new}\Delta t \end{cases}$ (Eulerian approach)

$\Delta t = \sqrt{\varepsilon / |\bar{F}/m|}$ optimal time step
- $|\bar{F}/m|$: acceleration
- ε: length (the spatial resolution you care about)

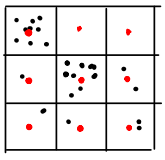
- Different schemes to evaluate $\bar{F}$

A) Particle-Particle model (PP)

$$\bar{F}_i = \sum_{j \neq i}^{N} \bar{F}_{ij} = \sum_{j \neq i}^{N} \frac{G m_i m_j}{|r_{ij}|^3} \bar{r}_{ij} \qquad (N^2)$$ Slow but precise

B) Particle-Mesh (PM)

- Divide the volume in cells and get density ρ
- Resolution is fixed by the grid ($N \log N$)



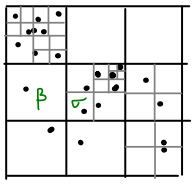
- Fourier space => FFT $\begin{cases} \rho_k(\bar{x}) = \int_{-\infty}^{\infty} \rho(\bar{x})\, e^{-i\bar{k}\bar{x}} \frac{d^3k}{(2\pi)^3} \\ \bar{\nabla}^2\varphi = 4\pi G\rho \rightarrow \tilde{\varphi}_k = \rho_k / k^2 \end{cases}$ Newtonian, Green's function $1/k^2$
- Back to real space to get $\varphi(\bar{x})$
- Apply force to all particles and repeat

C) P^3M (PP-PM)

- Hybrid approach
- Large range force use PM, short range force use PP $\Rightarrow$ $\vec{F} = \vec{F}^{PP}_{short} + \vec{F}^{PM}_{long}$
- Same precision of PP but faster

D) Tree-Code (PT)

- Use Gauss theorem
- Collective force of a distant collection of particles $\longrightarrow$ Baricenter !
- More precise than PM but a bit slower
- computation time scales as $\sim N \log N$

1) Build the tree 

split until you have 1 particle per cell

2) Compute baricenter and mass of all nodes

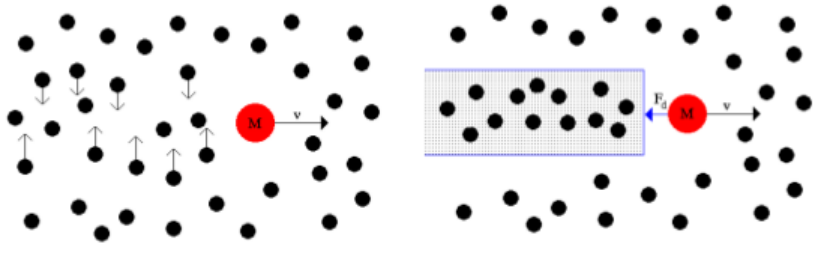
3) " force of all nodes on a particle
↳ eg. particle β feels force from 6 cells (the largest ones)
" σ " " " 9 "

E) Tree PM (Gadget)

$$\varphi_{\vec{k}} = \varphi^{short}_{\vec{k}} + \varphi^{long}_{\vec{k}}$$

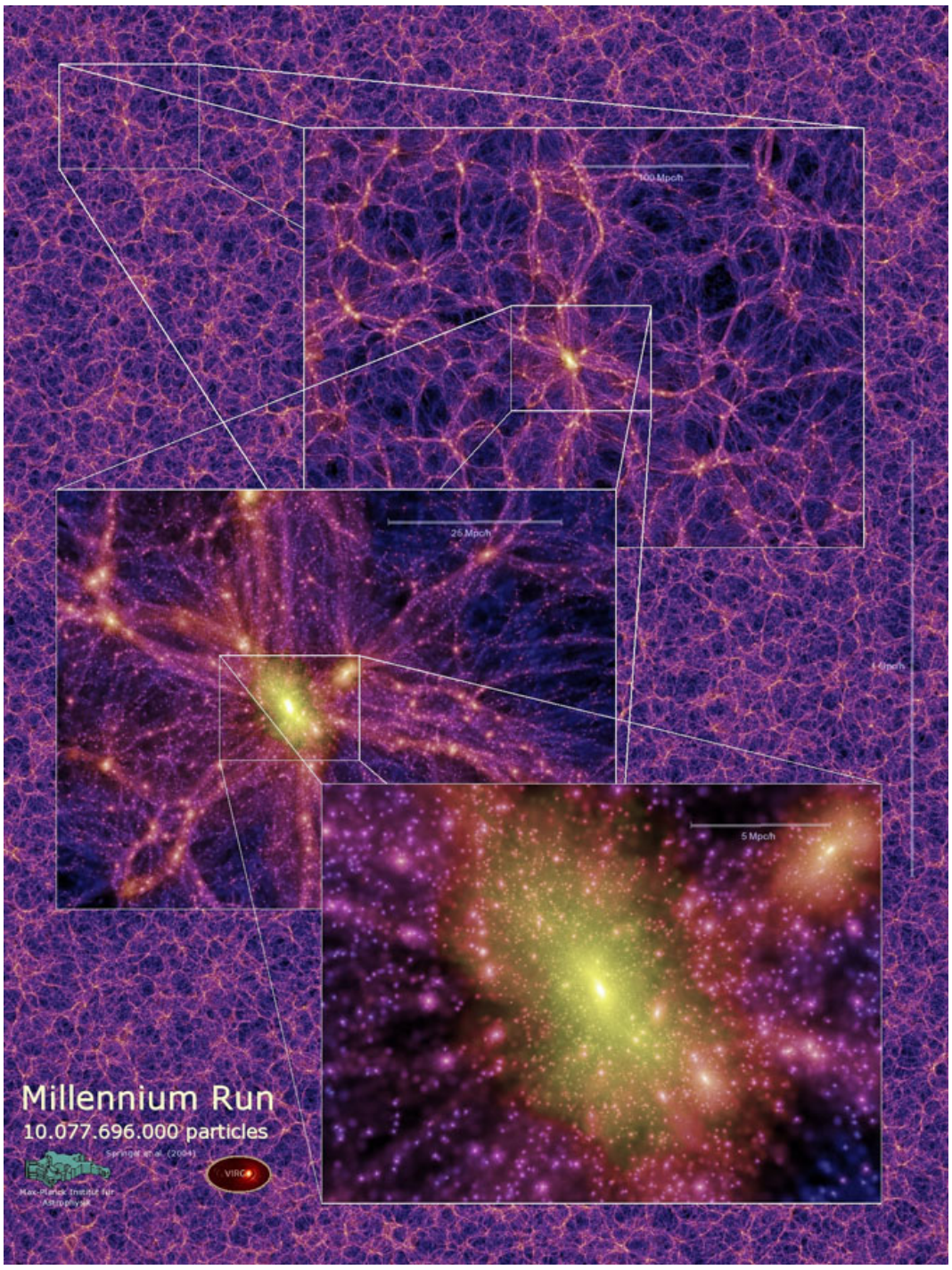
$$\varphi^{long}_{\vec{k}} = \varphi_k \exp\left(-\vec{k}^2 \vec{r}_s\right) \quad PM$$

$$\varphi^{short}_{k} = \varphi_k \left[1 - \exp\left(-\vec{k}^2 \vec{r}_s\right)\right] \quad Tree$$



source: J. Schombert

They capture non-linear effects such as dynamical friction


100 Mpc/h
25 Mpc/h
5 Mpc/h
Millennium Run
10.077.696.000 particles
Springel et al. (2004)
Max-Planck-Institut für Astrophysik
VIRGO

DM only

Key results

Dark Matter haloes:

- NFW profile / Einasto profile

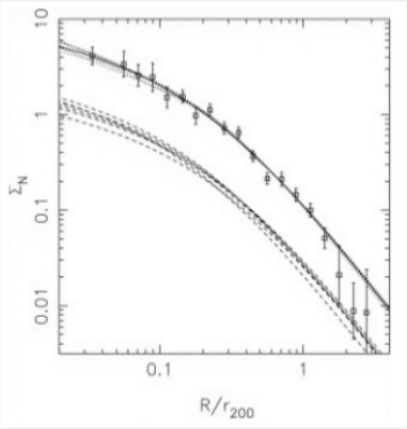
- Navarro, Frenk + White: haloes in n-body simulation show a profile:

$$\rho \propto \frac{1}{x(1+x^2)} \quad \text{with} \quad x \equiv \frac{r}{r_c} \quad \text{and} \quad r_c = cr_{\rm vir} \qquad (10)$$

- universal density profile, applicable to haloes of all masses
- fitting formula breaks down:
  - infinite core density
  - total mass diverges logarithmically

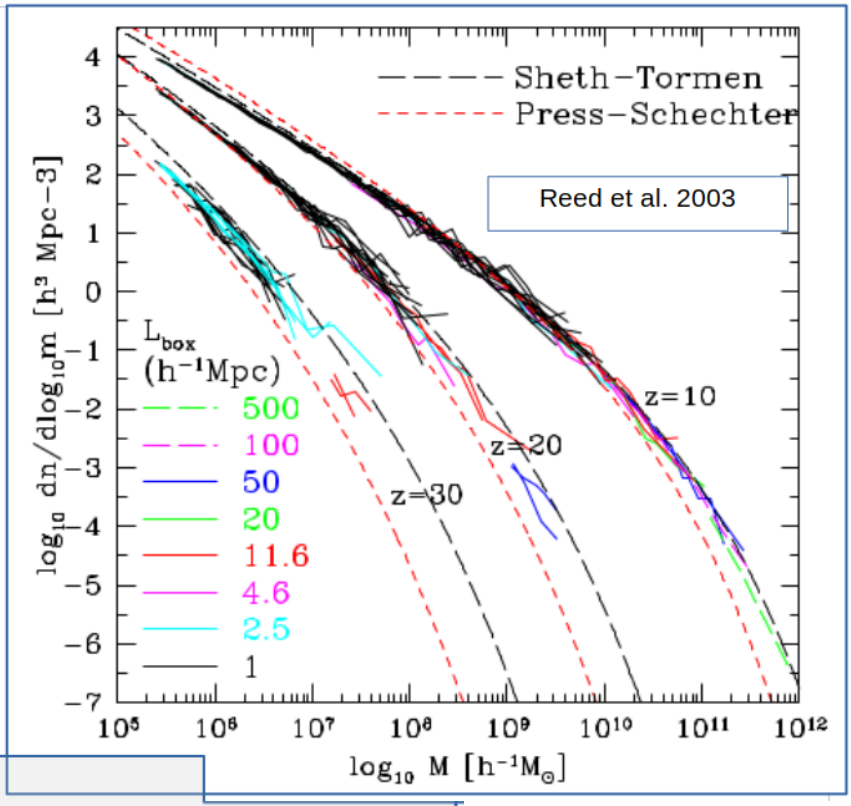
→ Einasto profile

Flat rotational curves



The Mass functions

- Press & Schechter (1974)
- Sheth & Tormen (1999)
- Jenkins et al. (2001)
- Tinker et al. (2008)



Press & Schechter (1974)

$$f_{\rm P-S}(\sigma) = \sqrt{\frac{2}{\pi}}\frac{\delta_c}{\sigma}\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right]$$

Sheth & Tormen (1999)

$$f_{\rm S-T}(\sigma) = A\sqrt{\frac{2a}{\pi}}\left[1+\left(\frac{\sigma^2}{a\delta_c^2}\right)^p\right]\frac{\delta_c}{\sigma}\exp\left[-\frac{a\delta_c^2}{2\sigma^2}\right]. \qquad (5)$$

The choice of values $A = 0.3222$, $a = 0.707$ and $p = 0.3$ provides a significantly better fit to mass functions determined

Baryons: hydrodynamical simulations

- Baryons: collisional ⇒ $P \neq 0$ need to account for it

$$\begin{cases} \frac{\delta \rho}{\delta t} + \frac{1}{a}\vec{\nabla}(\rho \vec{v}) = 0 & \text{matter conservation} \\ \frac{\delta(\rho \vec{v})}{\delta t} + \frac{1}{a}\vec{\nabla}(\rho \vec{v}^2 + \vec{P}) = -\frac{\dot{a}}{a}\rho \vec{v} - \frac{1}{a}P\vec{\nabla}\rho & \text{momentum} \\ \frac{\delta \vec{E}}{\delta t} + \frac{1}{a}[(\vec{E}+P)\vec{v}] = -2\frac{\dot{a}}{a}E - \frac{1}{a}\rho \vec{v}\vec{\nabla}P & \text{energy} \end{cases}$$

- They are a set of conservation laws: $\frac{\delta w}{\delta t}(\vec{x},t) = -\vec{\nabla}\vec{f}(\vec{x},t)$ (1) w = field: $\rho, \varepsilon, \dots$
- 2 ways to solve them: A, B $\vec{f}$ = flux of: $\rho, \varepsilon,$

A) Eulerian approach:

$$w(t) = w(t_0) + \left.\frac{\delta w}{\delta t}\right|_{t_0}(t-t_0) + \dots$$

$$\simeq w(t_0) - \left.\vec{\nabla}\vec{f}\right|_{t_0}(t-t_0)$$

$$\simeq w(t_0) - \left.\frac{f_{i+1}-f_i}{\Delta x}\right|_{t_0}(t-t_0)$$

↙ keep 1° order, use (1)

↙ discretize

B) Lagrangian method: Smooth Particle Hydrodynamic (SPH)

- mass extended particles with: $\vec{x}_i, \vec{v}_i, \rho_i, \varepsilon_i$ (energy density)
- solve the flux equations at the position of each particle

$$\langle f(\vec{x}) \rangle = \int f(\vec{x}') K_h(\vec{x}-\vec{x}')\, d^3x'$$

$$\simeq \sum_i^N f(\vec{x}_i) K_h(\vec{x}-\vec{x}_i)\left(\frac{m_i}{\rho_i}\right) dV$$

$K_h(\vec{x}-\vec{x}')$ smoothing kernel
h = smoothing scale
dV: mass resolution is fixed

i.e. approximate field with a spline
<-> superposition of all particles

Density field (example)

$$\langle \rho(\bar{r}) \rangle = \sum_i \rho_i W_h(\bar{r}-\bar{r}_i) \frac{m_i}{\rho_i} = \sum_i m_i W_h(\bar{r}-\bar{r}_i) \qquad w = \rho$$

$$\frac{\delta w}{\delta t}(\bar{x},t) = -\bar{\nabla}\bar{f}(\bar{x},t)$$

Derivatives

$$\delta_j \langle \bar{f}(\bar{r}) \rangle = \sum_i \frac{m_i}{\rho_i} \bar{f}(\bar{r}_i)\, \delta_j W_h(\bar{r}-\bar{r}_i) \qquad j\text{-th direction}$$

symmetric representation of derivative to reduce numerical errors: (numerical trick)

$$\frac{\bar{\nabla}\bar{f}}{\rho} = \bar{\nabla}\left(\frac{\bar{f}}{\rho}\right) + \frac{\bar{f}}{\rho^2}\bar{\nabla}\rho \qquad \bar{\nabla}\bar{f} = \rho\,\bar{\nabla}\left(\frac{\bar{f}}{\rho}\right) + \frac{\bar{f}}{\rho}\bar{\nabla}\rho \qquad \text{(product rule of derivatives)}$$

$$\Rightarrow \quad \bar{\nabla}\langle \bar{f}_j \rangle = \rho_j \sum_i \frac{m_i}{\rho_i^2} \bar{f}(\bar{r}_i)\,\bar{\nabla} W_h(\bar{r}-\bar{r}_i) + \frac{\bar{f}_j}{\rho_j}\sum_i m_i \bar{\nabla} W_h(\bar{r}-\bar{r}_i)$$

$$= \rho_j \sum_i m_i \left(\frac{\bar{f}_i}{\rho_i^2} + \frac{\bar{f}_j}{\rho_j^2}\right)\bar{\nabla} W_h(\bar{r}-\bar{r}_i)$$

Include presence of shocks

- Shocks are not captured by hydro eq.s
- Include artificial viscosity, Π_{ij}, to transfer kinetic energy in to heat
  i.e. coherent velocities → random (thermal)
  (in shocks energy ↑ but we assume S = const. ⇒ Π_{ij} allows S ↑)

Smooth Particle Hydrodynamics (SPH)

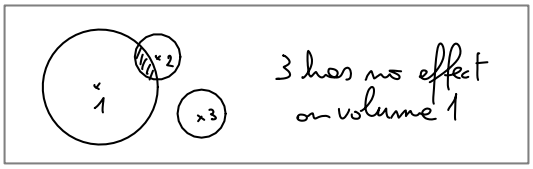
$$\begin{cases} \dfrac{d\bar{v}_j}{dt} = -\sum_i m_i \left(\dfrac{P_i}{\rho_i^2} + \dfrac{P_j}{\rho_j^2} + \Pi_{ij}\right)\bar{\nabla} W_h(\bar{r}_j - \bar{r}_i) & \text{Euler} \\ \dfrac{d\varepsilon_j}{dt} = \dfrac{1}{2}\sum_i m_i \left(\dfrac{P_i}{\rho_i^2} + \dfrac{P_j}{\rho_j^2} + \Pi_{ij}\right)\bar{\nabla} W_h(\bar{r}_j - \bar{r}_i)\cdot(\bar{v}_j - \bar{v}_i) & \text{Energy} \\ \text{continuity eq. is automatically satisfied by particles conservation} \\ \Rightarrow \text{no need to evaluate it} & \text{Continuity} \end{cases}$$

- Numerical issues with Kelvin-Helmotz instability

Kernel

· Function with Compact support $K(\vec{x}) = 0$ for $|\vec{x}| > h$

eg. $K_h(x) = \frac{8}{\pi h^3} \cdot \begin{cases} 1 - 6\left(\frac{x}{h}\right)^2 - 6\left(\frac{x}{h}\right)^3 & 0 \leq \frac{x}{h} < 0{,}5 \\ 2\left(1 - \frac{x}{h}\right)^3 & 0{,}5 \leq \frac{x}{h} < 1 \\ 0 & 1 \leq x/h \end{cases}$



⇒ Sum over neighbours (in a sphere of radious h), not over the entire volume

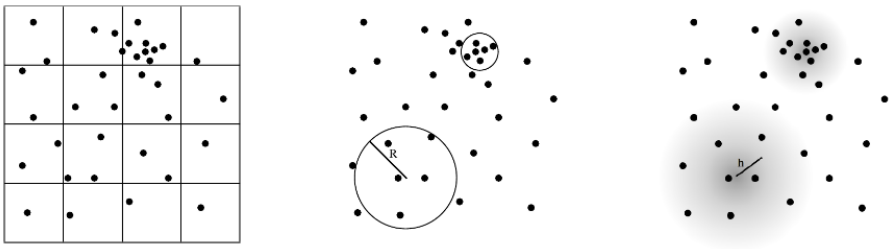


Figure 4.2.: Different methods to compute a continuous density from point mass particles are: the particle-mesh method (left), constructing a local volume according to the local density of particles (middle), and the approach adopted in smoothed particle hydrodynamics (right).
Image credit: Price (2012b)

"Other" ways to solve hydro equations

eg. Mesh, Adaptive-mesh
eg. Voronoy tessellization (eg. AREPO code)

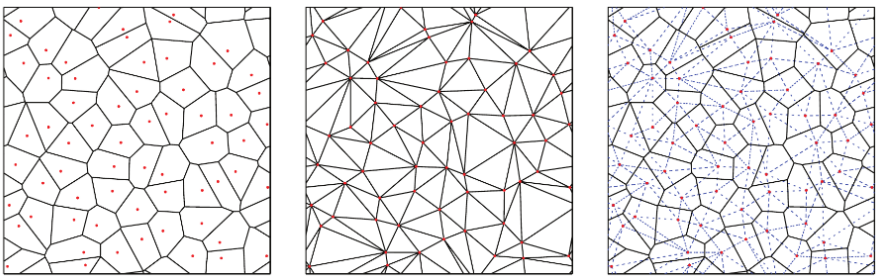
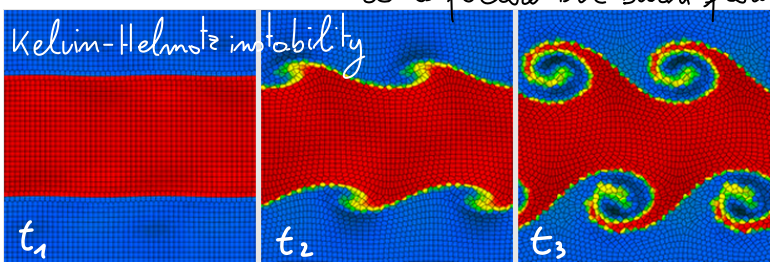
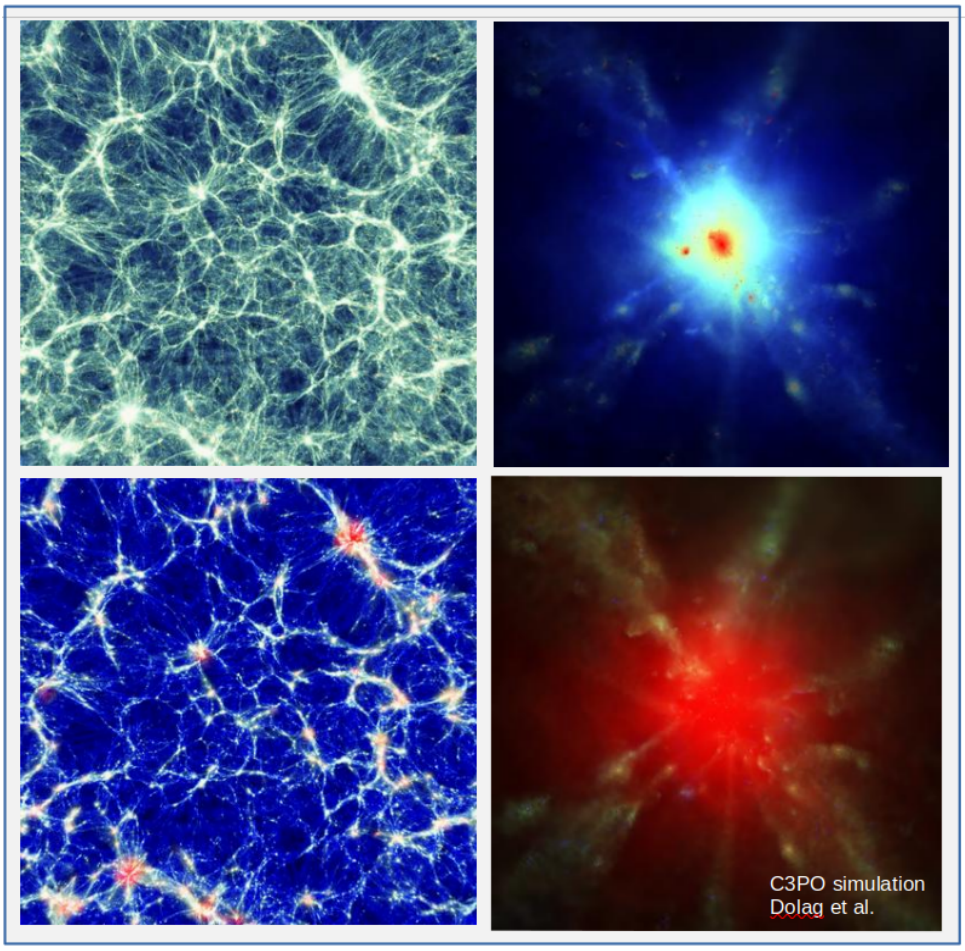


Figure 4.1.: Voronoi (left) and Delaunay (middle) tessellation of a two-dimensional periodic box. The Delaunay tessellation provides a complete set of perpendicular bisectors needed for the construction of the irregular moving-mesh.
Image credit: Springel (2010a)

cells follow the bulk flow



Very realistic representation of the data



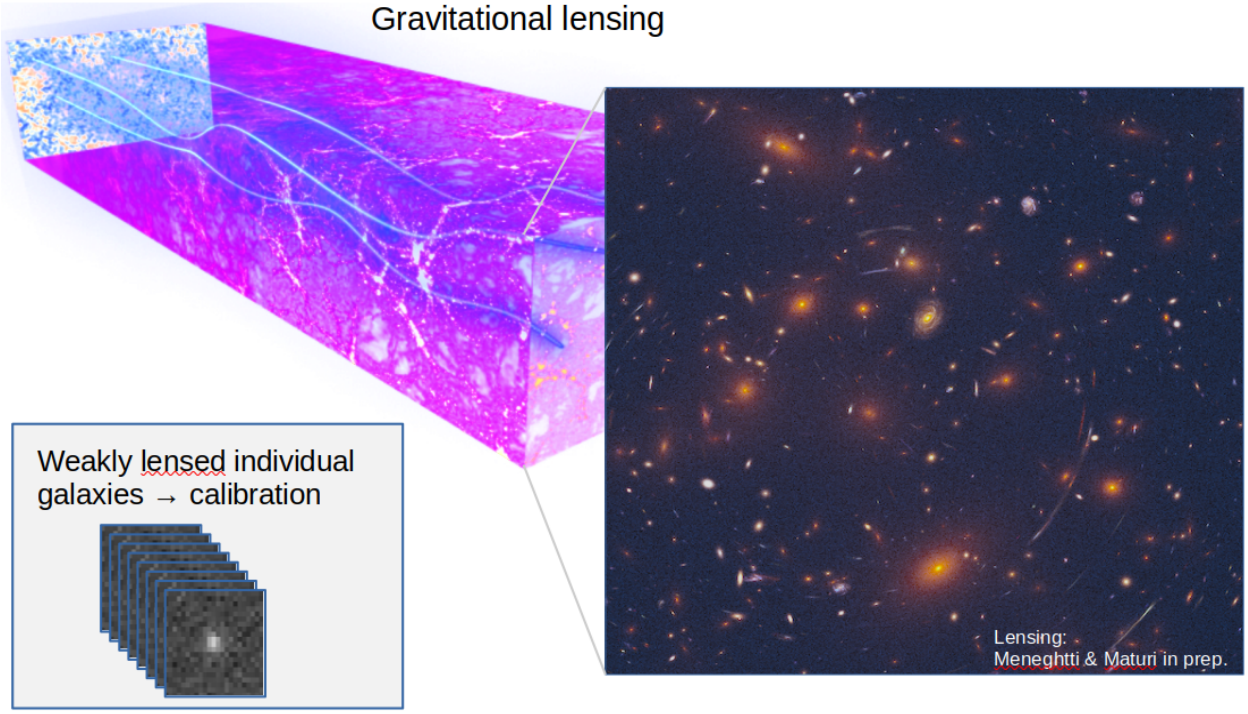
Observables (1)

- density
- temperature
- turboulence
- gravitational lensing
- magnetic fields
- ...

Observables (2)

Ray tracing simulations

Gravitational lensing

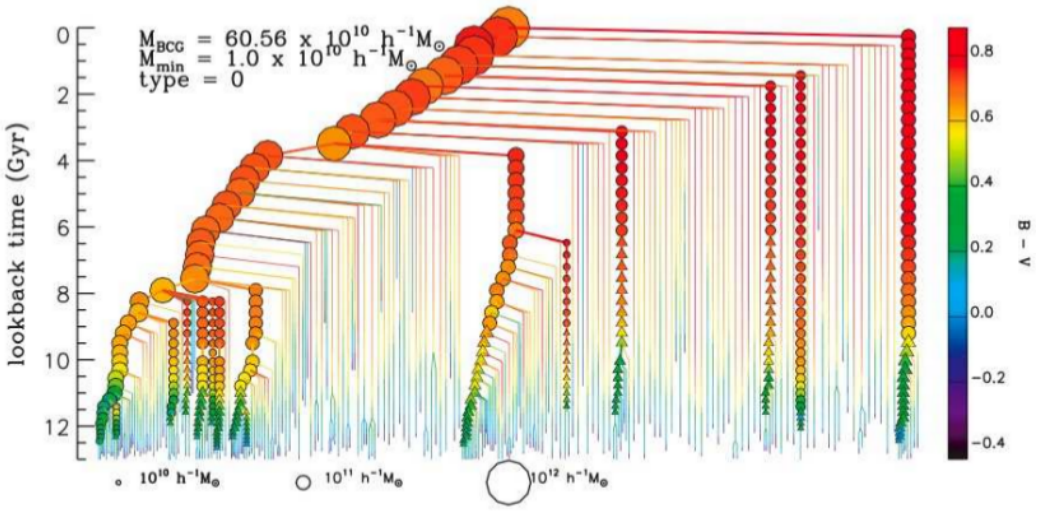


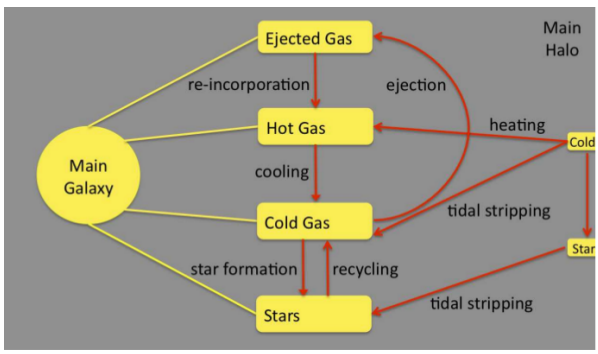
Semi-Analytical models

- Baryons give raise to further complex physics
- given contain local density, $\rho(\bar{x},t)$, pressure, $P(\bar{x},t)$, temperature, $T(\bar{x},t)$ (from hydro eq.s)
  baryons can "collapse in galaxies" → Stars, AGNs, SN, ...
- Add sub-grid physics : SN, AGN feedback, chemistry, shocks, ...
  ↳ they are a set of analytical expressions to account for that
  ↳ for example, include formation of AGNs ⇒ include energy injection in the medium

Semi-analytical models: a sketch

- Each halo has a mass and a position
- Accordingly attach a physical/phenomenological recipe
- Follow the story of each haloe (Milky-satellites)



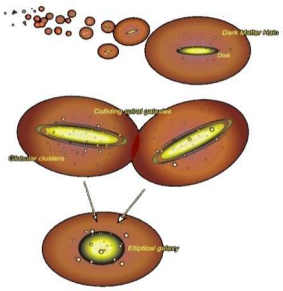


Merger trees and semi-analytic modelling

Matteo Maturi

Heidelberg University

Cosmology WS 2023/24



Summary

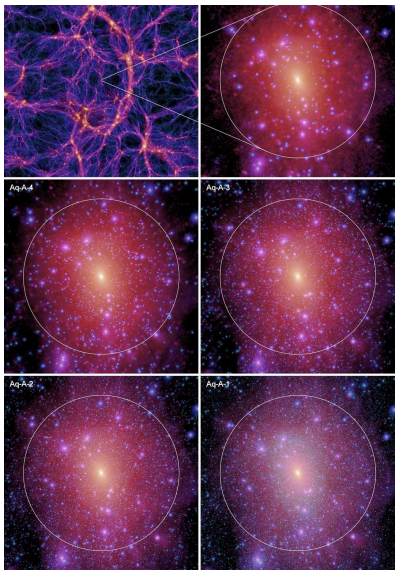
Subject

- The satellites of the Milky Way - Insights from semi-analytic modelling in a LambdaCDM cosmology
  Starkenburg E., Helmi A., De Lucia G., Li Y.-S., Navarro J. F., Font A. S., Frenk C. S., Springel V., Vera-Ciro C. A. and White S. D. M.
- The Spatial Distriburion of Galactic Satellites in the ΛCDM Cosmology
  Wang J., Frenk C. S. and Cooper A. P.

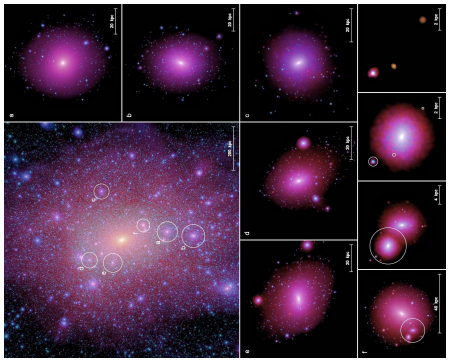
Content

- The satellites: Where? Star content? Metallicity properties? Comparison with hydrosim.
- Aquarious simulation
- Intro to semianalytocal models
- Papers semi-analytical model
- Papers results

Aquarius simulations



resimulations

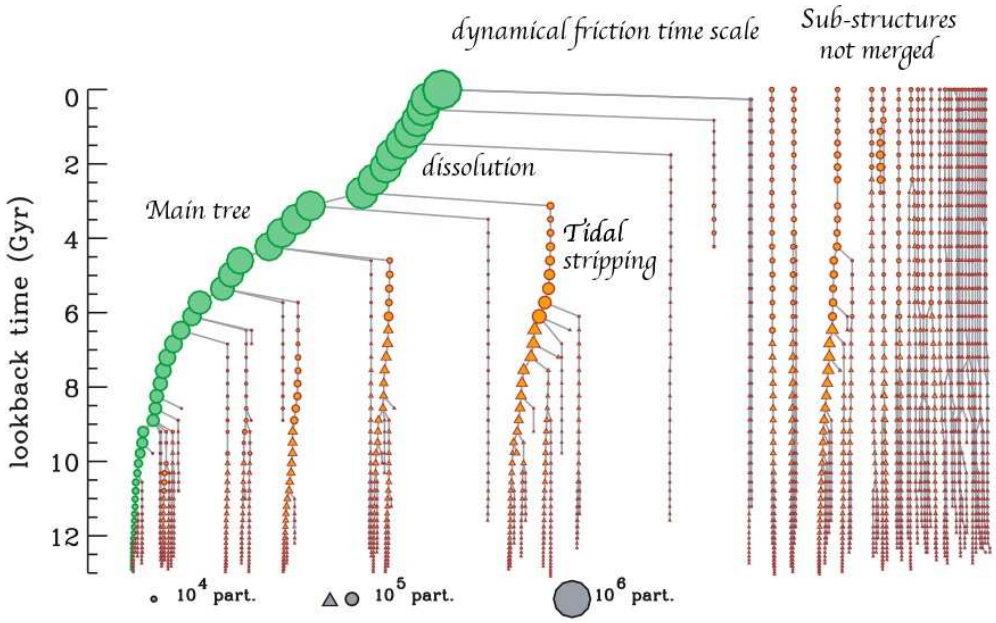


substructures in substructures

- DM only simulation (cube $100h^{-1}Mpc$) (=> full LCDM context)
- Resimulation of 6 isolated Milky-like galaxies
- Different resolutions up to about 200 million particles
- One halo at 1.5 billion particles (IGNORED HERE).

The merger tree: DM

1. **Identify** haloes down to 20 particles with FoF
2. Follow the mergin history of haloes $\Rightarrow$ **Merging tree**
3. Identify the **'first' progenitor**: $M_i = N_i + max(M_{i_1}, M_{i_2}, ..., M_{i_n})$,
   (N_i number of self-bound particles of halo i, M_{i_j} progenitor of the halo i)



Physics between haloes

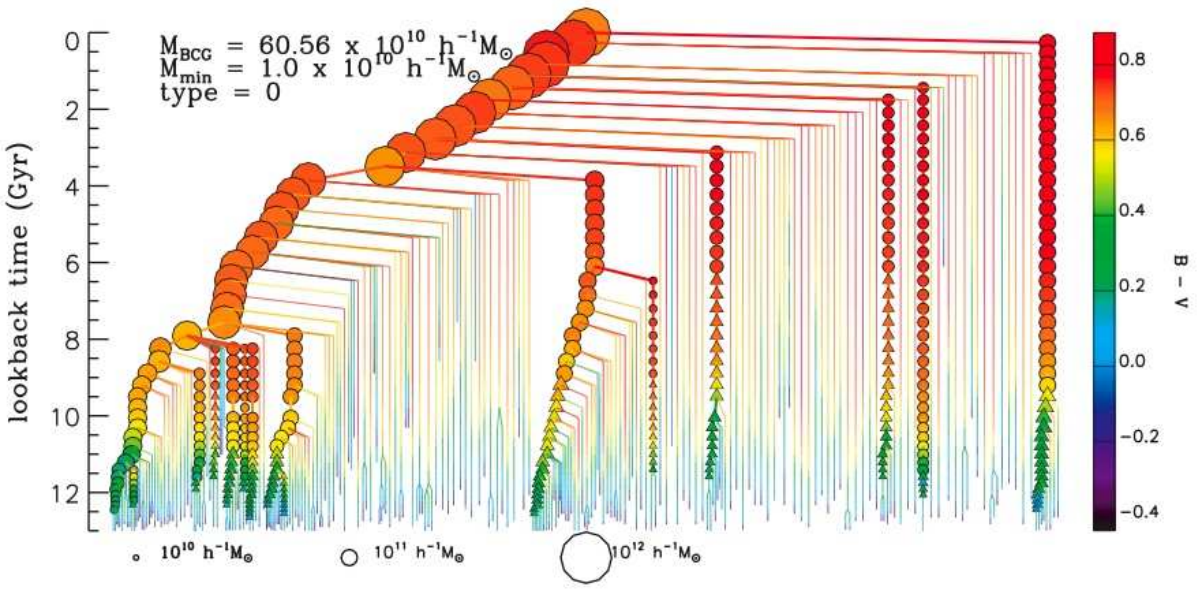
Gravitational interactions between haloes
– Merging of haloes
– Tidal truncation, tidal stripping (merging haloes loos peces)
– Dynamical friction (merging haloes slow down and get absorbed)
– Gravitational dissolution (haloes get smaller than 20 part.)
– Physics within haloes (sub-grid)...



Here it starts the spherical-cow domain

Semi-analytical models: a sketch

- Each halo has a mass and a position
- Accordingly attach a physical/phenomenological recipe
- Follow the story of each haloe (Milky-satellites)



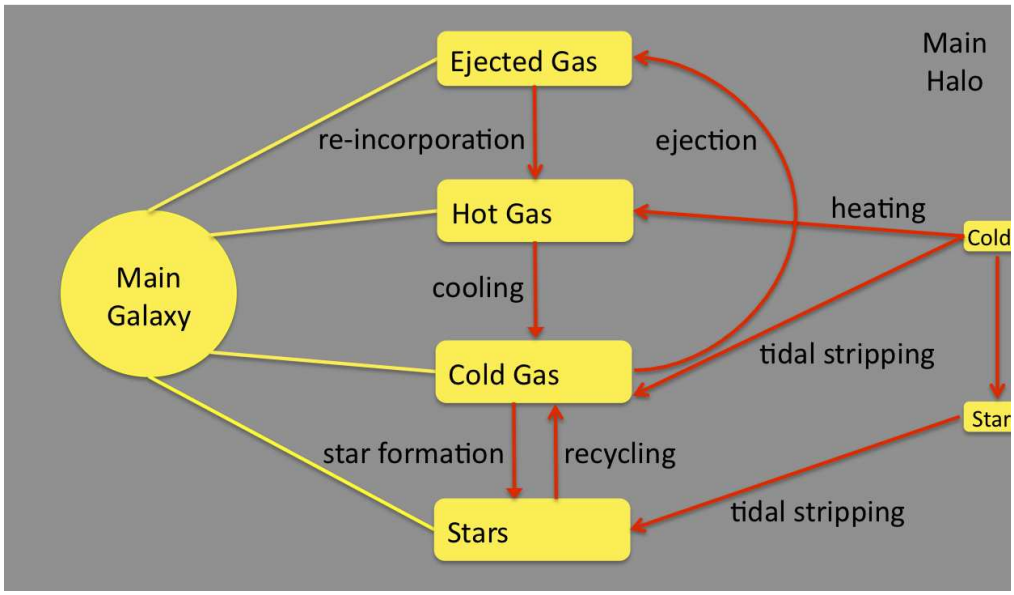
Paper Semi-analytic model

- Reionization
  - it reduces the content of barions in haloes (filtering mass)
  - Use global reionization ($15 > z > 11.5$) stronger than usual
  - Anyway local reionization by local sources is full at $z = 10$
- Cooling
  - depends on haloes metallicity and temperature
  - no cooling below $10^4 K$
  - helps star formation
- Star formation
  - cool gas in discs above a certain mass goes into stars
  - star formation is proportional to that amout of gas
- Passive stellar evolution
  - IMF: what is the mass distribution of stars when they form
  - Stellar evolutionary tracks (Padova 1994)
- SN feedback
  - Gas heating and ejection of gas
  - fraction of ejected gas depends on the halo potential ($\propto V_{200}^{-2}$)
  - gas can be reaquired

Semianalytical model

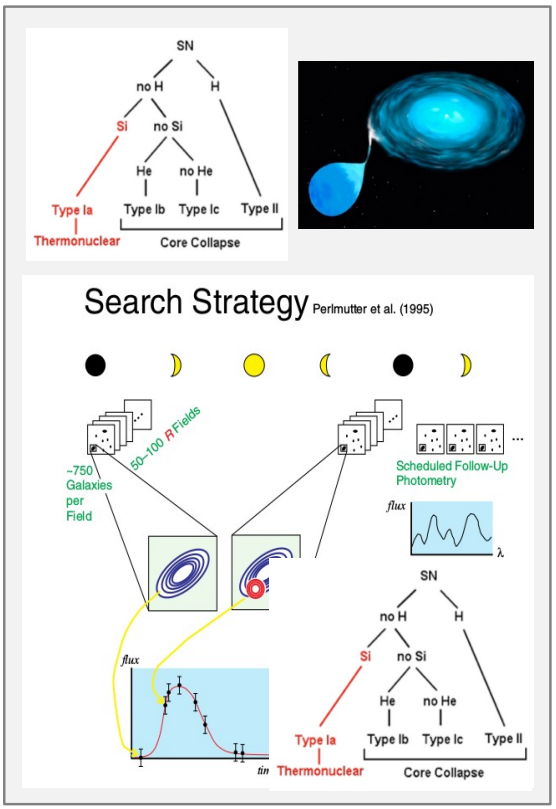
- Metals
  - NO full chemical chain
  - follows the star formation events
  - fine only for Type-II SN
- If haloe turns into a satellite
  - Ram pressure stripping of hot gas component
  - Ejected gas component
  - Star stripping: stars goes to central galaxy
    do not affect luminosity function, because a given star remains so
- Tydal stripping $\rightarrow$ goes to central galaxy
  - when particles of one haloe get < 20 particles they are fllowed by their (sub bixel) recipe (orphans)
  - important for luminosity function
  - BUT: how do they follow their spatial distribution???

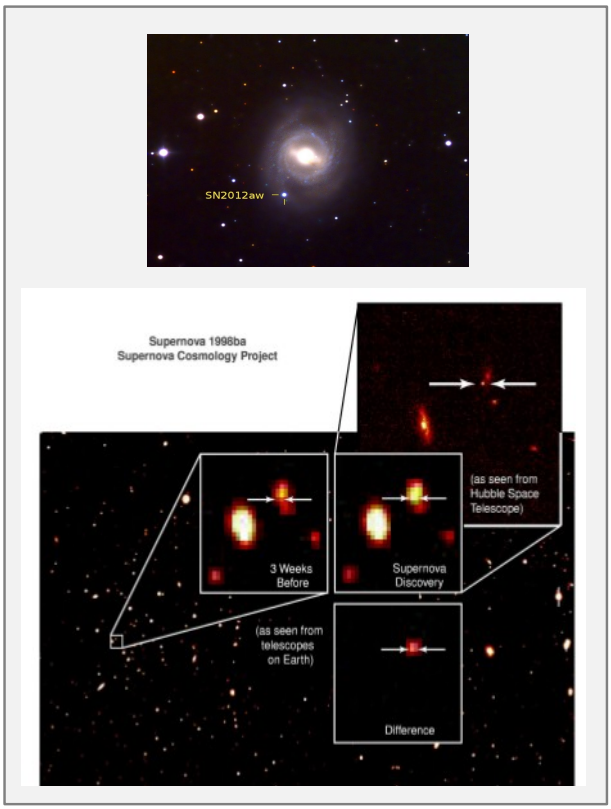
A more detailed skatch



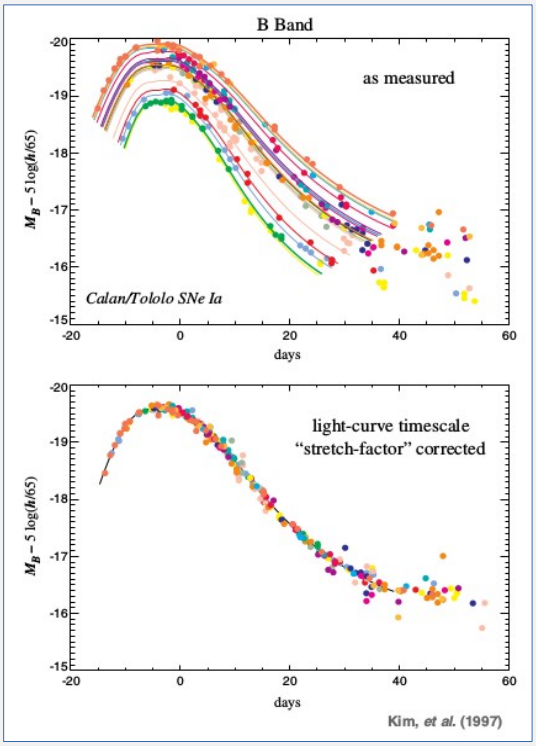
Luminosity distance

Supernovae Ia





Standardizable not standard candles!



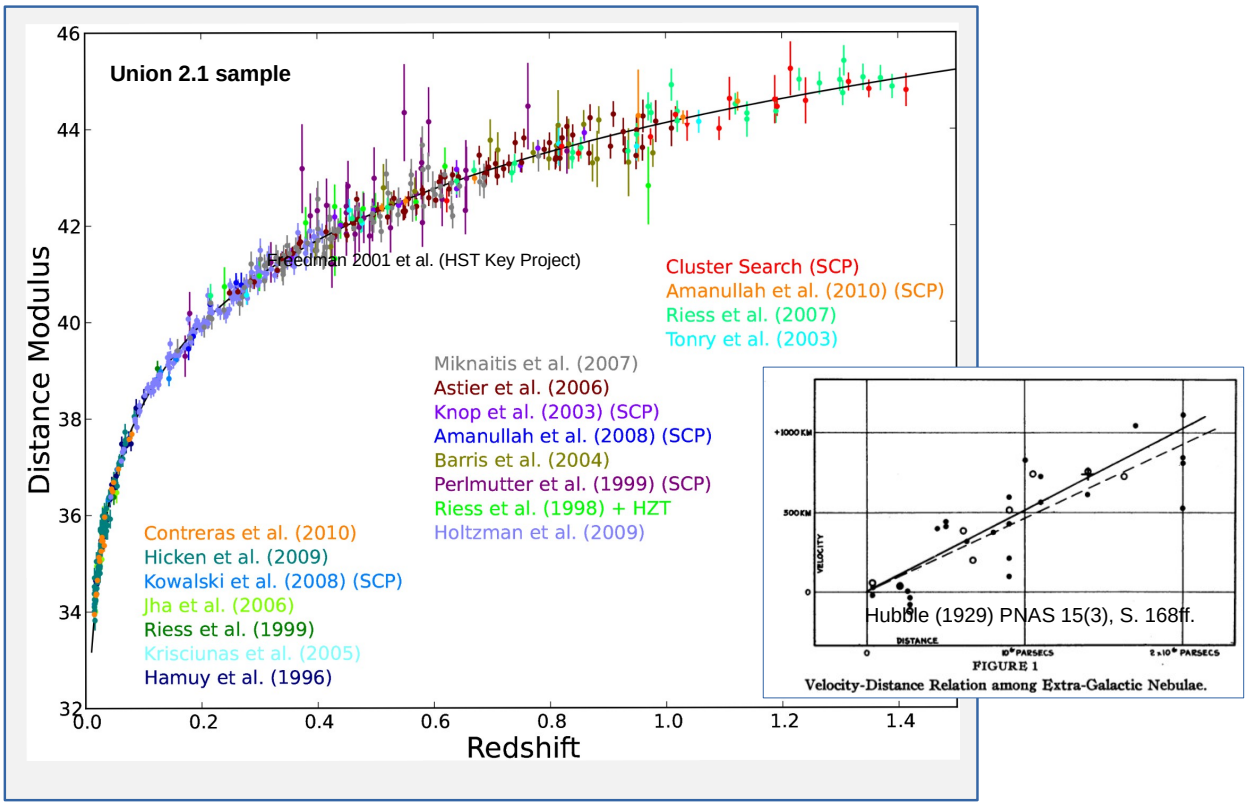
Calibrate:
- correlation between peak and width of the light curve
- empirical correlation

Hope:
- assume that all supernovae are the same for all z... (metallicity evolves with time...)

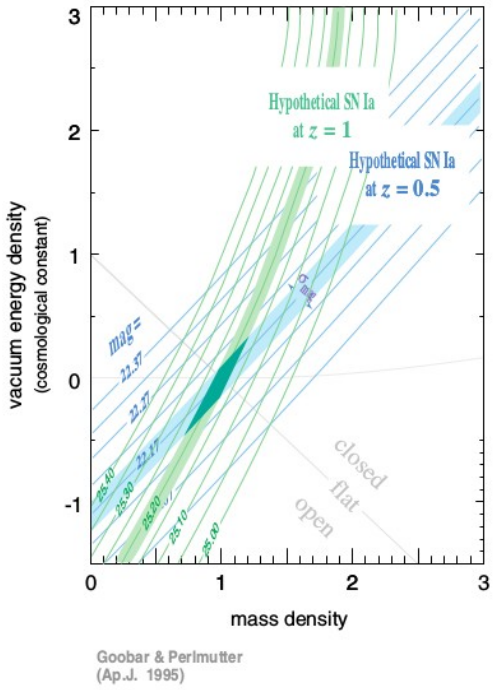
Try to control:
- absorbtion along the line of sight
- absorption within the host galaxy
- scatter due to gravitational lensing (marginal)

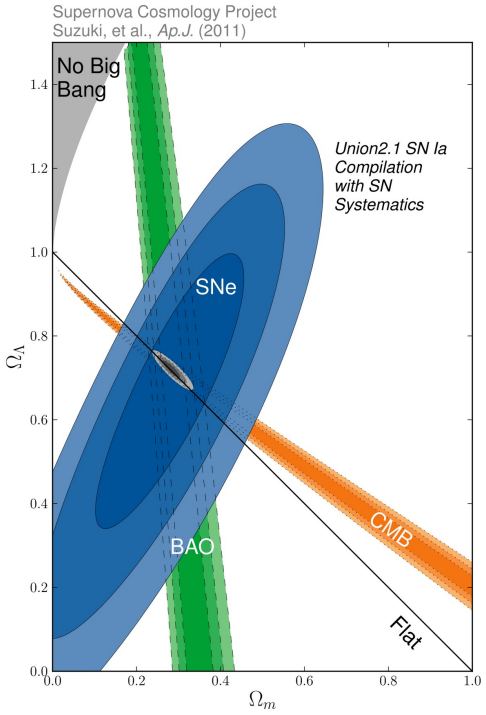
- introduce nuisance parameters when fitting

Super Novae


Union 2.1 sample
Distance Modulus
Redshift
Freedman 2001 et al. (HST Key Project)
Cluster Search (SCP)
Amanullah et al. (2010) (SCP)
Riess et al. (2007)
Tonry et al. (2003)
Miknaitis et al. (2007)
Astier et al. (2006)
Knop et al. (2003) (SCP)
Amanullah et al. (2008) (SCP)
Barris et al. (2004)
Perlmutter et al. (1999) (SCP)
Riess et al. (1998) + HZT
Holtzman et al. (2009)
Contreras et al. (2010)
Hicken et al. (2009)
Kowalski et al. (2008) (SCP)
Jha et al. (2006)
Riess et al. (1999)
Krisciunas et al. (2005)
Hamuy et al. (1996)
Hubble (1929) PNAS 15(3), S. 168ff.
FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.

Understanding the “banana plots”





Model free expansion rate

$$D_{\rm L}(a) = \frac{c}{H_0}\frac{1}{a}\int_a^1 \frac{dx}{x^2 E(x)} \equiv \frac{c}{H_0}\frac{1}{a}\int_a^1 \frac{dx}{x^2} e(x),$$

Luminosity function
(Volterra integral of the second kind)

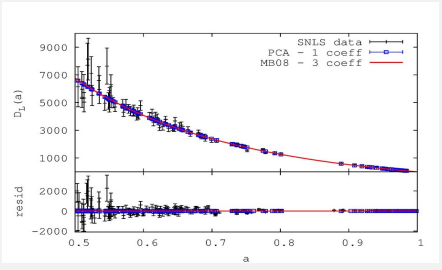
$$e(a) = -a^3 D'_{\rm L}(a) + a\int_1^a \frac{dx}{x^2} e(x).$$

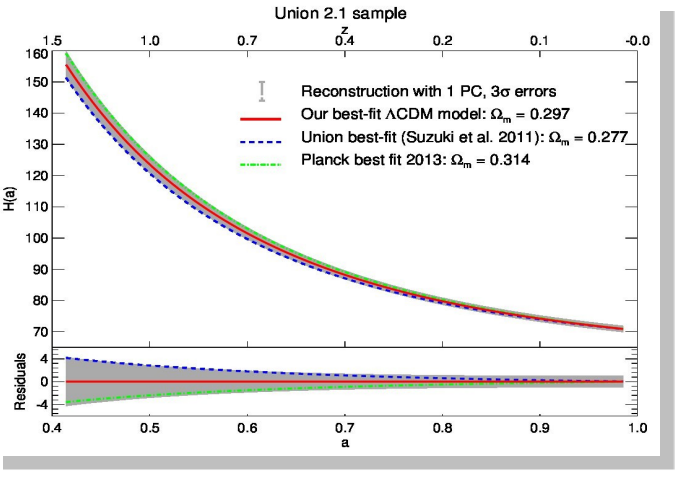
$$e(a) = \sum_{i=0}^{\infty} a^i f_i(a)$$

Solved in terms of a Neumann series

$$D_{\rm L}(a) = \sum_{j=0}^{J} c_j p_j(a).$$

$$D'_{\rm L}(a) = \sum_{j=0}^{L} c_j w'_j(a).$$





Gravitational lensing

Matteo Maturi

Heidelberg University

Cosmology WS 2023/24

Introduction

The key point

- Photons travel along null geodesics
- Matter fluctuations perturb the metric
- Photons follow perturbed null geodesics
- This is gravitational lensing

History: the beginnings

Photons as massive partiles: John Mitchell (1784) / Johann Von Soldner (1801) $\Rightarrow \quad \alpha = \frac{2GM}{c^2 r}$

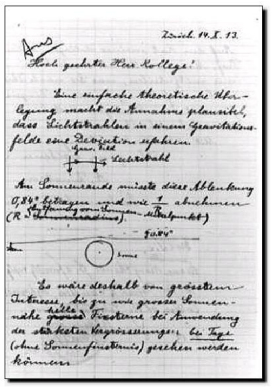
Equivalence principle: Einstein (1915) $\Rightarrow \quad \alpha = \frac{4GM}{c^2 r}$

Experimentl proof: Dyson, Eddington and Davidson (1920) $\Rightarrow \quad \alpha = 1'', 75$!

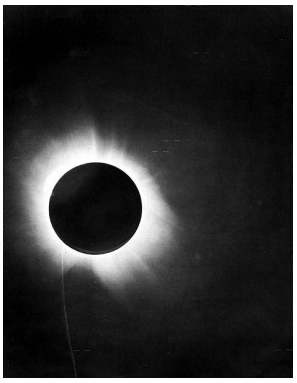
"Dear Mother, – Good news today. H.A. Lorentz has wired me that the British expedition has actually proved the light deflection near the Sun."
A. Einstein



A. Einstein



Einstein notes



Eddington Experiment

History: toward the modern times

Ideas about possible schenarios



"The thinker", Rodin

- Einstein rings (Chowolson, 1924)
- Multiple images
- Stellar microlensing (Mandl -Einstein, 1936)
- Galaxy (nebulae) lensing (Zwicky, 1937)
- M31 star by ML star (Liebes, 1964)
- Time delay (Refsdal, 1964)

The very first observations

Lensing by the Sun (Dyson, Eddington, Davidson 1920)
QSO 0957+561 (Walsh, Carswell, Weymann, 1979)
- 2 images d=6" zs=1.41 identical spectra
- galaxy lens discovery zl=0.36 (Stockton, 1980)
- VLBI → same core-jet structure

Giant Arcs in A370 CI2244 (Lynds & Peterosian 1986])
Rings (Hewitt et al. 1988)
Quasar microlensing (Irwin et al. 1989)
Weak lensing cluster (Tyson, Valdes, Wenk 1990)
Weak lensing group of galaxies - LSS (Brainderd, Blandford, Smail 1996)
Microlensing star - star (ML bulge) (Udalski et al. 1993)
Time delays QSO 095+561 (Kindic et al. 1997)

Gravitational lensing and optics

4-interval

$$ds^2 = c^2 dt^2 - d\vec{x}^2$$
$$ds^2 = \left(1 + \tfrac{2U}{c^2}\right) c^2 dt^2 - \left(1 - \tfrac{2U}{c^2}\right) d\vec{x}^2 = 0 \qquad \left[v^2 = \frac{dx^2}{dt^2}\right]$$

Diffraction index

$$n \equiv \frac{c}{v} = 1 + \frac{2}{c^2}|\Phi| \geq 1 \qquad \Delta t = \int_o^s \frac{2}{c^3}|\Phi|\, dz$$

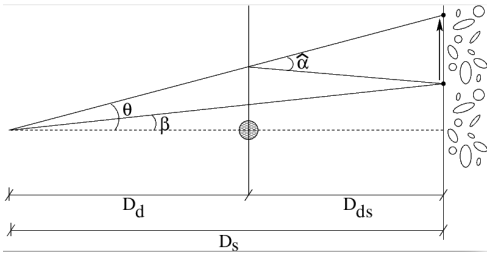
Deflection angle

$$\hat{\vec{\alpha}} = -\int \vec{\nabla}_\perp n\, dz = \frac{2}{c^2}\int \vec{\nabla}_\perp \Phi\, dz$$

The lens equation

$$\vec{\beta}(\vec{\theta}) = \vec{\theta} - \frac{D_{ds}}{D_s}\vec{\alpha}(\theta) = \vec{\theta} - \hat{\vec{\alpha}}(\theta)$$

Assuming: $\theta, \beta, \alpha \ll 1$



- Thin lens approximation: lens size much smaller than distances
- It relates the lens, the source and its image positions on the sky
- Describes how a deflection field acts on a background source.
- $\hat{\vec{\alpha}}$ is the **reduced deflection angle**

The lensing potential: deflection

- The *lensing potential* is defined as:

$$\psi(\vec{\theta}) \equiv \frac{2}{c^2}\frac{D_{\rm ds}}{D_{\rm d}D_{\rm s}}\int \Phi(D_d\vec{\theta}, z)\, dz$$

gravitational potential integrated along the line of sight (thin lens!)

- The reduced deflection angle $\vec{\alpha}$ is the gradient of the lens potential

$$\vec{\nabla}_\theta\psi(\vec{\theta}) = \frac{2}{c^2}\frac{D_{\rm ds}}{D_{\rm d}D_{\rm s}}\int \vec{\nabla}_\theta\Phi\, dz = \frac{D_{\rm ds}}{D_{\rm s}}\frac{2}{c^2}\int \vec{\nabla}_\perp\Phi\, dz = \frac{D_{\rm ds}}{D_{\rm s}}\vec{\alpha}(\vec{\theta})$$

$$\vec{\nabla}_\theta\psi(\vec{\theta}) = \vec{\tilde{\alpha}}(\vec{\theta})$$

- The convergence κ is the laplacian of the lens potential

$$\nabla^2_\theta\psi(\vec{\theta}) = \frac{2}{c^2}\frac{D_d D_{ds}}{D_s}\int \nabla^2_\theta\Phi\, dz = \frac{2}{c^2}\frac{D_d D_{ds}}{D_s}4\pi G\int \nabla^2_\theta\varrho\, dz = \frac{2}{c^2}\frac{D_d D_{ds}}{D_s}4\pi G\Sigma(\vec{\theta})$$

$$\nabla^2_\theta\psi(\vec{\theta}) = 2\kappa(\vec{\theta}) \qquad \kappa(\vec{\theta}) \equiv \Sigma(\vec{\theta})/\Sigma_{cr} \qquad \Sigma_{cr} \equiv \frac{c^2}{4\pi G}\frac{D_s}{D_d D_{ds}}$$

κ quantidies the surface mass density

Here, we used the Poisson eq. $\nabla^2\Phi = 4\pi G\varrho$ and defined the critical density Σ_{cr}

Lens mapping: shear and convergence

- The lens equation can be **linearized** for small α

$$\vec{\beta}(\vec{\theta}) = \vec{\theta} - \frac{D_{ds}}{D_s}\vec{\alpha}(\theta) = \vec{\theta} - \vec{\tilde{\alpha}}(\theta) \qquad \rightarrow \qquad \vec{\beta}(\vec{\theta}) \sim \vec{\beta}_0 + A(\vec{\theta}_0)(\vec{\theta} - \vec{\theta}_0)$$

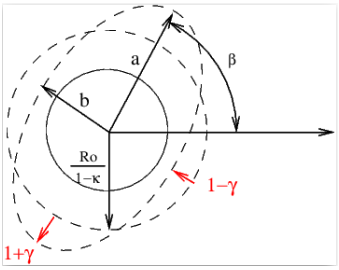
- The image *transformation* of the source is described by the Jacobian matrix of the lens equation

$$A \equiv \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial^2 \psi(\vec{\theta})}{\partial\theta_i \partial\theta_j}\right) = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix} = (1-\kappa)\begin{pmatrix} 1-g_1 & g_2 \\ g_2 & 1+g_1 \end{pmatrix}$$

- isotropic magnification (convergence) and anisotropic distortion (shear) terms

$$A = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

- Here, we introduced the **shear** $\gamma = \gamma_1 + i\gamma_2 = |\gamma| e^{2i\phi}$



$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \qquad \gamma_2 = \frac{1}{2}(\psi_{,12} + \psi_{,21})$$

$$g_i = \frac{\gamma_i}{1-k} \qquad \text{reduced shear}$$

Image magnification

- **Magnification** (defined as the ratio between solid angles)

$$\begin{aligned} a &= r(1-\kappa+\gamma)^{-1} \\ b &= r(1-\kappa-\gamma)^{-1} \end{aligned}$$

$$\begin{aligned} \Omega_I &= \pi ab = \pi r^2 / \det A \\ \Omega_0 &= \pi r^2 = \pi \end{aligned} \qquad \mu = \frac{\Omega_I}{\Omega_0} = \frac{1}{\det A} = \frac{1}{(1-\kappa)^2 - \gamma^2}$$

- **Magnification tensor**:

$$M = A^{-1}$$

Its eigenvalues measure the amplification along the tangential and radial directions:

$$\mu_t = \frac{1}{\lambda_t} = \frac{1}{1-\kappa-\gamma} \qquad \mu_r = \frac{1}{\lambda_r} = \frac{1}{1-\kappa+\gamma}$$

- **Strong lensing and the critical curves**:
magnification diverges along the curves where $\lambda_t = 0$ $\lambda_r = 0$

The mass sheet degeneracy

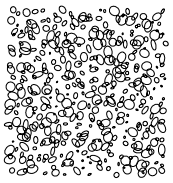
- Homogeneous mass distributions do not introduce any image distortion
- Clear invariance transformation of the lens mapping:

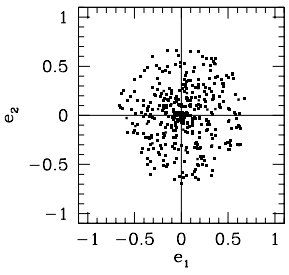
$$A \to \lambda A = \lambda(1-\kappa)\begin{pmatrix} 1-g_1 & g_2 \\ g_2 & 1+g_1 \end{pmatrix} = (1-\kappa')\begin{pmatrix} 1-g_1 & g_2 \\ g_2 & 1+g_1 \end{pmatrix}$$

- The new convergence is $\kappa' = 1 - \lambda(1-\kappa)$
- The new shear is $\gamma' = \lambda\gamma$
- The new reduced shear (what we observe) is unchanged $g' = g$
- One way to break this degeneracy is to combine shear and magnification estimates

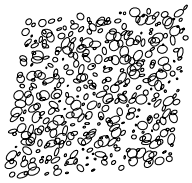
Looking at the sky: galaxies as background sources

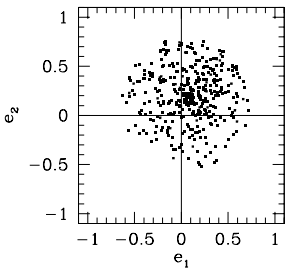
Distant galaxies	$(z \sim 1-2)$
Typical size	$r \sim 1'' - 10''$
Typical density	$n \sim 10 - 100 \, \frac{gal}{arcmin^2}$



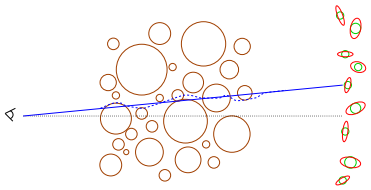


Distorted ellipticity	$\epsilon = \frac{\epsilon^{(s)} + g}{1 + g^* \epsilon^{(s)}}$
Hypothesis:	$\langle \epsilon^{(s)} \rangle = 0 \quad \rightarrow \quad \langle \epsilon \rangle = g$





Gravitational lensing by LSS



Refregier 2003, ARA&A,41,645R

- Linear structures
- Small density contrast: $\delta \equiv \frac{\rho - \rho_b}{\rho_b} \ll 1$
- Characterized by large scales
- Fill the whole line of side: sum over all contributions

- So far we concentrated our attention on single lenses
- But the universe is filled with distributed matter
- Its distribution depends on cosmology $\rightarrow$ let's study it!
- The effective convergence is given by the sum of lens planes

$$\kappa_{\rm eff}(\vec{\theta}, w) = \frac{3H_0^2\Omega_{\rm m}}{2c^2} \int_0^w dw' \frac{f_K(w-w')f_K(w')}{f_K(w)} \frac{\delta[f_K(w')\vec{\theta}, w']}{a(w')} \,.$$

Reminder:

$f_K(w)$: The comoving angular–diameter distance
$\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$: density contrast
$a(w)$: scale factor of the universe at distance w.
$2\kappa(\vec{\theta}) = \nabla^2_\theta \psi(\vec{\theta})$: convergence
$\psi(\vec{\theta}) \equiv \frac{2}{c^2} \frac{D_{\rm ds}}{D_{\rm d} D_{\rm s}} \int \Phi(D_d\vec{\theta}, z)\, dz$ and $\nabla^2\Phi = 4\pi G\varrho$: lending potential

Cosmic shear power spectrum

- Sources are distributed in redshift: weight over the source–distance distribution $G(w)$,

$$\bar{\kappa}_{\rm eff}(\vec{\theta}) = \int_0^{w_{\rm H}} dw\, G(w)\, \kappa_{\rm eff}(\vec{\theta}, w)$$

- The power spectrum of the effective convergence $P_\kappa(k)$ is related to the matter power spectrum $P_\delta(k)$ by the Limber's equation,

$$P_\kappa(k) = \frac{9H_0^2\Omega_{\rm m}^2}{4c^2} \int_0^{w_{\rm H}} dw\, \frac{\bar{W}^2(w)}{a^2(w)} P_\delta\left(\frac{k}{f_K(w)}, w\right)$$

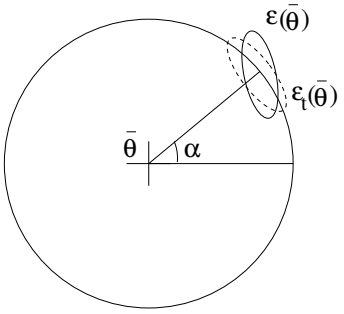
the weight function $\bar{W}(w)$ is given by

$$\bar{W}(w) = \int_w^{w_{\rm H}} dw'\, G(w')\, \frac{f_K(w'-w)}{f_K(w')}$$

The aperture mass

$$M_{\rm ap}(\vec{\theta}) = \int d^2\theta' \, \gamma_t(\vec{\theta}')Q(\vec{\theta} - \vec{\theta}')$$

- Q is defined to be compensated
- Info on surface mass density (K)
- Detection of clusters (candidates)



Lens mapping

- Assumption: sources are much smaller than the scale on which lensing changes

⇒ Linearize $\bar\beta = \bar\theta - \bar{\hat\alpha}(\bar\theta) = \bar\theta - \bar\nabla\psi(\bar\theta)$ at 1° order $\delta\bar\beta \simeq A\,\delta\bar\theta$

$\delta\bar\theta \simeq A^{-1}\delta\bar\beta$ (where A is invertible)

- Jacobian matrix of the lens mapping

$$A_{ij} = \frac{\delta\beta_i}{\delta\theta_j} = \frac{\delta\theta_i}{\delta\theta_j} - \frac{\delta}{\delta\theta_j}\frac{\delta}{\delta\theta_i}\psi(\bar\theta) = \delta_{ij} - \psi_{ij}(\bar\theta) \qquad \psi_{ij} \equiv \frac{\delta^2\psi(\bar\theta)}{\delta\theta_i\,\delta\theta_j}$$

if $\psi_{ij} = 0$ ⇒ $A = I$ identity ⇒ no lensing

- Decompose in isotropic and Anisotropic (trace free) components
(1) (2)

(1) $\mathrm{Tr}(A) = 1-\psi_{11} + 1-\psi_{22} = 2 - \psi_{11} - \psi_{22} = 2 - \bar\nabla^2\psi = 2(1-\kappa)$ $\qquad \kappa \equiv \frac{1}{2}\bar\nabla^2\psi$

(2) $\Gamma \equiv -\left(A - \frac{1}{2}\mathrm{Tr}(A)\cdot I_2\right)$

$$\Gamma_{11} = -\left[\cancel{1}-\psi_{11} - \frac{1}{2}(\cancel{2}-\psi_{11}-\psi_{22})\right] = \frac{1}{2}(\psi_{11}-\psi_{22}) \equiv \gamma_1$$

$$\Gamma_{22} = -\left[\cancel{1}-\psi_{22} - \frac{1}{2}(\cancel{2}-\psi_{11}-\psi_{22})\right] = \frac{1}{2}(\psi_{22}-\psi_{11}) = -\gamma_1$$

$$\Gamma_{12} = \psi_{12} = \psi_{21} = \frac{1}{2}(\psi_{12}+\psi_{21}) \equiv \gamma_2$$

spin 2 (pseudovector)

⇒ κ = convergence, $\qquad \gamma \equiv \gamma_1 + i\gamma_2 = |\gamma| e^{i2\theta}$ Complex shear

$$\Rightarrow A = (1-\kappa)I - \Gamma = (1-\kappa)\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}-\gamma_1 & -\gamma_2\\ -\gamma_2 & \gamma_1\end{pmatrix} = \begin{pmatrix}1-\kappa-\gamma_1 & -\gamma_2\\ -\gamma_2 & 1-\kappa+\gamma_1\end{pmatrix}$$

$0 \to \bigcirc \qquad 0 \to 0$

- in weak lensing regime: $\det(A) = (1-\kappa)^2 - \gamma^2 \simeq 1$

- But what we see is $\bar\theta$ and not $\bar\beta$!

⇒ inverse lensing $\delta\theta = A^{-1}\delta\beta$ which is what we "see": $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix}1-\kappa+\gamma_1 & \gamma_2\\ \gamma_2 & 1-\kappa-\gamma_1\end{pmatrix}$

$\delta\theta$ observable; $\delta\beta$ invertible if $\det(A)\neq 0$; $\frac{1}{\det(A)} = \mu$ magnification

Cosmic shear

- $\bar{x}'(\bar{\theta}, w) = f_k(w)\,\bar{\theta}$ Unperturbed light bundle k = curvature
  - comoving distance
  - angular diameter distance
- $\bar{x}(\bar{\theta}, w) = f_k(w)\,(\bar{\theta} - \bar{\alpha}_{eff})$ Perturbed light bundle, $\bar{\alpha}_{eff}$ effective deflection angle
- $\bar{\alpha}_{eff}(\bar{\theta}, w) = \dfrac{f_k(w)\bar{\theta} - \bar{x}(\bar{\theta}, w)}{f_k(w)} = \bar{\nabla}_\theta \psi = \dfrac{2}{c^2}\displaystyle\int_0^{z_s} \dfrac{D_{ds}}{D_s}\,\bar{\nabla}_\perp \varphi\, dz' = \dfrac{2}{c^2}\int_0^{w_s} \dfrac{f_k(w-w')}{f_k(w)}\,\bar{\nabla}_\perp \varphi\left[f_k(w')\bar{\theta}, w'\right] dw'$
- $\kappa_{eff}(\bar{\theta}, w) = \dfrac{1}{2}\bar{\nabla}_\theta \bar{\alpha}_{eff}(\bar{\theta}, w) = \dfrac{1}{c^2}\displaystyle\int_0^w \dfrac{f_k(w-w')\,f_k(w')}{f_k(w)}\,\bar{\nabla}_\perp^2 \varphi\left[f_k(w')\bar{\theta}, w'\right] dw'$ (lens plane; $*$)
  $= \dfrac{3H_0^2\Omega_0}{2c^2}\displaystyle\int_0^w \dfrac{f_k(w-w')\,f_k(w')}{a(w')\,f_k(w)}\,\delta\left[f_k(w')\bar{\theta}, w'\right] dw'$ (source plane)

  Effective convergence

  $*$ $\bar{\nabla}_\perp^2 \varphi = \dfrac{3H_0^2\Omega_0}{2a}\,\delta$ $\delta \equiv \dfrac{\rho - \langle\rho\rangle}{\langle\rho\rangle}$ 3-D density contrast

With distributed sources

$\bar{\kappa}_{eff}(\bar{\theta}) = \displaystyle\int_0^{w_H} dw\, G(w)\,\kappa_{eff}(\bar{\theta}, w)$ G: sources redshift distribution

$= \dfrac{3H_0^2\Omega_0}{2c^2}\displaystyle\int_0^{w_H} dw\, G(w) \int_0^w \dfrac{f_k(w-w')\,f_k(w')}{a(w')\,f_k(w)}\,\delta\left[f_k(w')\bar{\theta}, w'\right] dw'$

$= \dfrac{3H_0^2\Omega_0}{2c^2}\displaystyle\int_0^w \left[\int_0^{w_H} G(w)\,\dfrac{f_k(w-w')}{f_k(w)}\,dw\right] f_k(w')\,\dfrac{\delta\left[f_k(w')\bar{\theta}, w'\right]}{a(w')}\,dw'$

$= \dfrac{3H_0^2\Omega_0}{2c^2}\displaystyle\int_0^{w_H} W(w')\,f_k(w')\,\dfrac{\delta\left[f_k(w')\bar{\theta}, w'\right]}{a(w')}\,dw'$ $W(w') \equiv \displaystyle\int_{w'}^{w_H} G(w)\,\dfrac{f_k(w-w')}{f_k(w)}\,dw$

- Projection: Limber's eq. $\delta_{3D} \to \delta_{2D}$

$P_\kappa(\ell) = \dfrac{9H_0^4\Omega_0^2}{4c^4}\displaystyle\int_0^{w_H} \dfrac{W^2(w')}{a^2(w')}\,P_\delta\left(\dfrac{\ell}{f_k(w')}, w'\right)$ $P_\kappa(\ell) = P_\delta(\ell)$ become they are both 2nd order derivatives of φ

Another related statistic:

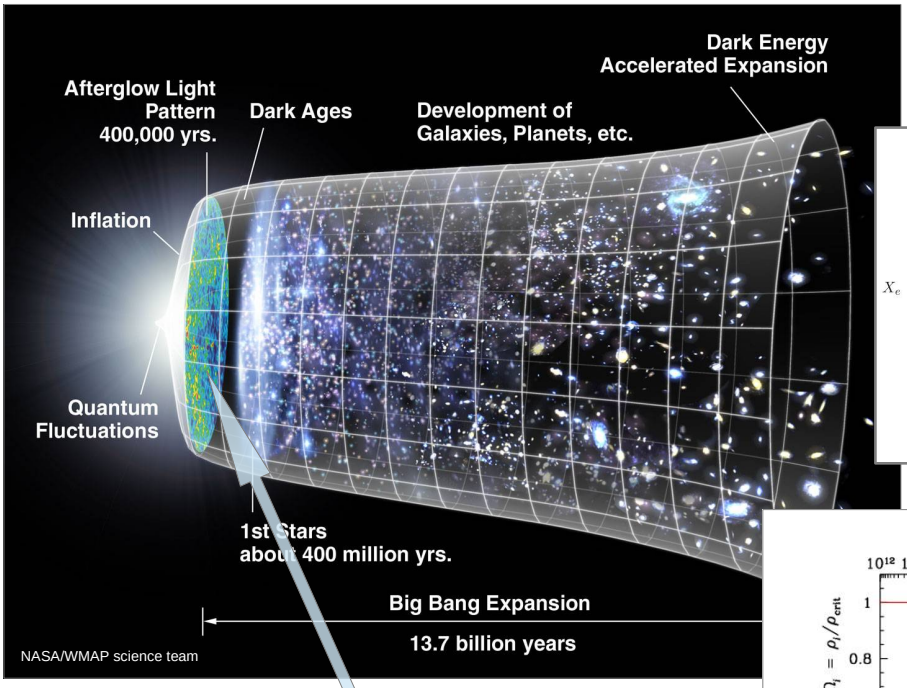
- Aperture mass dispersion: $\langle M_{ap}^2\rangle(\theta) = 2\pi\displaystyle\int_0^\infty \ell\, d\ell\, P_\kappa(\ell)\, J^2(\ell\theta)$ $J(x) = \dfrac{12}{\pi x^2} J_4(x)$ (4^{th} Bessel function)

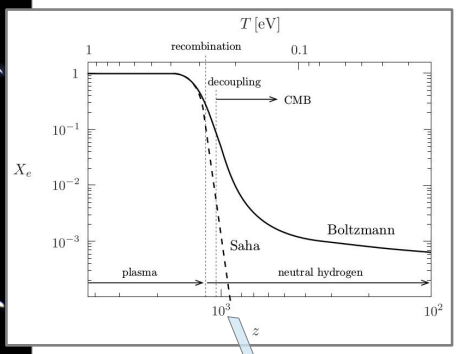


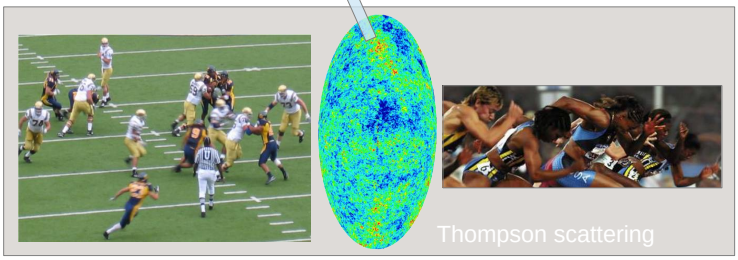
Cosmic Microwave Background: intro and the power spectrum

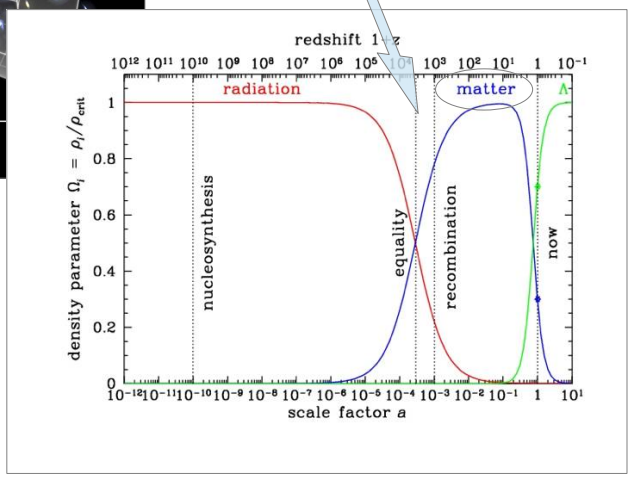
Matteo Maturi

Heidelberg University









26/01/24 *Matteo Maturi* *2*

Few simple facts



Some history:

- Last scattering surface => z=1100
- Up to recombination baryons and photons are tightly coupled => same T

$a(1+z)^{-1}$	=>	universe is 1000 smaller
$T_r \propto a^-1 = (1.z)$	=>	" " 1000 hotter
$T \propto a^{3/2} = (1+z)^{-3/2}$	=>	" " photons traveled for 99,997% age of the universe
$\rho \propto_0 0.04(1+z)^3$	=>	" " 10^9 times denser

- Linear perturbations (analytical)
  Simple physics (adiabatic contractions)
  |=> It is a clean observable = Powerful!

- Strong prove for Dark Matter:

  Amplitude with linear growth from a=1backword, fluctations turn out as: $\frac{\delta(a=1)}{D_+(a_{CMB})} \geq a_{CMB}^{-1} \approx 10^{-3}$

  But in reality they are smaller $\delta(a_{CMB}) \sim 10^{-5}$. Why? *(flat)*
  -- DM interacts only through gravity
  => do not participate to the oscillations visible in the CMB
  => today we see the large contribution on the cosmic structures which is not visible in the CMB temperature anysotropies
  -- the earlier estimate does not account for this

- Important scale : $R_h \sim 1.7^o$ how can T be so uniform on much larger scales?

- <T> = 2.725K => $\pi \approx 1mm$

26/01/24 *Matteo Maturi* *4*

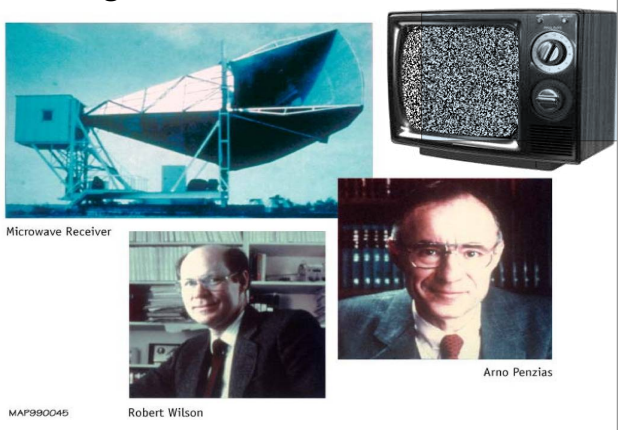
The discovery of the CMB

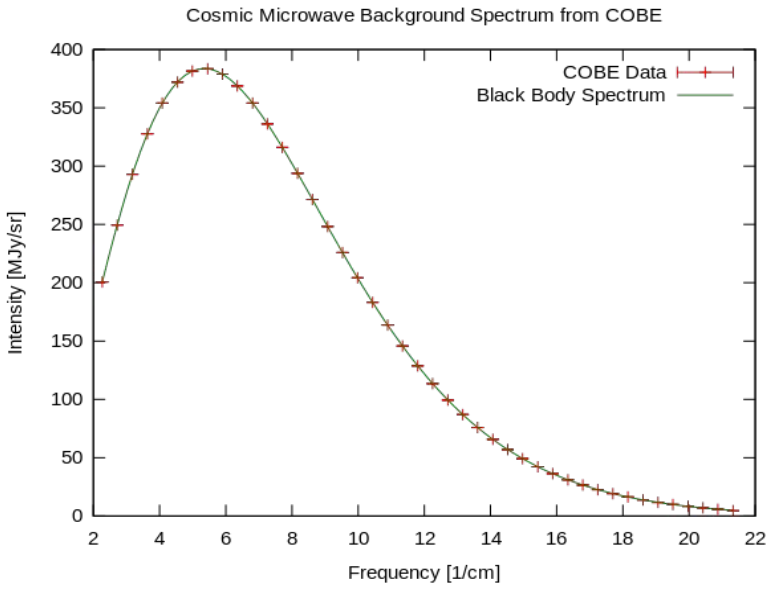


Some history:

- **1946** *Robert Dicke*. Radiation of cosmic matter, predic: T < 20K (not yet CMB)
- **1946** *George Gamow*. Predict T ≈ 50K assuming H = 3 10^6 years ant actual temperature of IGM
- **1948** *Ralph Alpher & Robert Herman*. Estimate T ≈ 5K
- **1949** *Ralph Alpher & Robert Herman*. Correct to T ≈ 28K
- **1953** *George Gamow*. Estimate T ≈ 7K .
- **1956** *George Gamow*. Estimate T ≈ 6K .
- **1960s** *Robert Dicke.* Estimate T ≈ 40K name it MBR (Microwave Background Radiation)!
- **1965** *Arno Penzias & Robert Woodrow Wilson*. Measure T ≈ 3K as BigBang signature (CMB)!

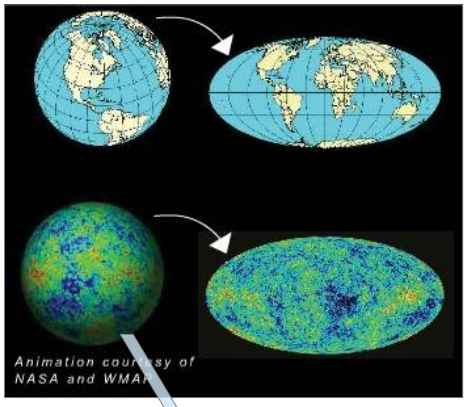
"Fixing antennas"...

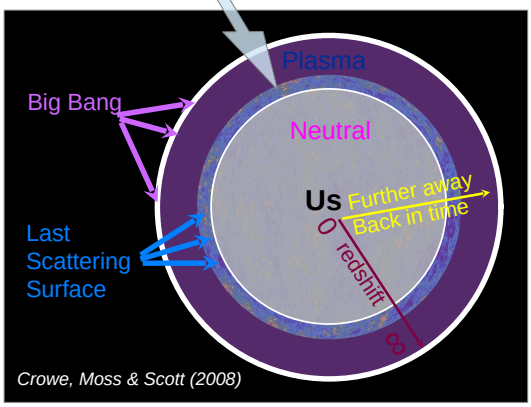


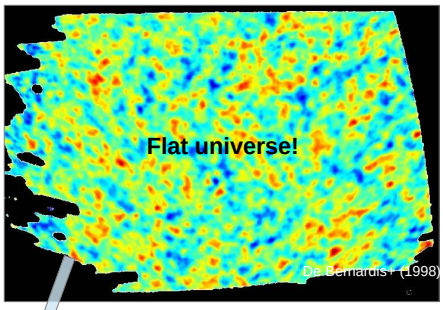


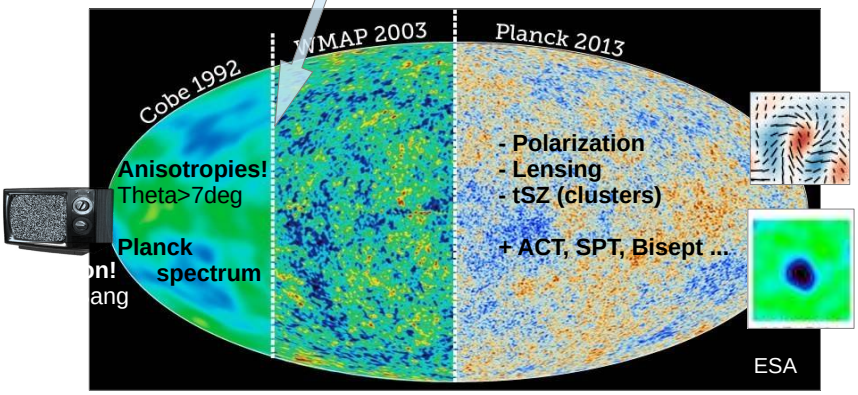
26/01/24 *Matt*

CMB observations



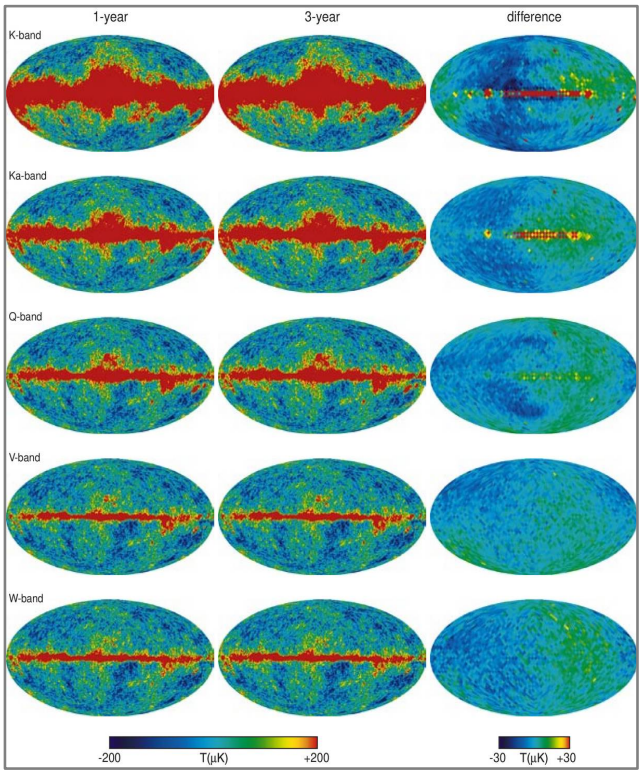






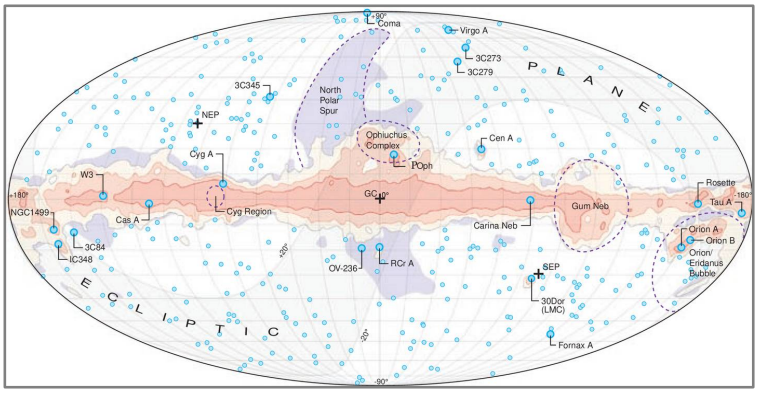
26/01/24 *Matteo Maturi* *6*

Not easy... Galactic foreground



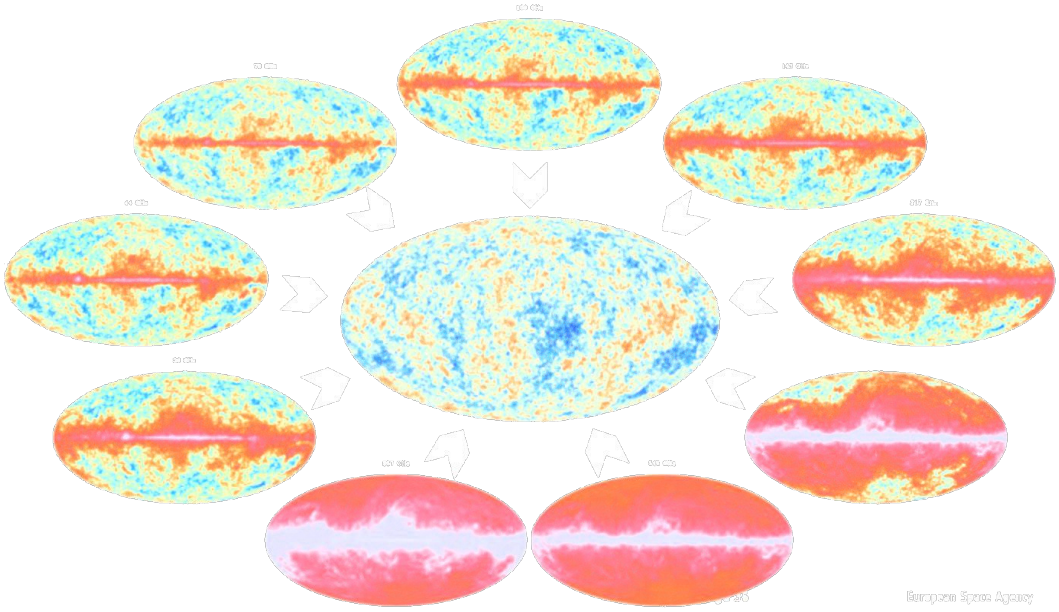
Objects:
- gas
- dust
- localized sources:
  SN remnants, star forming regions...

Process:
- synchrotron
- free-free
- thermal (black body)



26/01/24 *Matteo Maturi* *7*

But possible... Components separation



$$\delta T_{R-J}(\nu) = T_{sync}\left(\frac{\nu}{\nu_K}\right)^{\beta_{sync}(\nu)} + T_{free}\left(\frac{\nu}{\nu_K}\right)^{\beta_{free}(\nu)} + \delta T_{CMB}\, a(\nu) + T_{dust}\left(\frac{\nu}{\nu_K}\right)^{\beta_{dust}(\nu)}$$

Dealing with a sphere:
Decomposition in Spherical harmonics



We see Temperature fluctuations projected on a 2D surface (sky)

$$\Theta(\hat{\mathbf{n}}) = \frac{T(\hat{\mathbf{n}}) - \langle T \rangle}{\langle T \rangle} = \frac{\delta T}{T}(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

Temperature fluctuations

$$Y_{lm} = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(cos\theta) e^{im\phi}$$

Spherical harmonics

P_l^m are the Legendre polynomials

$$a_{lm} = \int_{\theta=-\pi}^{\pi} \int_{\phi=0}^{2\pi} \Theta(\hat{\mathbf{n}}) Y^*_{lm}(\hat{\mathbf{n}}) d\Omega$$

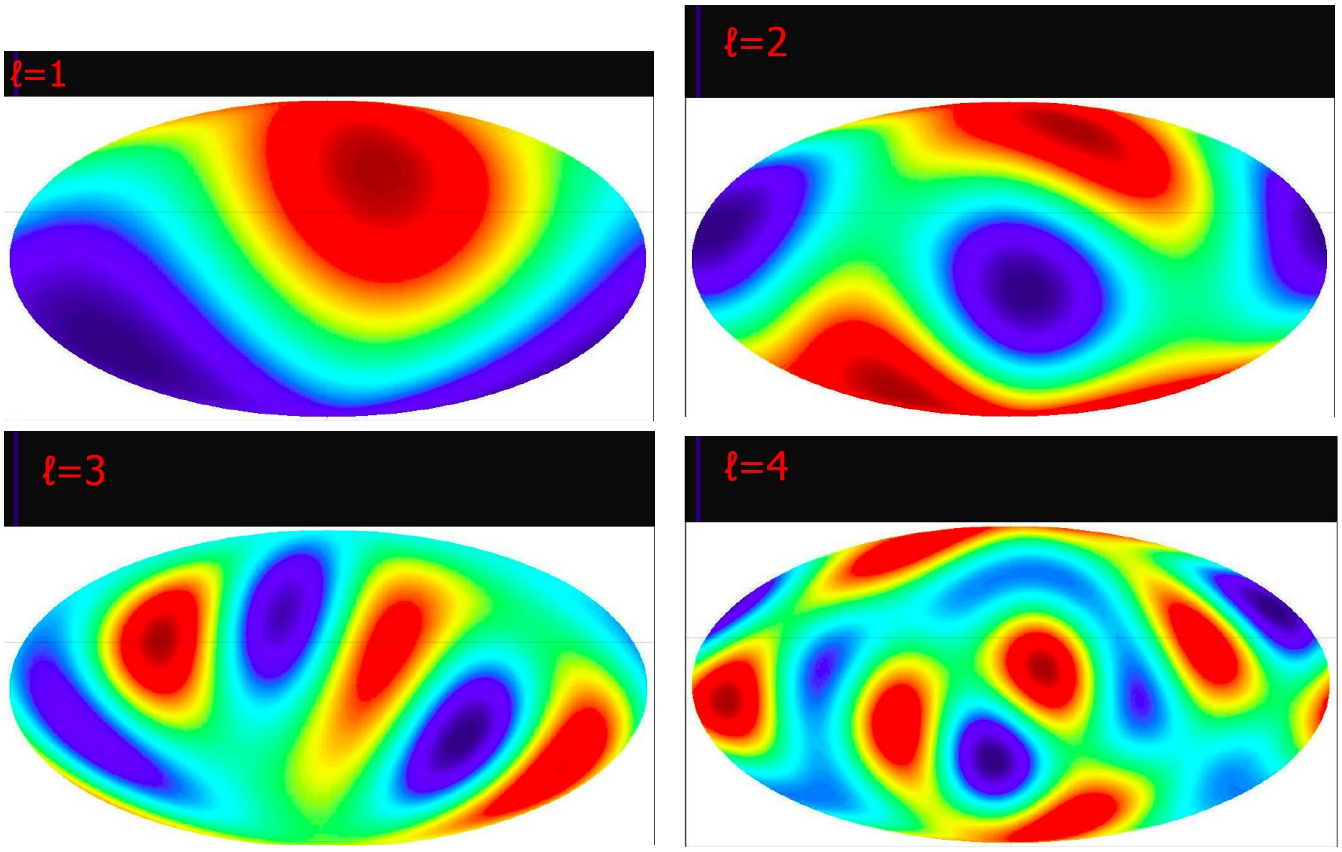
Coefficients

$$\langle a_{lm} a^*_{l'm'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

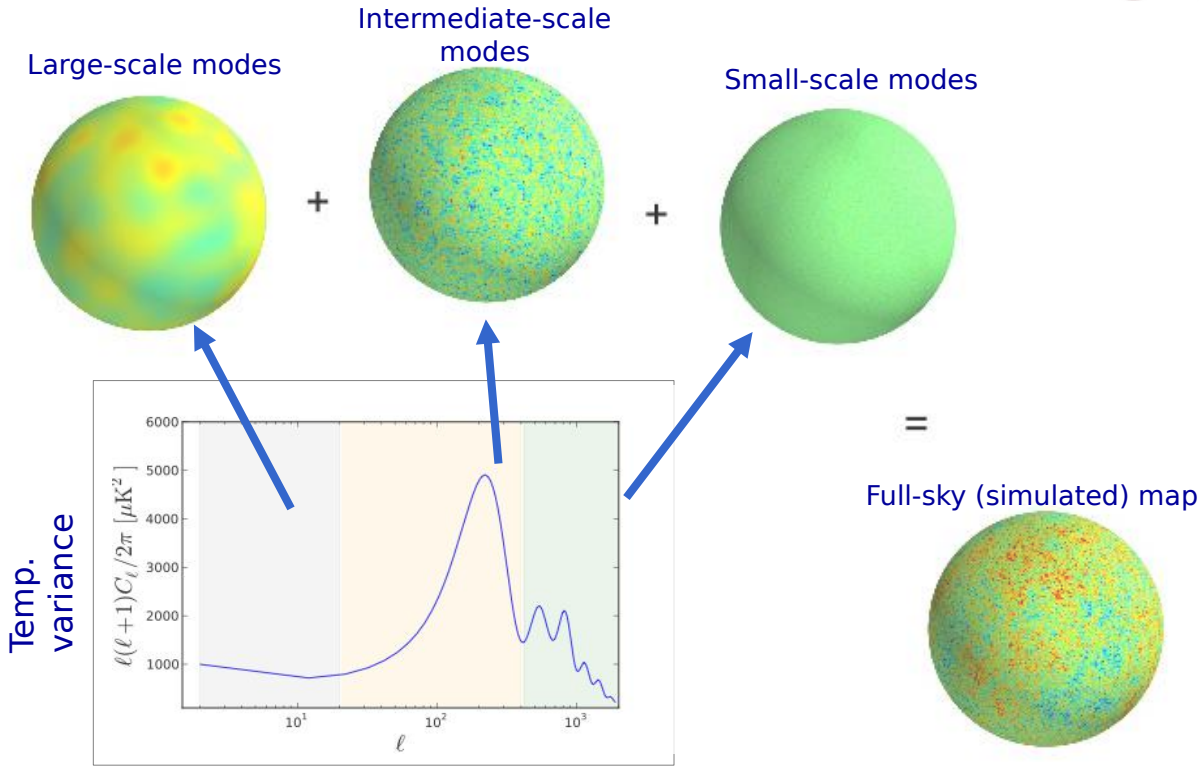
Power spectrum

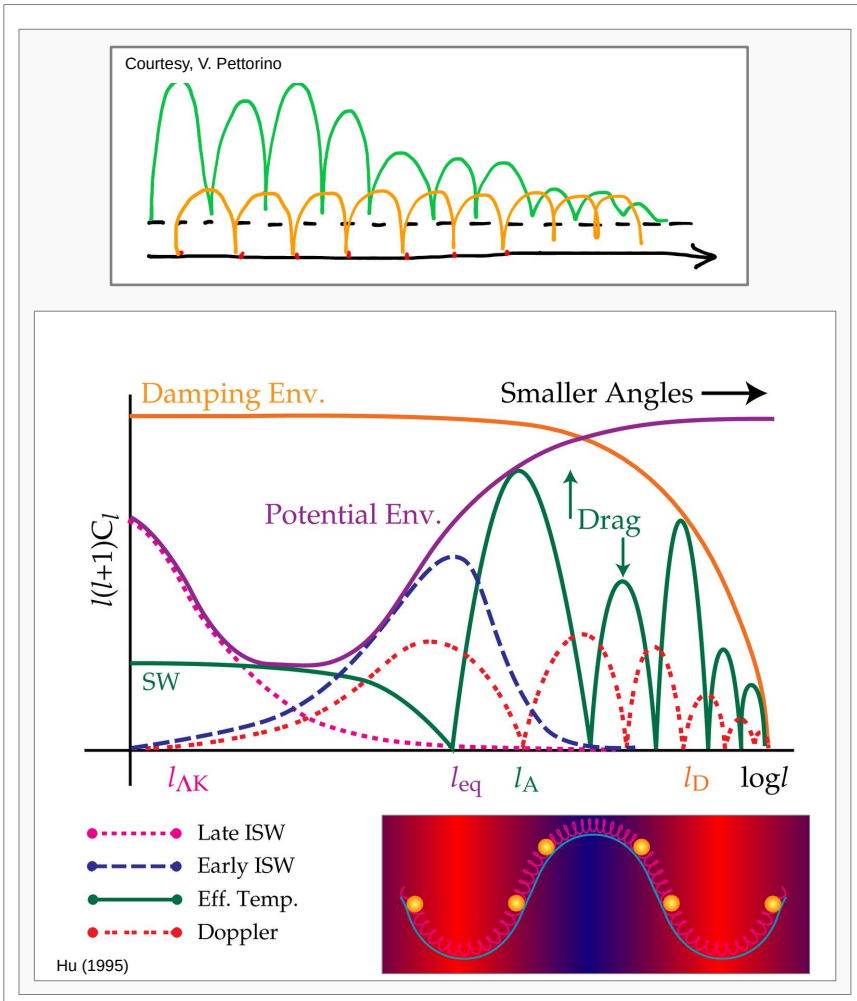
ℓ is called 'multipole' and represents a given angular scale in the sky $\theta \sim \pi/\ell$

How the modes look like?



26/01/24 *Matteo Maturi* *10*


Large-scale modes
Intermediate-scale modes
Small-scale modes
+
+
=
Temp. variance
$\ell(\ell+1)C_\ell/2\pi\ [\mu K^2]$
6000
5000
4000
3000
2000
1000
10^1
10^2
10^3
ℓ
Full-sky (simulated) map
l = oscillations per ~180 degrees
Courtesy of Douglas Scott
26/01/24
Matteo Maturi
11



UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Power spectrum

Compression and expansion ($\lambda > R_H$)
- ~~Velocity > Doppler effect~~
- Gravitational t. delay
- Gravitational redshift > SW effect

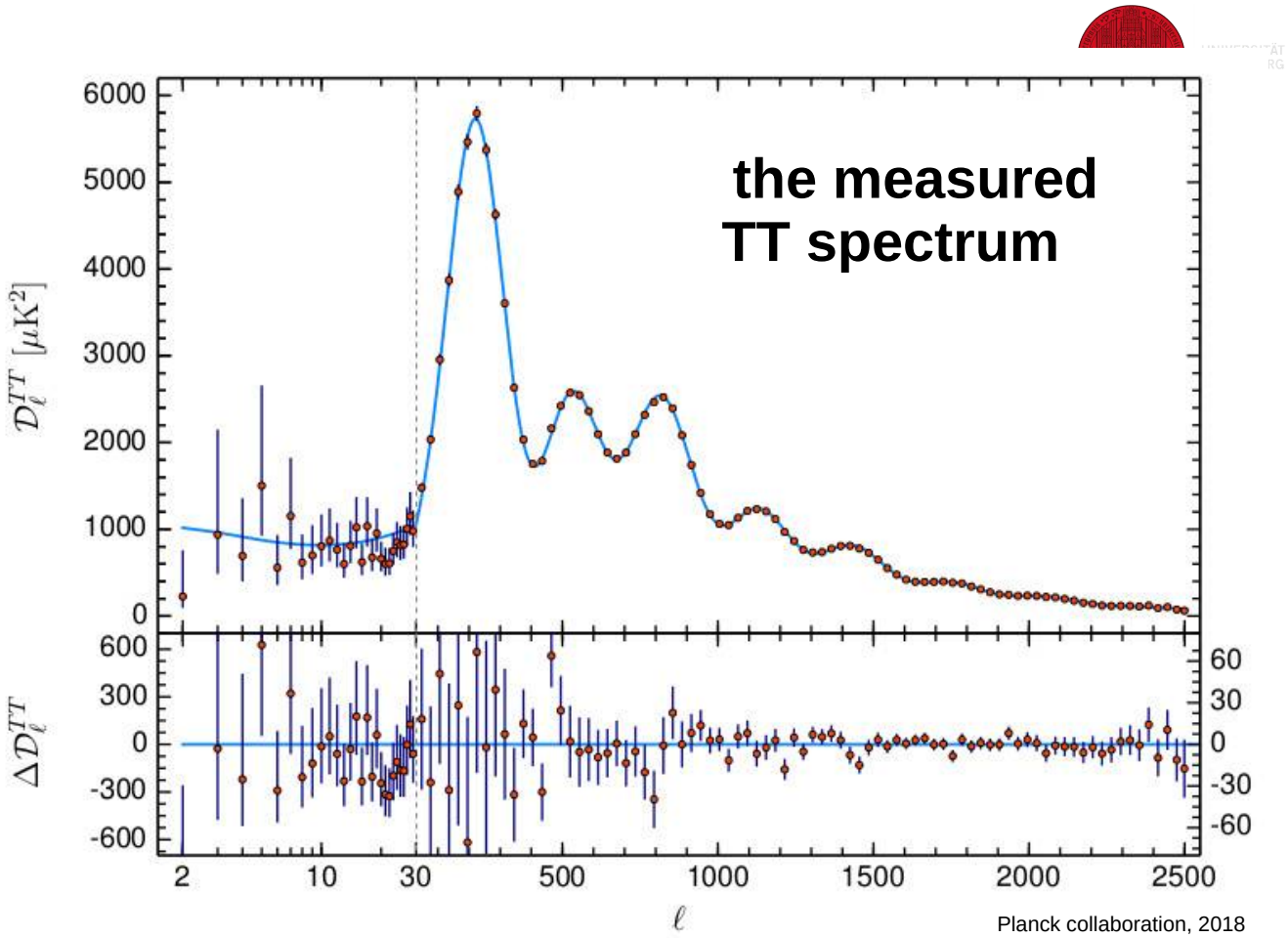
Harmonic oscillationsi ($\lambda < R_H$)
- pressure is active
- velocity and compression

Baryon dragging

Damping suppression

Velocity and density add in quadrature

26/01/24 *Matteo Maturi* *16*



26/01/24 *Matteo Maturi* *21*

CMB temperature fluctuations

- Observable: $T_0 = \langle T(\bar{\theta})\rangle = 2{,}725\,K$ average temperature
  $\delta T(\bar{\theta}) = T(\bar{\theta}) - T_0$ fluctuations $\bar{\theta} = \{R.A., Dec.\}$

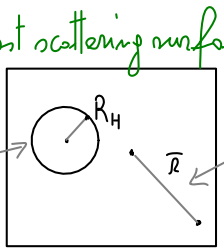
- Fields coupled to photons
  1) Gravity: $\varphi(\bar{r})$ gravitational blue/red-shift, time delays
  2) Density: $\delta(\bar{r})$ adiabatic expansion/compression
  3) Velocity: $\bar{v}(\bar{r})$ doppler effect

- Conceptually

$$\frac{\Delta T}{T}(\theta,\varphi) = \underbrace{\frac{\varphi(\bar{n})}{3c^2}}_{\text{gravity}} + \underbrace{\frac{1}{3}\delta(\bar{n})}_{\text{density}} - \underbrace{\frac{\bar{n}\,\bar{v}(\bar{n})}{c}}_{\text{velocity}} + \underbrace{\frac{2}{c^2}\int \dot{\varphi}(\bar{n},t)\,dt}_{\varphi\text{ evolution}} + \underbrace{tSZ + kSZ}_{\text{galaxy clusters}} \;\overset{\text{Convolution}}{\Big|}\; \underbrace{\text{lensing}}_{LSS} + \text{Reionizati.}$$

- Anisotropies
  - primary $z \geq z_{rec}$ info on primordial conditions
  - secondary $z < z_{rec}$ " " structures

- 2 Regimes:



$n < R_H(a_{CMB}) \Rightarrow P \neq 0$
Acoustic Oscillations

$n > R_H(a_{dec}) \Rightarrow P = 0$
gravity only, No oscillations

(1) $n > R_H(a_{CMB})$: Gravity

- $\bar{v}$: $\bar{v}(\bar{n}) = \bar{v}_\odot + \bar{v}_{dec}(\bar{n})$ $\bar{v}_\odot$ (earth) not interesting $\bar{v}_{dec} \approx 0$ for $n > R_H$

- φ:
  - Grav. Redshift $\frac{\delta T}{T_0} = \frac{\delta\nu}{\nu} = \frac{\delta\varphi}{c^2}$ $\delta\varphi < 0$ (overdensity) cooler (red-shift); $\delta\varphi > 0$ (under density) hotter (blue-shift)
  - Time delay $\frac{\delta t}{t} = -\frac{\delta\varphi}{c^2}$ $\frac{\delta T}{T} = \frac{\delta\nu}{\nu} = -\frac{\delta z}{z} = \frac{\delta a}{a} = \frac{\dot{a}\,\delta t}{a} = \frac{2}{3} t^{-1/3} t^{-2/3}\,\delta t = \frac{2}{3}\frac{\delta t}{t} = -\frac{2}{3}\frac{\delta\varphi}{c^2}$

$a \propto t^{2/3}$ matter domination (Friedmann)

$\Rightarrow$ Sachs-Wolfe (SW) effect $\frac{\delta T}{T} = \frac{\delta\varphi}{c^2} - \frac{2}{3}\frac{\delta\varphi}{c^2}$ $\boxed{\frac{\delta T}{T} = \frac{\delta\varphi}{3c^2}}$

(2) $r < R_H(z_{CMB})$: Acoustic oscillations

- baryons are coupled to photons and affect hydro-eq. of photons (momentum density)
- DM coupled only through gravity φ in Poisson and Euler eq.s

- In comoving coordinates (Photon gas, linear regime)

$$\begin{cases} \dot{n} + \bar{\nabla}(n_0 \bar{u}) = 0 & \text{Continuity} \\ \dot{\bar{u}} + 2H\bar{u} = -\frac{c^2 \bar{\nabla}\delta P}{a^2(\mu_0 + P_0)} - \frac{\bar{\nabla}\delta\varphi}{a^2} & \text{Euler} \\ \bar{\nabla}^2\varphi = 4\pi G a^2 \delta\rho & \text{Poisson} \\ P = \frac{1}{3}\rho c^2 = \frac{\mu}{3} & \text{eq. of state} \end{cases}$$

n = photons number density comoving
$\mu = \rho c^2$ photons energy density
$P = \frac{\mu}{3}$ photons pressure
$\bar{u}$! = streaming velocity of the perturbation

- "Ignore" expansion "H=0" + For convenience $a \doteq 1$: (i.e. ignore damping term for simplicity)

$$\frac{\dot{n}}{n_0} + n_0 \bar{\nabla}\bar{u} = 0 \qquad \dot{\bar{u}} = -\frac{c^2\bar{\nabla}\delta P}{\mu_0 + P_0} - \bar{\nabla}\delta\varphi \qquad \bar{\nabla}^2\varphi = 4\pi G\delta\rho$$

- Photon gas
  - $n \propto T^3$: $\frac{\delta n}{n_0} = 3\frac{\delta T}{T_0}$
  - $\mu \propto T^4$: $\frac{\delta\mu}{\mu_0} = 4\frac{\delta T}{T_0}$
  - $P = \frac{\mu}{3} \propto T^4$: $\frac{\delta P}{P_0} = 4\frac{\delta T}{T_0}$
  - $\Theta \equiv \frac{\delta T}{T_0}$
  - $\mu_0 + P_0 = 4P_0$

$$3\dot{\Theta} + \bar{\nabla}\bar{u} = 0 \qquad \dot{\bar{u}} = -\frac{c^2\bar{\nabla}\delta P}{4P_0} - \bar{\nabla}\delta\varphi = -c^2\bar{\nabla}\Theta - \bar{\nabla}\delta\varphi$$

$\dot{}$, $\bar{\nabla}$

$$3\ddot{\Theta} + \bar{\nabla}\dot{\bar{u}} = 0 \implies \boxed{\ddot{\Theta} - \frac{c^2}{3}\bar{\nabla}^2\Theta - \frac{1}{3}\bar{\nabla}^2\delta\varphi = 0}$$

- To Fourier space + add gravitational redshift by hand

$$\ddot{\hat{\Theta}} + \frac{c^2}{3}k^2\hat{\Theta} + \frac{1}{3}k^2\delta\hat{\varphi} + \frac{\delta\ddot{\hat{\varphi}}}{c^2} = 0$$

$\frac{\delta\ddot{\hat{\varphi}}}{c^2}$: grav. redshift

$(\bar{\nabla}^2 \to -k^2)$ $\left(\frac{\delta\ddot{\hat{\varphi}}}{c^2}\right)$

- Define $\hat{\Theta} \equiv \hat{\Theta} + \frac{\delta\hat{\varphi}}{c^2}$ effective temperature fluctuations

$$\ddot{\hat{\Theta}} + \frac{c^2}{3}k^2\left(\hat{\Theta} - \frac{\delta\hat{\varphi}}{c^2}\right) + \frac{1}{3}k^2\delta\hat{\varphi} = 0 \qquad \boxed{\ddot{\hat{\Theta}} + \frac{c^2}{3}k^2\hat{\Theta} = 0} \implies \text{for } \dot{\hat{\Theta}}_0 = 0 \quad \hat{\Theta}(t_{rec}) = \hat{\Theta}_0 \cos\left[\frac{ck}{\sqrt{3}}t_{rec}\right]$$

t - t_{enter}

$R_s = \frac{c_s t_{rec}}{\sqrt{3}}$ (c_s) sound horizon at t_{rec} $= \hat{\Theta}_0 \cos(R_s k)$

- At decoupling $\dot{P} = 0$ $\Rightarrow$ oscillations get frozen

DM, B : ϕ compresses
γ, B : P counter-action
} BAO

$c_s = \frac{c}{\sqrt{3}} \approx 0{,}6\,c$ sound speed high because $\eta \equiv \frac{n_\gamma}{n_B}$ is very large

$R_s(t) = c_s t$ Sound horizon: defines maximum size of oscillations

(3) Silk Damping

- free streaming of photons
- Diffusion scale, λ_D, above which the power is damped

$\lambda \approx (n_e \sigma_T)^{-1}$ mean free path of photons



$dN = n_e \sigma_T c\, dt$ number of collisions in dt

$\lambda_D \approx \sqrt{N}\,\lambda$ diffusion scale

$\lambda_D^2 = \int \lambda^2 dN = \int \frac{c\,dt}{n_e \sigma_T}$ diffusion scale

$C_\ell \to C_\ell \cdot \exp\left(-\frac{k^2}{k_D^2}\right)$ $\ell_D \equiv \frac{2\pi}{\lambda_D}$ Silk damping: exponential cut off at high ℓ

- small scales baryonic structures are damped

Various contributions to the TT Power Spectrum

Power spectrum

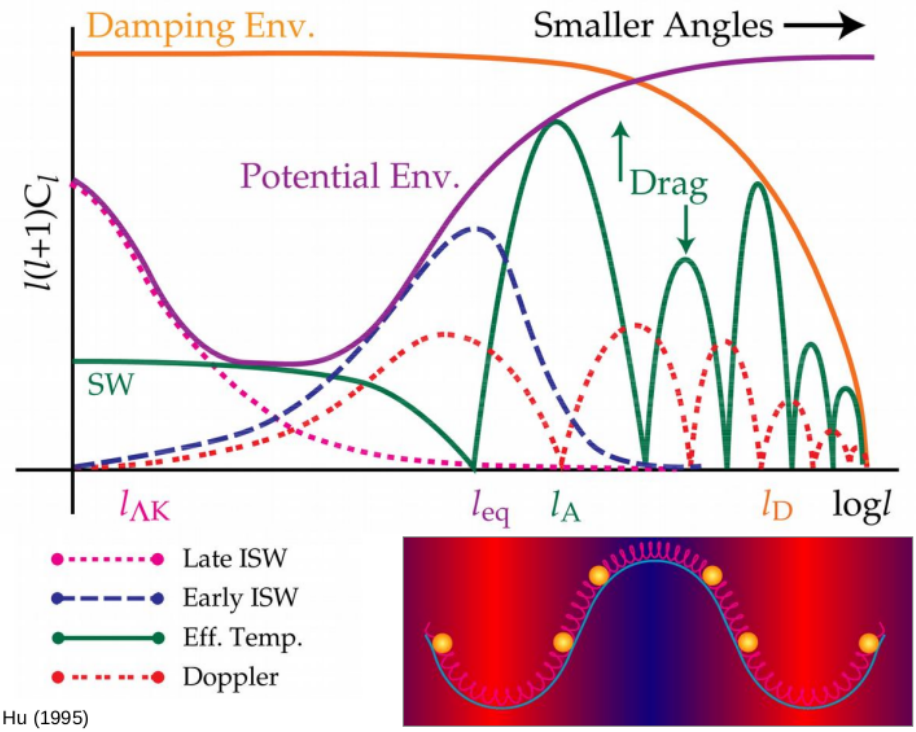
Compression and expansion ($\lambda > R_H$)
- ~~Velocity~~ > Doppler effect
- Gravitational t. delay
- Gravitational redshift > SW effect

Harmonic oscillationsi ($\lambda < R_H$)
- pressure is active
- velocity and compression

Baryon dragging

Damping suppression

Velocity and density add in quadrature

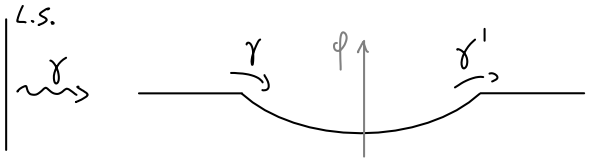


Hu (1995)

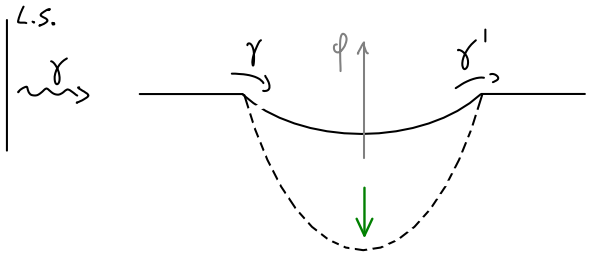
(4) Secondary temperature fluctuations $z < 1000$

- **Integrated Sachs-Wolfe effect** (ISW) $\quad \frac{\delta T}{\delta t} = \int \dot{\varphi}[r(t), t]\, dt$

$\downarrow \frac{\delta \varphi}{\delta \tau}$ conformal time

$\dot{\varphi} = 0$: 

$\gamma = \gamma'$ No frequency change

$\dot{\varphi} \neq 0$: 

$\gamma \neq \gamma'$ γ' has a different frequency

ISW

Early $\quad z \approx z_{rec} \quad \varphi = \varphi_{DM} + \varphi_B + \varphi_R \quad \varphi_R \to 0$ suddenly

$\Rightarrow \dot{\varphi} < 0 \Rightarrow \frac{\Delta T}{T} < 0 \quad l \lesssim 1°$ peack

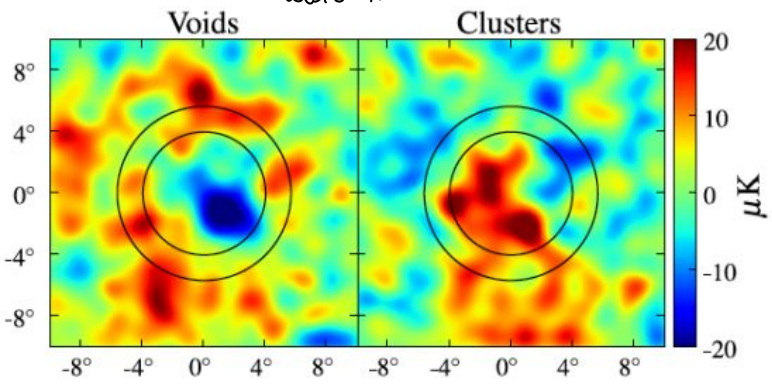
Late $\quad z \ll z_{rec} \quad \Lambda > 0 \quad \varphi \to \varphi + \varphi_\Lambda$

very large scales $\Rightarrow \Lambda$ affects low k

RS (Rees-Sciama) $\longrightarrow$ Non-linear scales $\Rightarrow$ collapse and cluster move

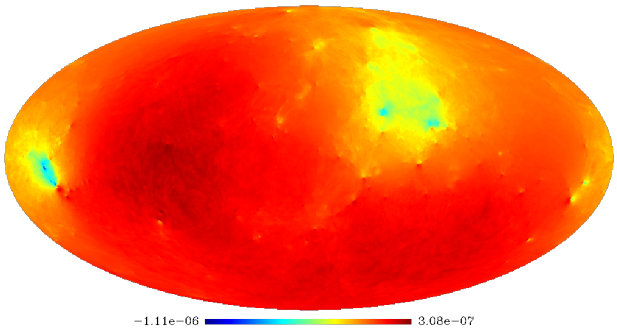
$\Rightarrow$ very high k (small scales)

Late-ISW



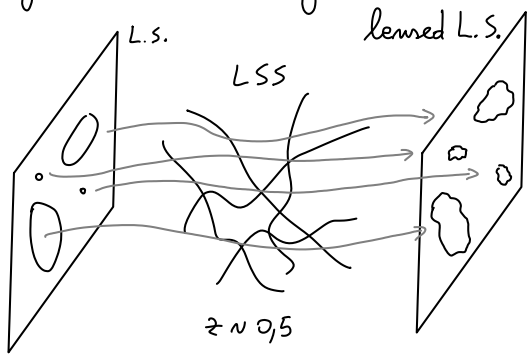
Correlation with voids & superclusters. Benjamin+ (2008)

RS



Maturi+ (2007)

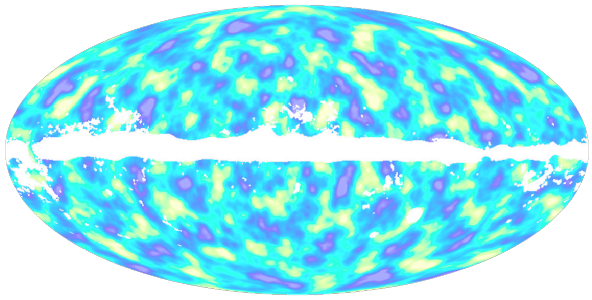
Gravitational lensing



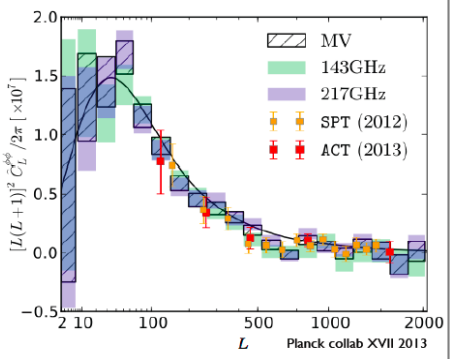
- deflection angle $|\vec{\alpha}| \ll 1$
  $T(\vec{x}) \to \tilde{T}(\vec{x}+\vec{\alpha}) \simeq T(\vec{x}) + \vec{\nabla} T(\vec{x})\vec{\alpha}$
- Change arrival position of photons
- Above certain angular scales is random
  => "Smoothing": peaks are less sharp
- small effect, only 1 ~ 2%
- for $\Omega_M \uparrow$ and or $\sigma_8 \uparrow$ stronger effect

Also... CMB as a lensed source

weak lensing reconstruction: measure matter distribution
power spectrum of reconstructed lensing potential

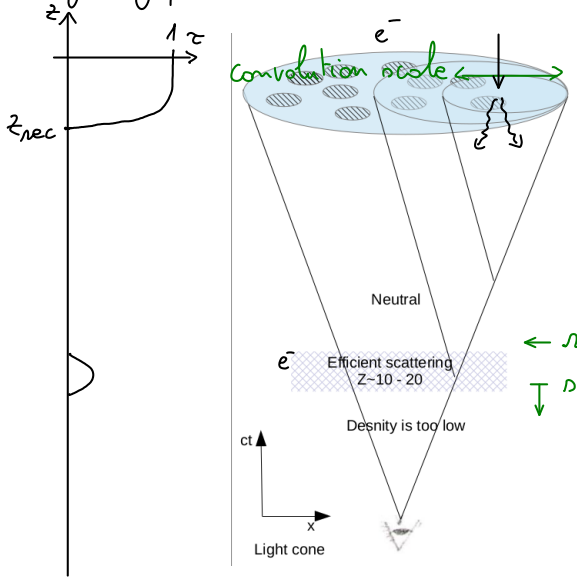


$$\tilde{\Theta}(\hat{n}) = \Theta(\hat{n} + \nabla\psi)$$
$$\tilde{C}_l^{\Theta} \approx \int dl C_{l-l'}^{\psi} C_{l'}^{\Theta}$$



Global Reionization $z \simeq 10$ (?)

- Galaxy formation => stars => UV radiation => ionization of Inter Galactic Medium (IGM)



- Thomson scattering of CMB photons with free e^-
  "Fog" convolves the peaks
  smaller scale fluctuations are diluted away
  amplitude of high l peaks is lowered

← reionization
↧ scattering is not relevant (low n_{e^-})

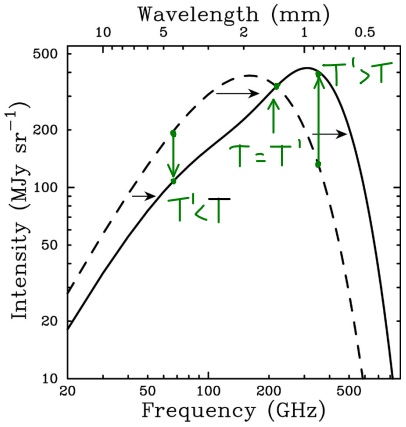
- The typical smoothing scale depends on z_{re}:
  lower z_{re} -> larger convolution scale
  => CMB measures constrain $z_{re} > 6$

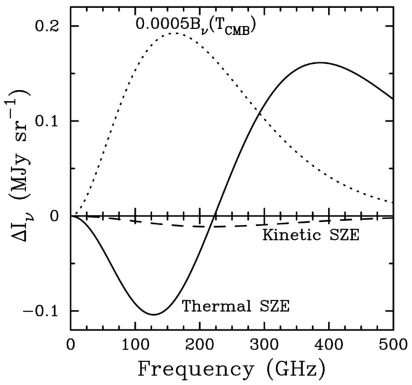
- Sunjaev-Zel'dovich effect

Within galaxy clusters: Intra Cluster Medium (ICM) is fully ionized => free e^-

Kinematic SZ:
- Clusters have a peculiar motion
- scattering creates a doppler shift
- depends on n_e (small effect)

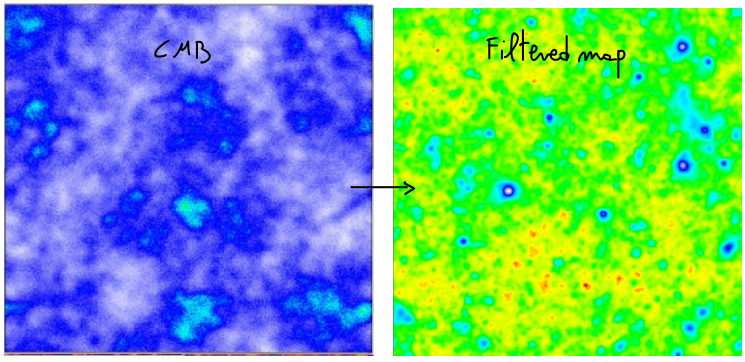
Thermal SZ:
- ICM has high T
  => Inverse Compton scattering: e^- give energy to photons
- CMB black-body spectrum "shifted" toward higher ν
- Temperature change in photons is frequency dependent

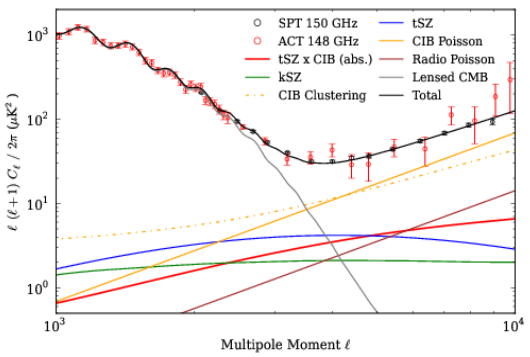




Distortion of the CMB Planck spectrum

- Clusters Mass function
- Pressure profile of galaxy clusters, mass estimates





Polarization

Thomson scattering produces linearly polarized light when incoming radiation have a quadrupole momentum

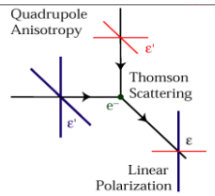
$$I_{ab} = \frac{1}{2} I\sigma_0 + P_{ab} \qquad P_{ab} = \frac{1}{2}(Q\sigma_3 + U\sigma_1 - V\sigma_2)$$

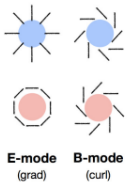
Total Intensity

Polarization tensor

linear linear circular

Stokes parameters





Experiments

COBE : Planck spectrum, anisotropies 1992

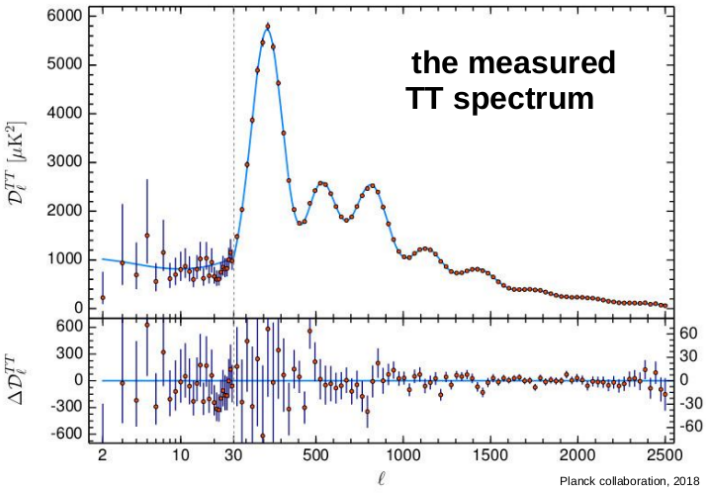
Boomerang : stratospheric baloon, curvature of the universe

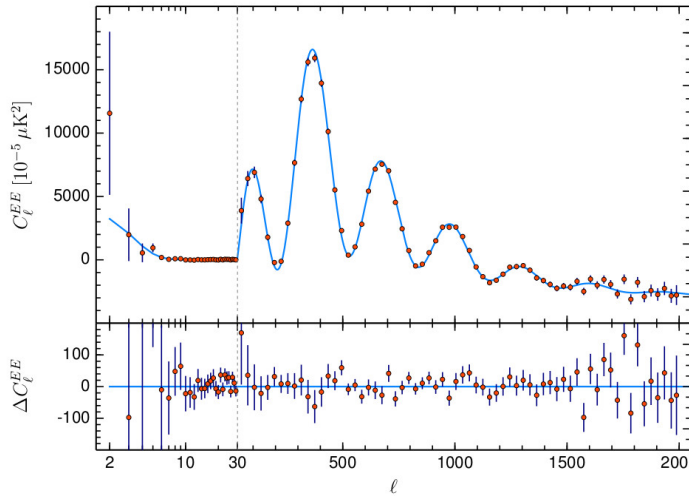
WMAP : higher resolution (~15')

PLANK : even higher resolution (~5') + Polarization

SPT : ground based single dish (~1')

ACT : ground based single dish (~1')

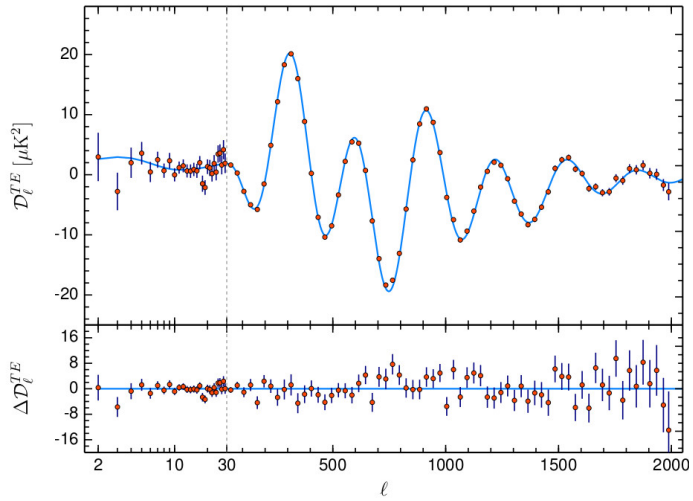




Polarization power spectra

$$C_l^{EE} = \frac{1}{2l+1}\sum_m |E_{lm}|^2 \qquad C_l^{BB} = \frac{1}{2l+1}\sum_m |B_{lm}|^2$$

$$C_l^{TE} = \frac{1}{2l+1}\sum_m \Theta_{lm} E_{lm} \qquad < \text{Cross correlation}$$

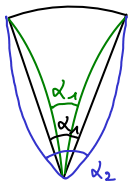


Main dependencies on the cosmological parameters

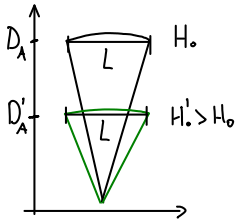
The features of the power spectrum depend on the cosmological parameters

1) Because of the change in the angular diameter distance the same physical scale L displays a different angle α

$$D_A = \frac{1}{H_0}\int \frac{dz}{E(z)}$$

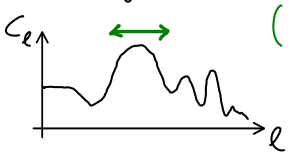
- Ω_0 

$(\Omega_0 \uparrow \quad \alpha \uparrow \quad \ell \downarrow)$

- H_0 

$(H_0 \uparrow \quad \alpha \uparrow \quad \ell \downarrow)$

$(\Omega_0 \downarrow)$ $(\Omega_0 \uparrow)$ rescaling toward higher/lower ℓ $(H_0 \downarrow)$ $(H_0 \uparrow)$

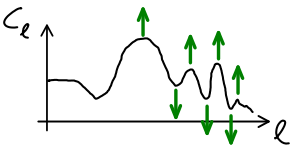


2) Change in the baryonic acoustic oscillations

- Ω_B $\quad \psi = \psi_{DM} + \psi_B + \psi_R$ $\quad (\Omega_B \uparrow \Rightarrow \psi \uparrow) \Rightarrow$ smaller dilation (even peaks) and larger contraction (odd peaks)

$P = P_R$ not affected

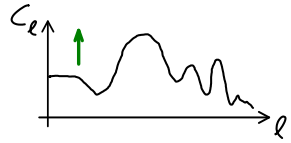
$\Rightarrow$ Change in the amplitude of the acoustic peaks



3) Change in the dynamic of the universe

- Ω_Λ $\quad (\Lambda \uparrow \Rightarrow \frac{\Delta T}{T} \uparrow)$ late ISW

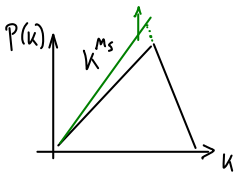
Λ affects the entire universe: large physical scales
low z : large angles
$(\ell \to 0)$ large scales

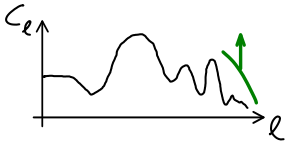


4) Change in the primordial power spectrum (inflation)

- n_s $\quad P(k) \propto k^{n_s} \cdot T^2(k)$ $\quad n_s \uparrow \Rightarrow$ for k small $P(k) \uparrow$

transfer func.

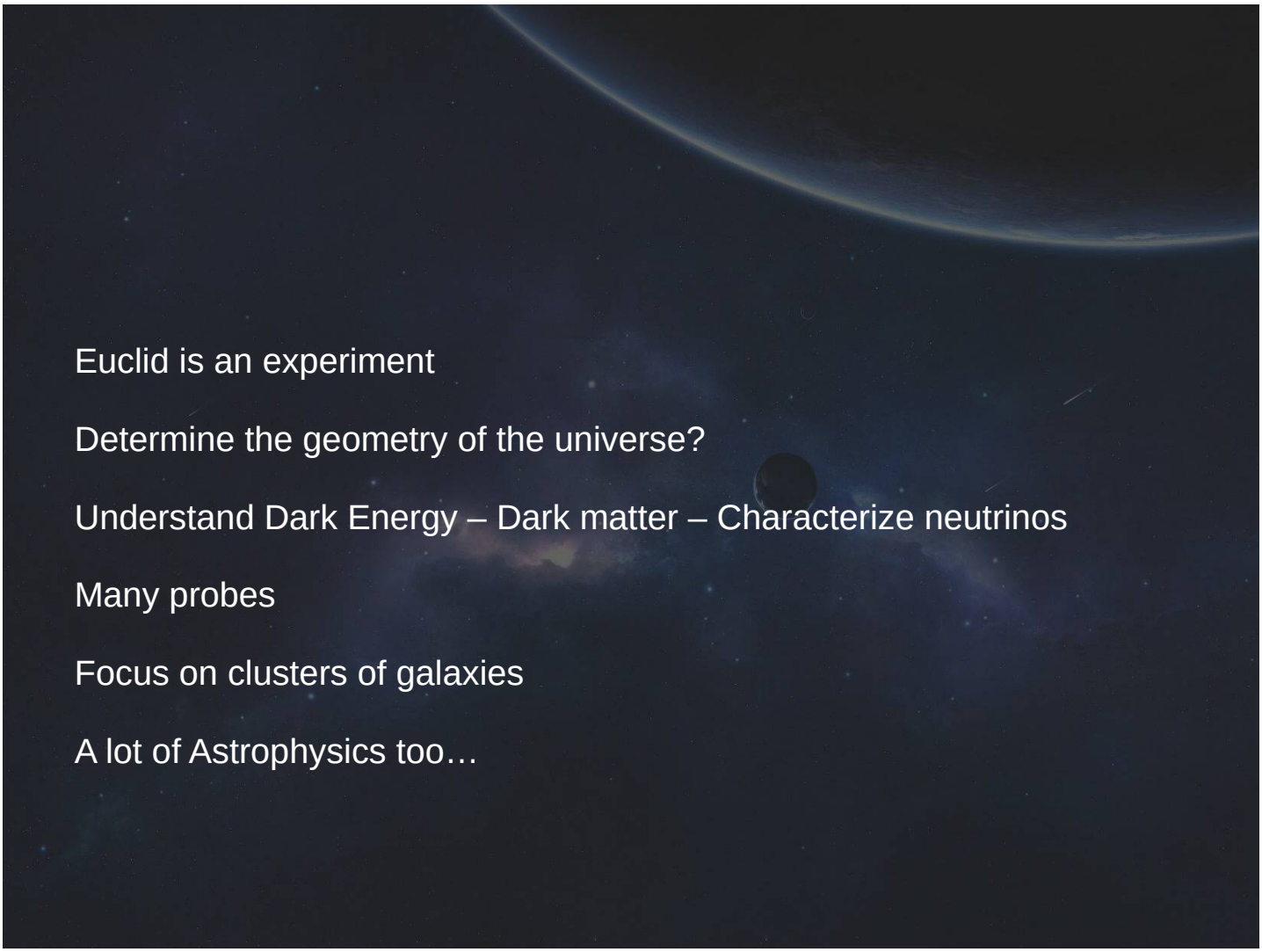


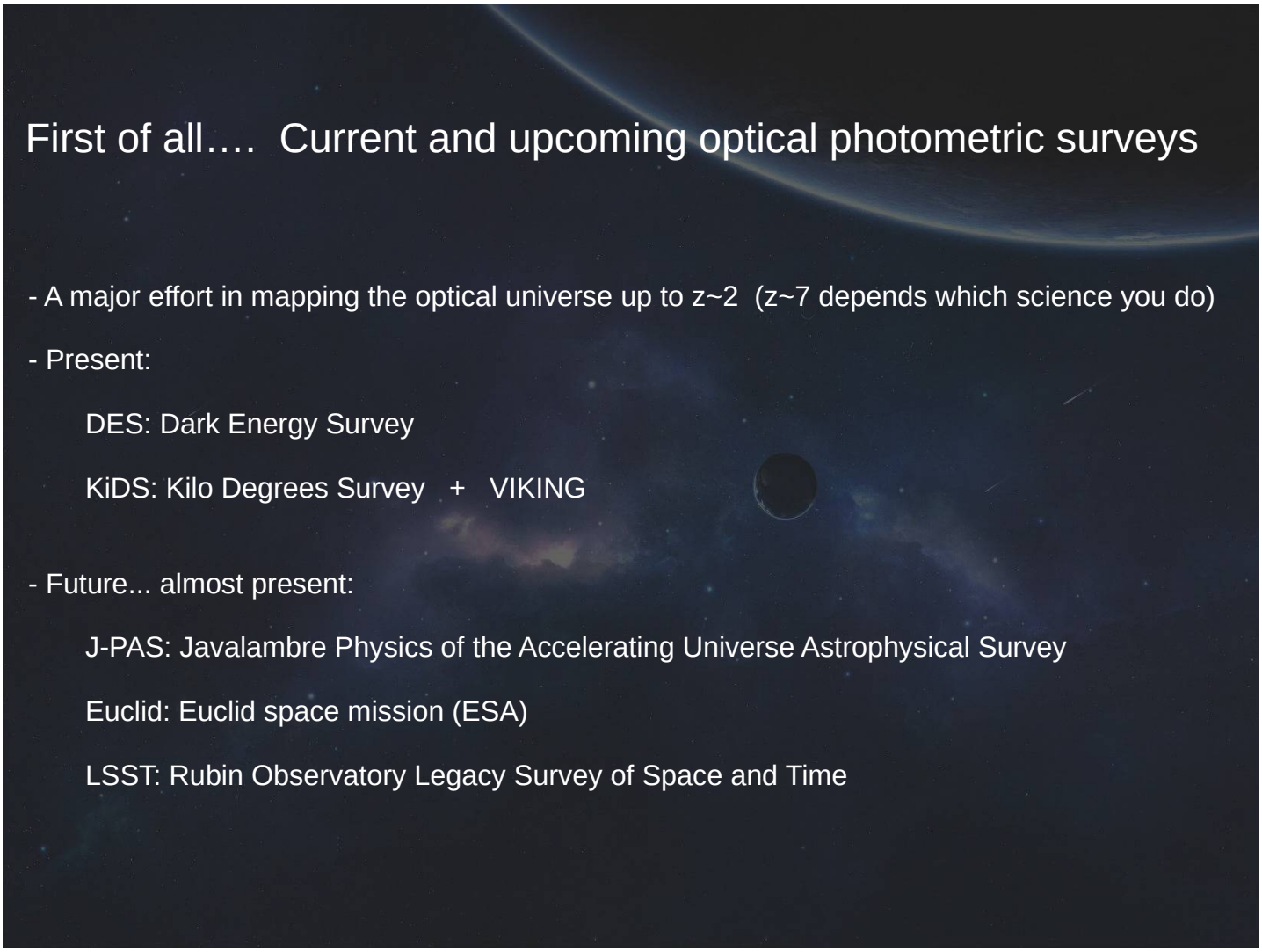


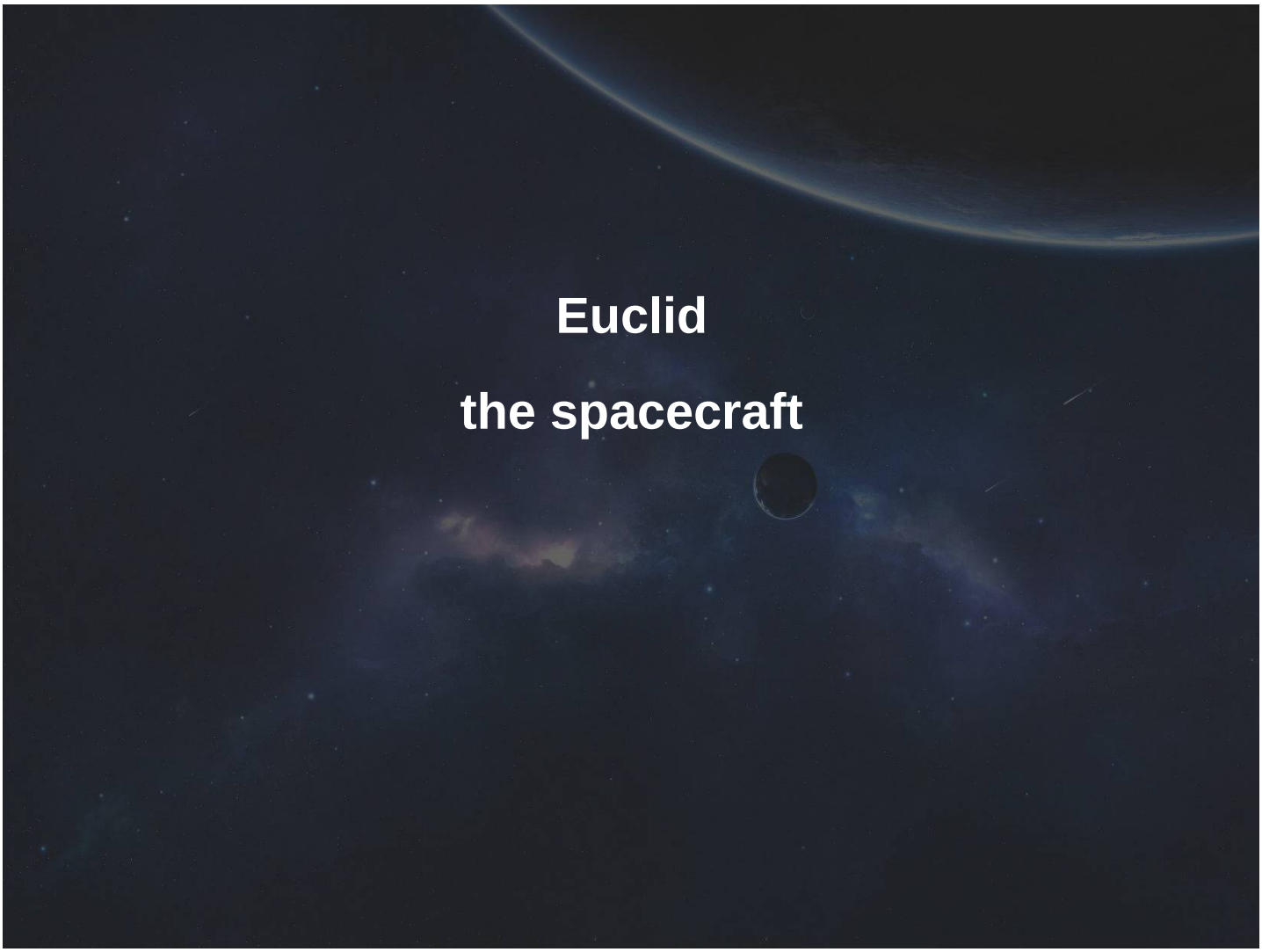
5) Gravitational lensing : $\Omega_M, \sigma_8 \uparrow \Rightarrow$ suppression of higher modes

6) Global reionization : $z_{re} \downarrow \Rightarrow$ suppression starts at lower ℓ

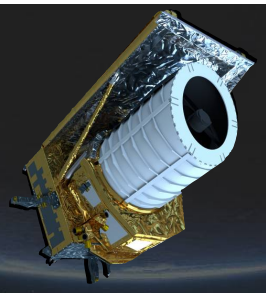

ESA Euclid Space mission
Matteo Maturi
UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386


Euclid is an experiment
Determine the geometry of the universe?
Understand Dark Energy – Dark matter – Characterize neutrinos
Many probes
Focus on clusters of galaxies
A lot of Astrophysics too…


First of all…. Current and upcoming optical photometric surveys
- A major effort in mapping the optical universe up to z~2 (z~7 depends which science you do)
- Present:
DES: Dark Energy Survey
KiDS: Kilo Degrees Survey + VIKING
- Future... almost present:
J-PAS: Javalambre Physics of the Accelerating Universe Astrophysical Survey
Euclid: Euclid space mission (ESA)
LSST: Rubin Observatory Legacy Survey of Space and Time


Euclid
the spacecraft

Euclid in a nutshell



Main driver of the mission:	cosmology, dark energy, dark matter and gravity
ESA space mission:	Cosmic Vision 2007 program
Coverage:	imaging and spectroscopy, optical and IR, full extragalactic sky
Experiment, probes	weak-gravitational lensing (cosmic shear, clusters masses) galaxy clustering measures (3D distribution of galaxies) clusters of galaxies redshift-space distortion galaxy evolution and formation, Stellar…
Science goal I'll focus on:	cosmology, astrophysics of galaxy clusters
Science team:	~1000 persons
It is a global effort:	a nice and global community involving (17 countries, ~260 institutes) a huge coordination effort all major astronomical surveys are involved all codes must be standardized and validated
Also a push for theorists:	need for better models, e.g. in structure formation need a better understanding of behavior of baryons

Euclid Top Level Science Requirements

Sector	Euclid Targets
Dark Energy	Measure the cosmic expansion history to better than 10% in redshift bins $0.7 < z < 2$. Look for deviations from $w = -1$, indicating a dynamical dark energy. Euclid *alone* to give $FoM_{DE} \geq 400$ (1-sigma errors on w_p, & w_a of 0.02 and 0.1 respectively)
Test Gravity	Measure the growth index, γ, with a precision better than 0.02 Measure the growth rate to better than 0.05 in redshift bins between $0.5 < z < 2$. Separately constrain the two relativistic potentials Ψ , Φ . Test the cosmological principle
Dark Matter	Detect dark matter halos on a mass scale between 10^8 and $>10^{15}$ M_{Sun} Measure the dark matter mass profiles on cluster and galactic scales Measure the sum of neutrino masses, the number of neutrino species and the neutrino hierarchy with an accuracy of a few hundredths of an eV
Initial Conditions	Measure the matter power spectrum on a large range of scales in order to extract values for the parameters σ_8 and n to a 1-sigma accuracy of 0.01. For extended models, improve constraints on n and α wrt to Planck alone by a factor 2. Measure a non-Gaussianity parameter : f_{NL} for local-type models with an error < +/-2.

Laureijs et al 2011

Neutrinos

Slides by J. Mellier, adaptation

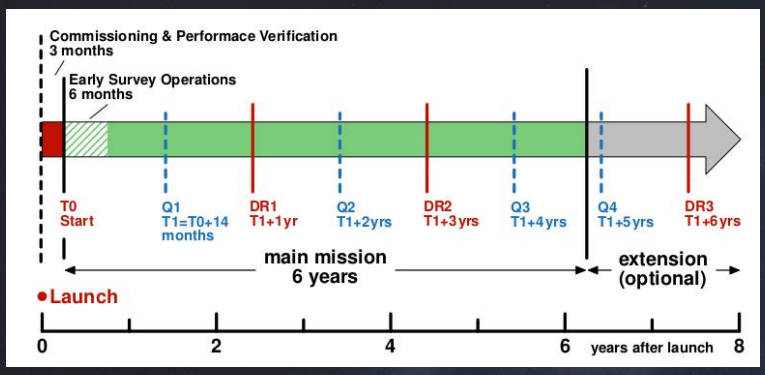
What is it, when, where, how

Space mission in L2, 6 years

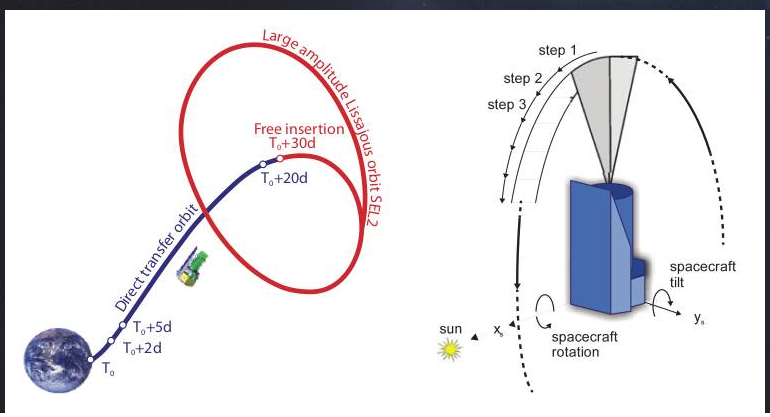
Launch: SpaceX, Falcon 9, July 2023

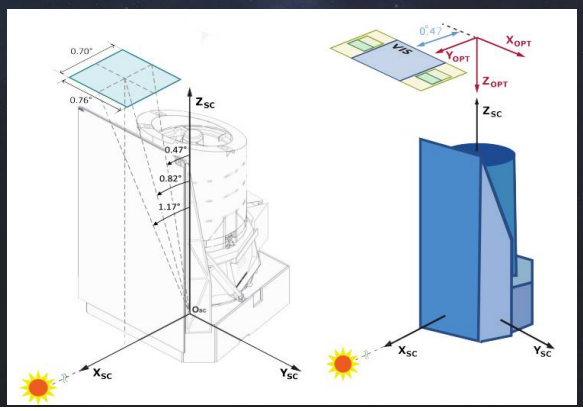
Instrument:
- 1.2m telescope
- VIS: one optical band: r+i+z
- NISP: infrared: Y,J,H

Scaramella et al. 2021





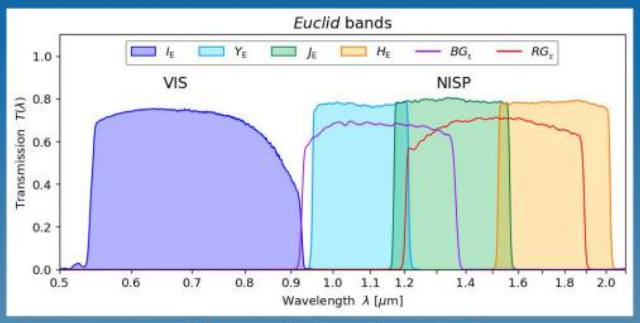




Euclid instruments

Imaging

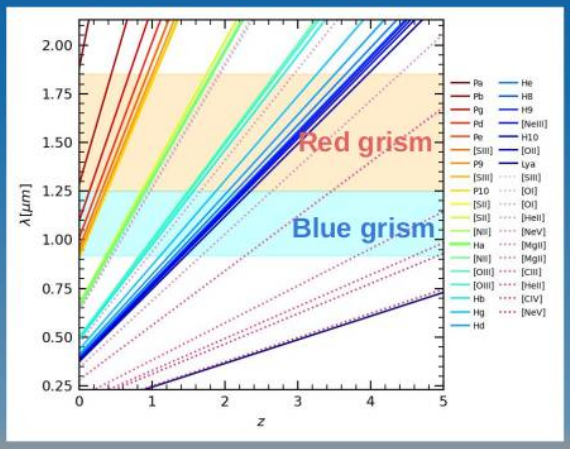
VIS: I_E [530-920nm] – pixel scale: 0.1 arcsec
NISP-P: $Y_E J_E H_E$ – pixel scale: 0.3 arcsec
FoV = 0.57 deg^2



https://www.euclid-ec.org/science/overview/

Slitless Spectroscopy

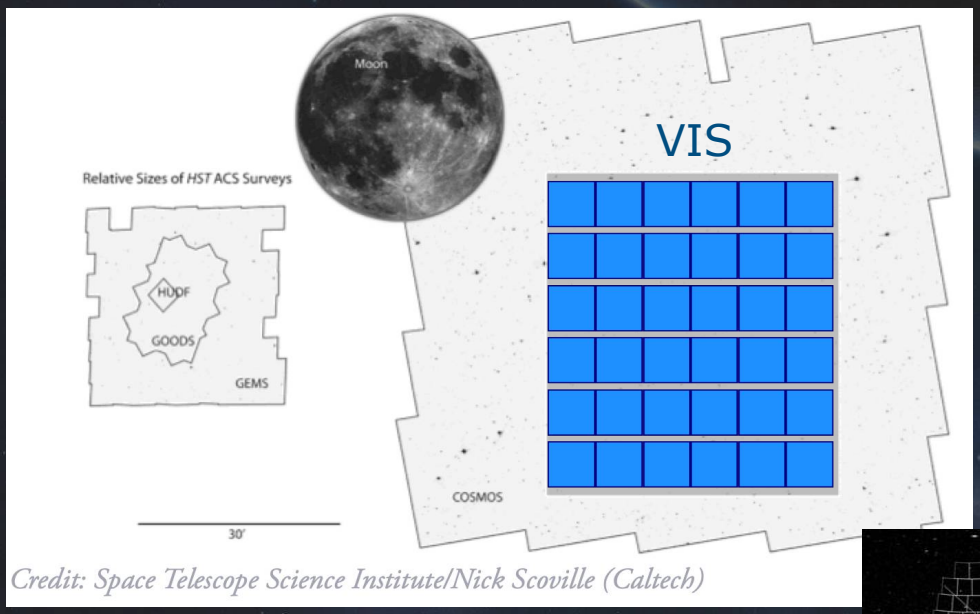
NISP-S:
Red grisms: 1.25–1.85 μm in Wide Survey
Blue grism: 0.92–1.25 μm in Deep Survey
λ/Δλ ~380 (for a 0.5″ source)



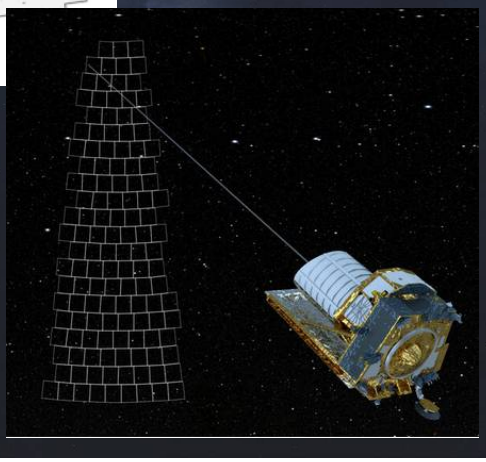
Porto Ercole, 29/09/2023 ✧ *A journey through galactic environments*

Slide from Micol Bolzanella

Wide field imager

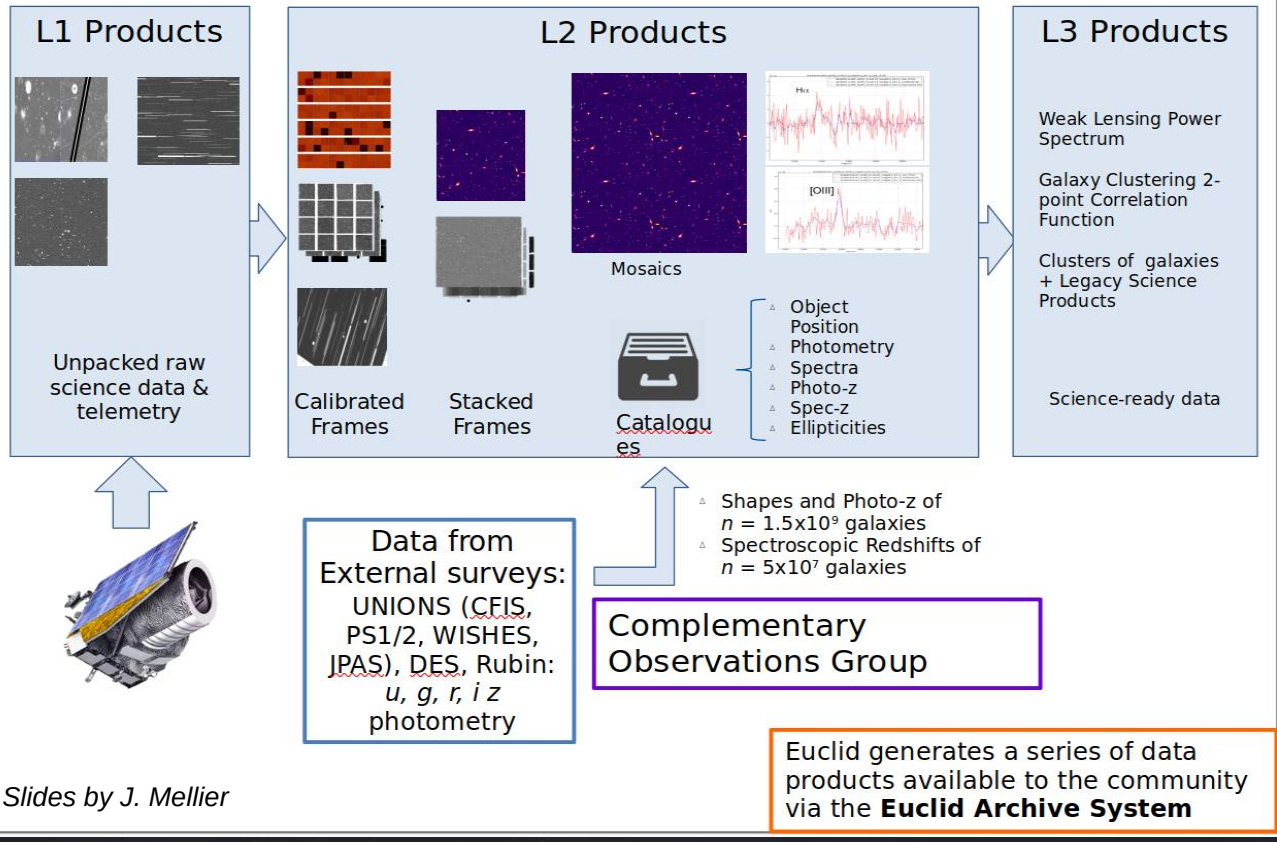


Credit: Space Telescope Science Institute/Nick Scoville (Caltech)



What do we get?

The Euclid Data Products

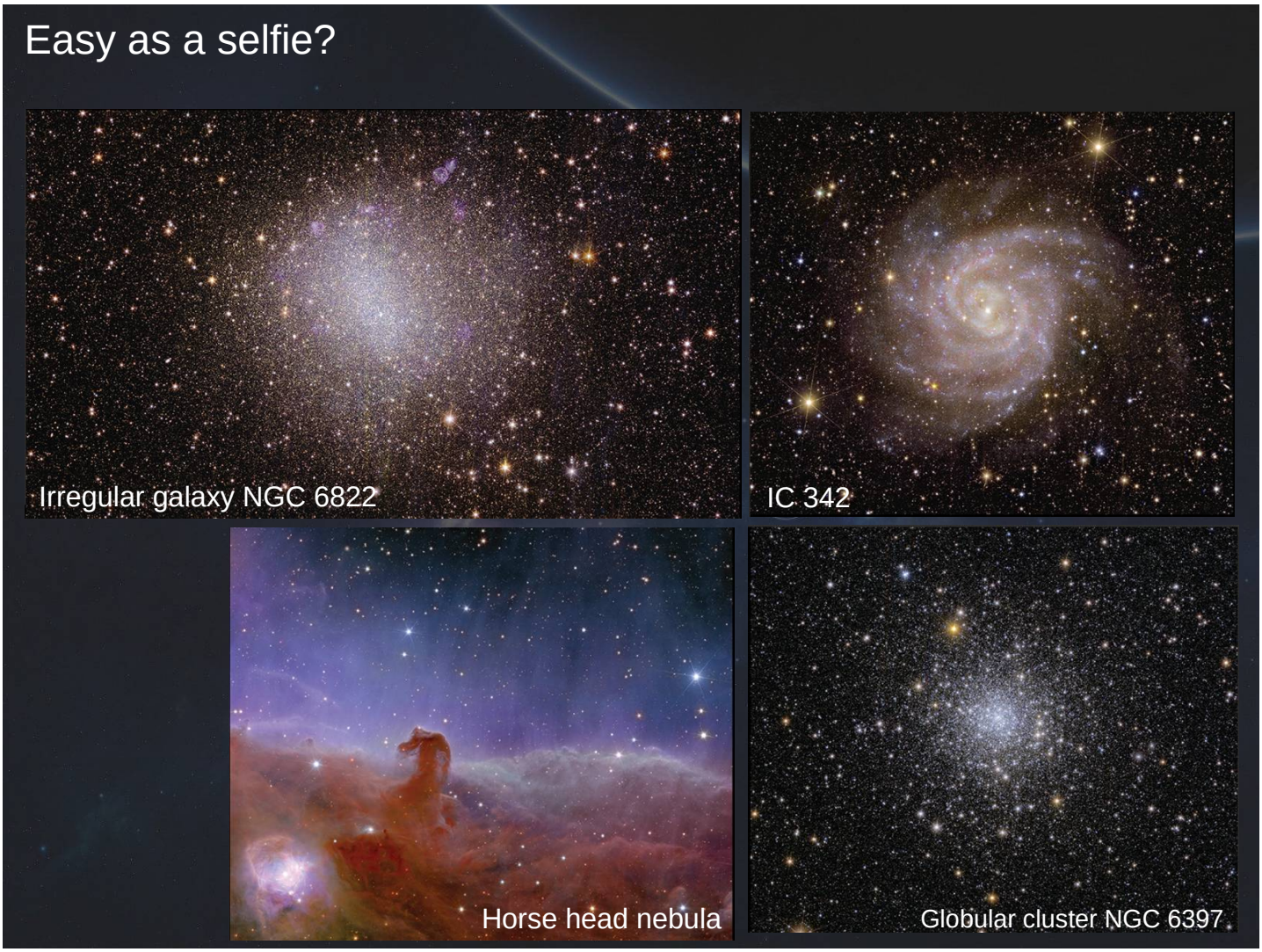


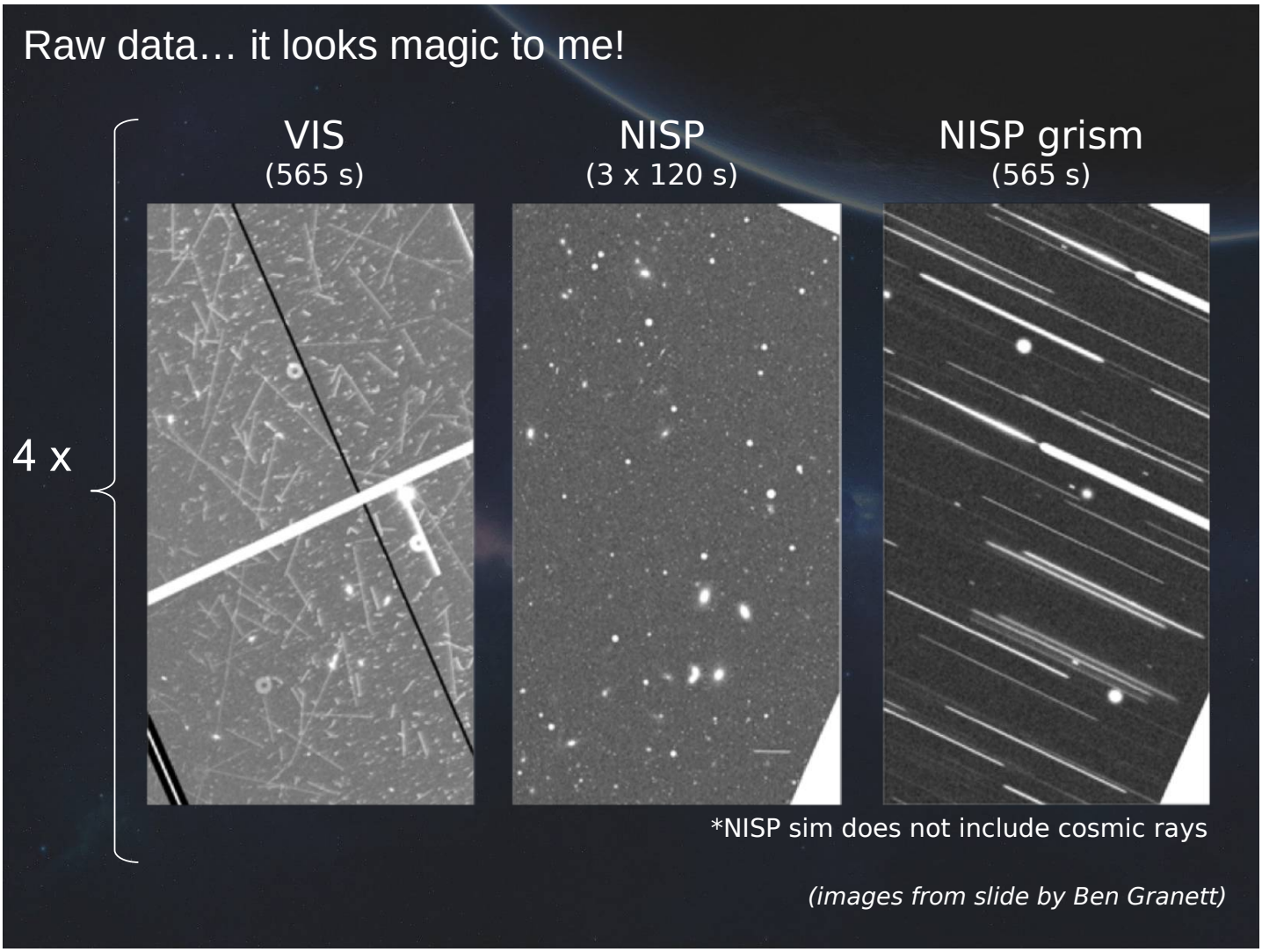
Slides by J. Mellier

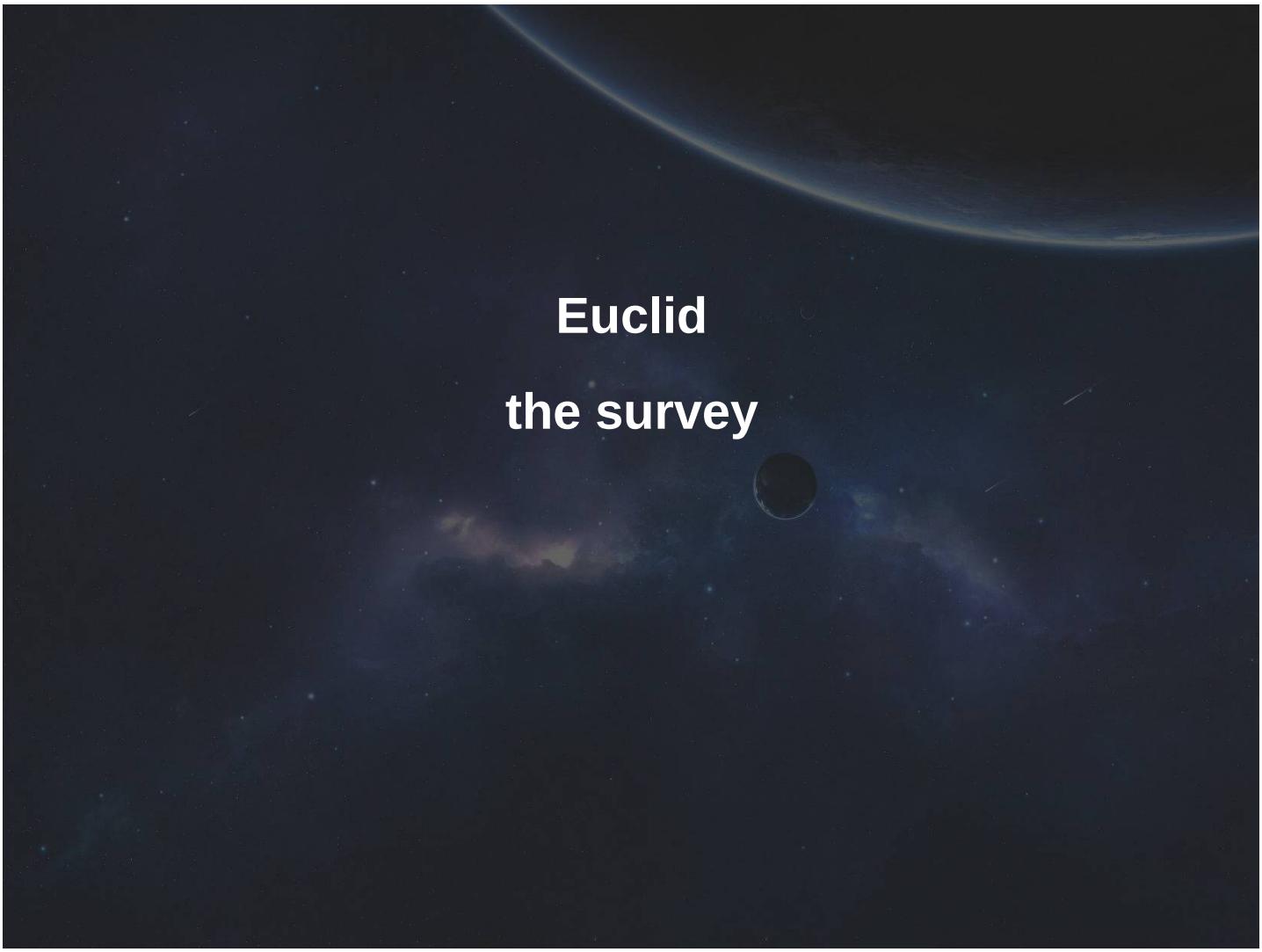
Easy as a selfie? Perseus cluster



Euclid's view on the Perseus Cluster of galaxies. (Credit: ESA/Euclid/Euclid Consortium/NASA/J.-C. Cuillandre (CEA Paris-Saclay)/G. Anselmi)


Easy as a selfie?
Irregular galaxy NGC 6822
IC 342
Horse head nebula
Globular cluster NGC 6397

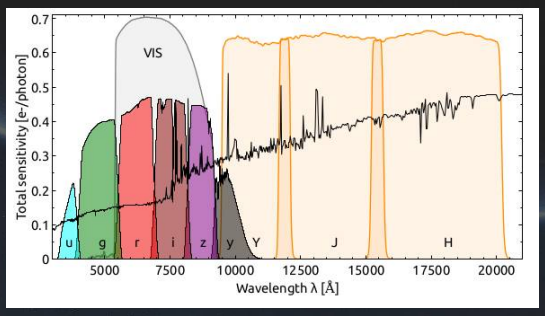

Raw data… it looks magic to me!
VIS
(565 s)
NISP
(3 x 120 s)
NISP grism
(565 s)
4 x
*NISP sim does not include cosmic rays
(images from slide by Ben Granett)


Euclid
the survey

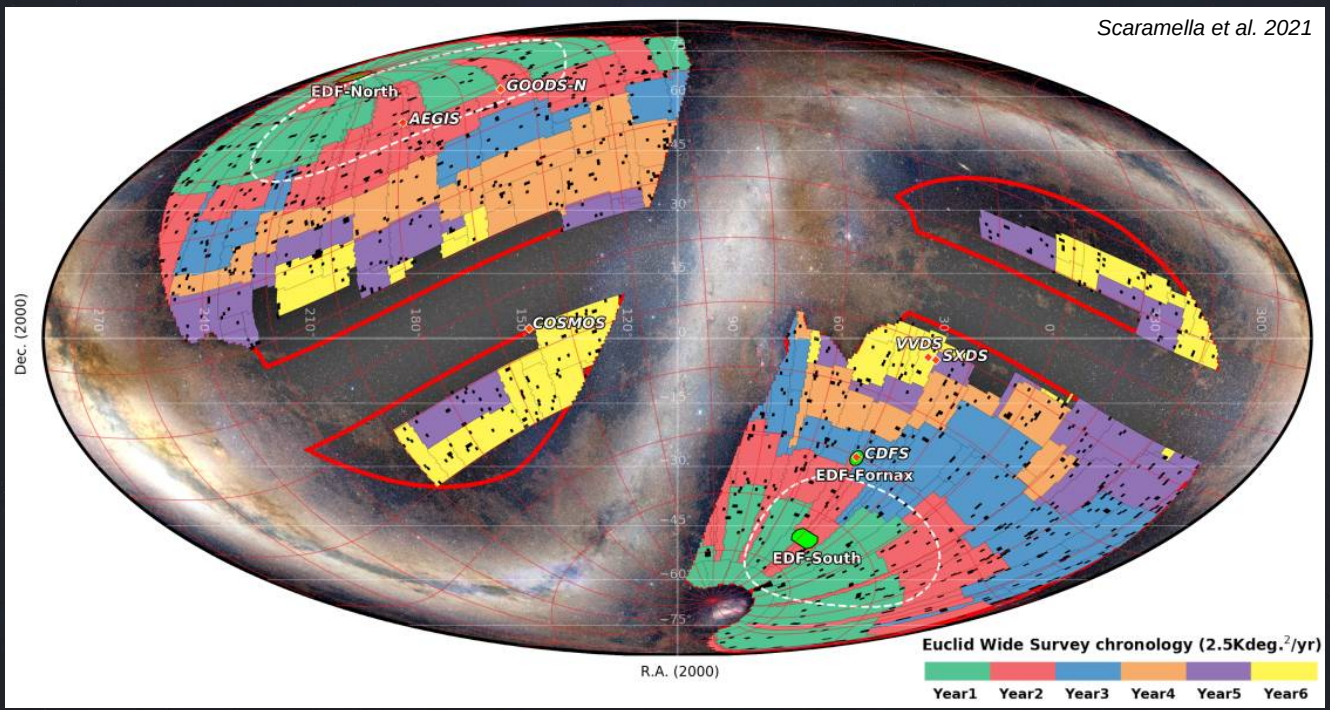
What will it see?

Wide field: 15.000 deg^2
- 12 billions sources (1.5 billion for lensing)
- 35 million spectroscopic redshifts of galaxies
- We expect 2 10^6 clusters (Sartoris et al. 2016)

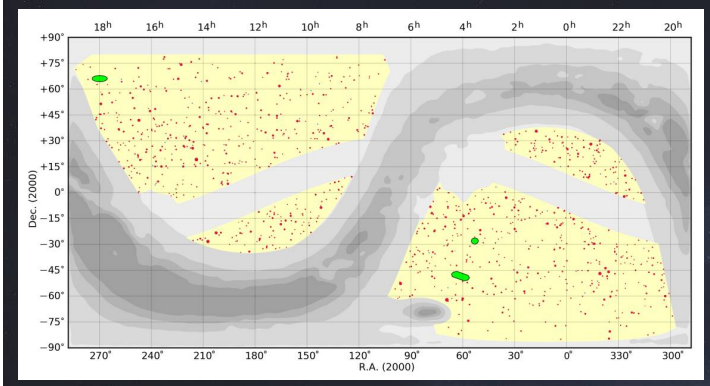
Deep field: 40 deg^2 (2 mag deeper)
- 10 million sources (1.5 million for lensing)
- 150 000 spectroscopic redshifts of galaxies

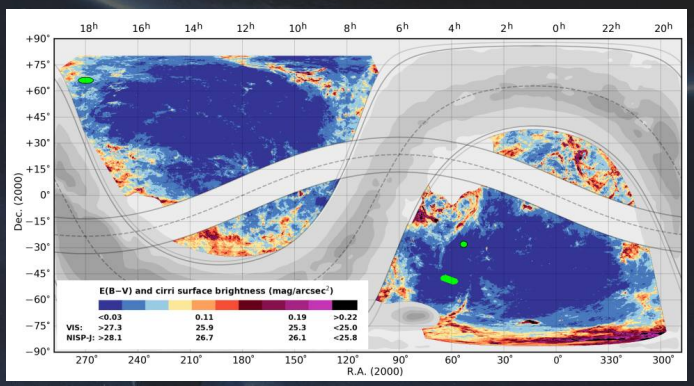


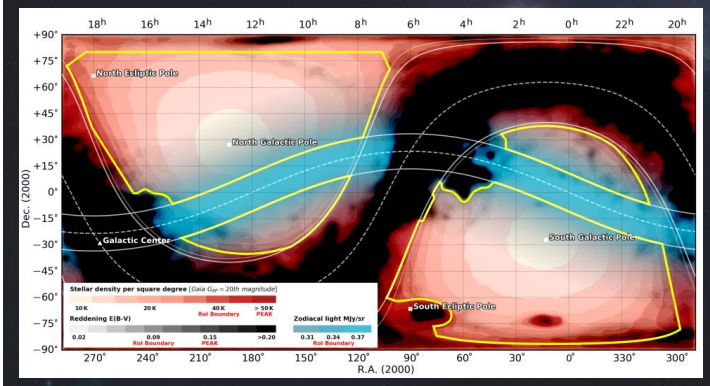
Scaramella et al. 2021

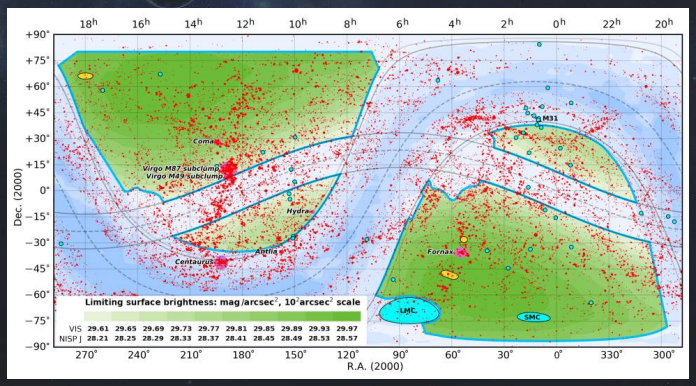


Why so patchy? A dirty universe

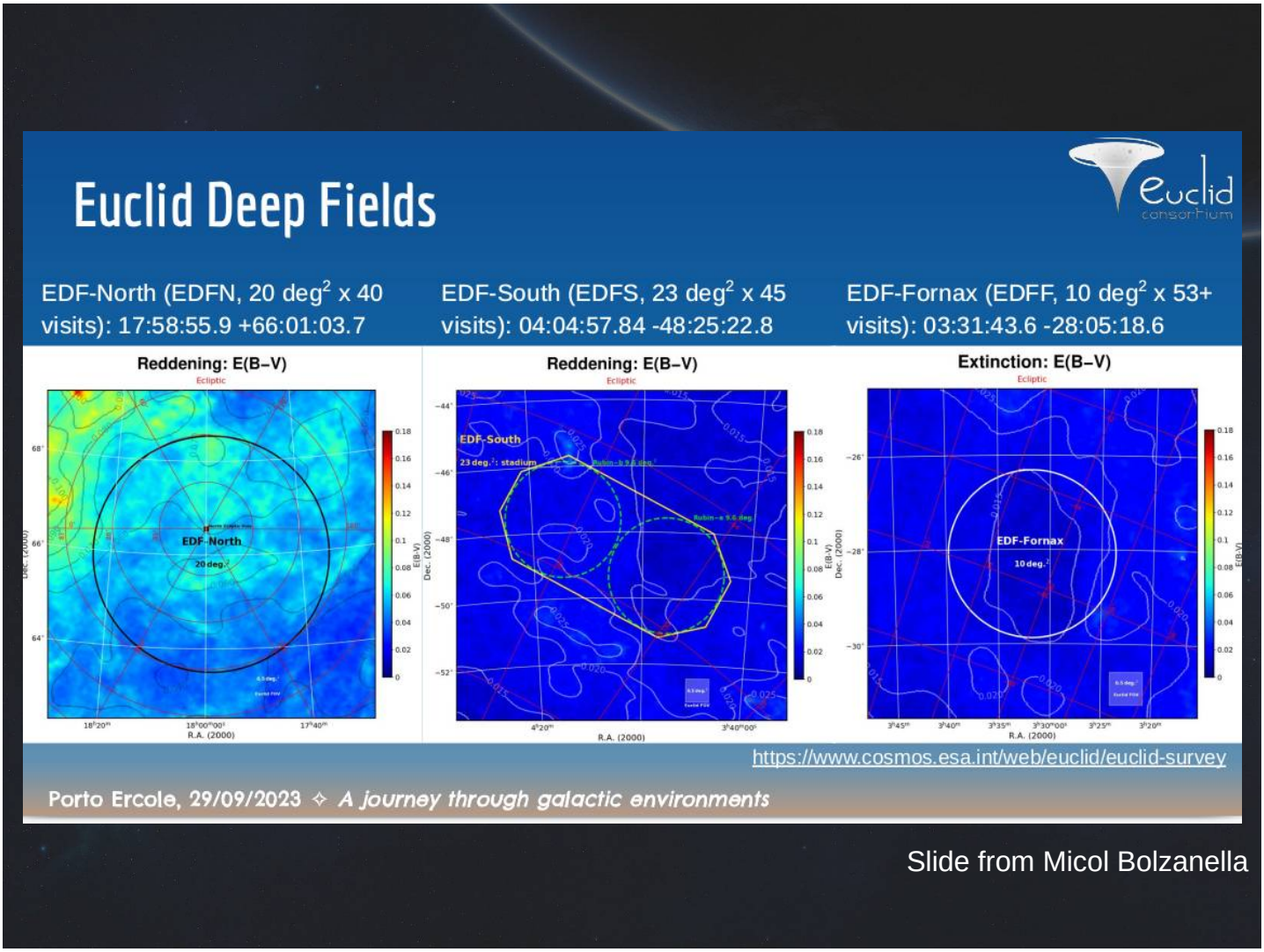








Scaramella et al. 2021



Slide from Micol Bolzanella

Euclid Auxiliary Fields

→ six well-known fields to calibrate photometric redshifts and colour-gradients + field for photometric self-calibration of the survey. Depth refers to increase in SNR compared to the Wide Survey

Field Name	RA	Dec	Area [deg²]	Depth [SNRxWide]
Self-calibration (inside EDFN)	268.813	+65.29	2.5	8×
CANDELS/AEGIS	214.827	+52.82	1.0	4×
GOODS-NORTH	189.250	+62.25	0.5	4×
COSMOS-Wide	150.119	+02.21	2.0	5×
VVDS-Deep	36.500	−04.50	0.5	5×
CDFS (inside EDFF)	53.117	−27.81	0.5	5×
SXDS	34.500	−05.00	2.0	5×

Porto Ercole, 29/09/2023 ✧ *A journey through galactic environments*

Slide from Micol Bolzanella


More than “Just”
Euclid

Infrastructure

Dedicated clusters for the data processing

Dedicated numerical simulations

ESA had to expand the ground network

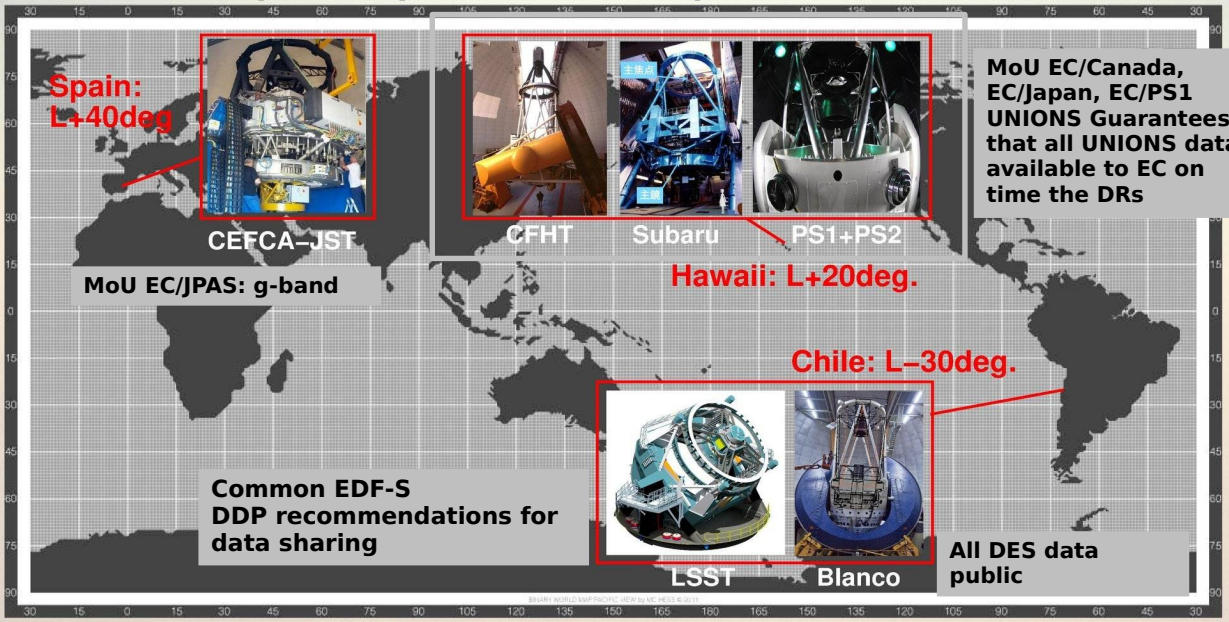




Not only Euclid data… heterogeneous data

From J.-C. Cuillandre

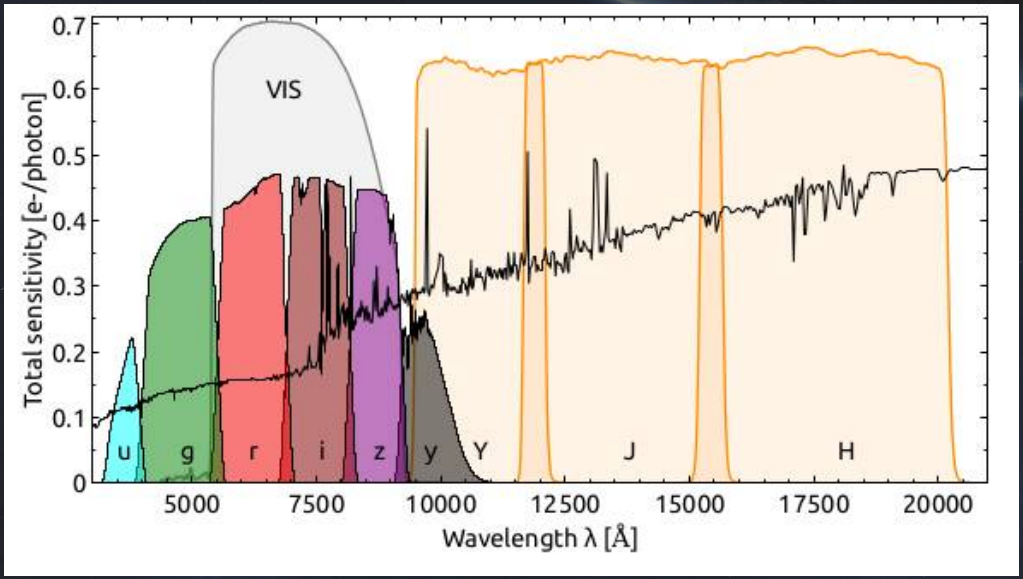
Present and upcoming wide–field imagers relevant to Euclid



Etendue

Facility	Year	Aper.	FOV	IQ[r]	CCD class	Type	Hemisphere
LSST	2022	6.6m	9.6 sq.deg.	0.8"	Deep depletion	Surveyor	South
Subaru	2013	8.2m	1.8 sq.deg.	0.6"	Fully depleted	Observatory	North
Blanco	2013	4.0m	3.0 sq.deg.	1.0"	Fully depleted	Observatory	South
JST	2020	2.5m	4.8 sq.deg.	0.8"	Deep depletion	Surveyor	North
PS1+PS2	2019	2x1.5m	7.0 sq.deg.	1.0"	Fully depleted	Surveyor	North
CFHT	2003	3.6m	1.0 sq.deg.	0.6"	EPI	Observatory	North

Why this combined effort?



Complementary data

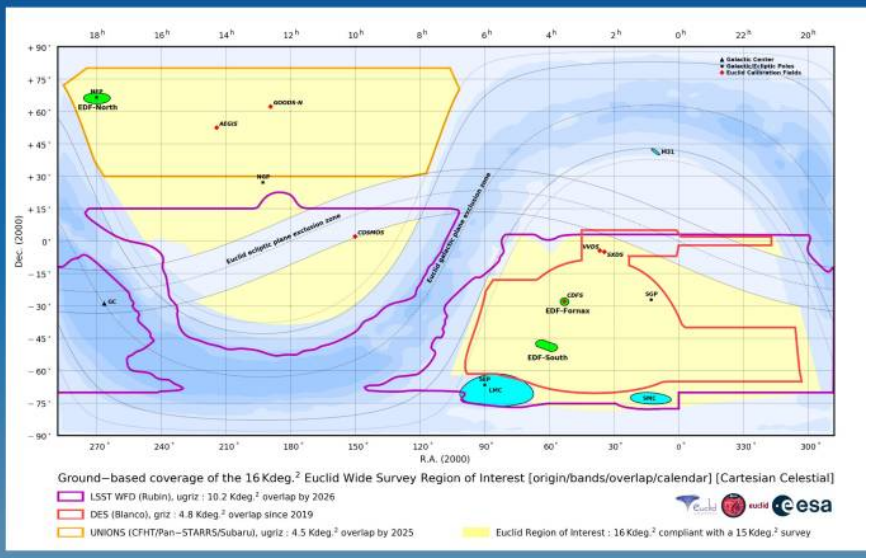
Sharing data also at the pixel level

More bands for photo-z - not all galaxies will have a spectra

Synergies with ground based telescopes

To achieve the requirements on photometric redshifts precision
$\sigma_z < 0.05(1+z)$
ground based photometry is needed:

- UNIONS in the north
- DES + LSST in the south
- 2 Deep Fields (EDF-S and EDF-F) are also Deep Drilling Fields of LSST



ESA/Euclid Consortium Survey Group – Jean-Charles Cuillandre

Porto Ercole, 29/09/2023 ✧ *A journey through galactic environments*

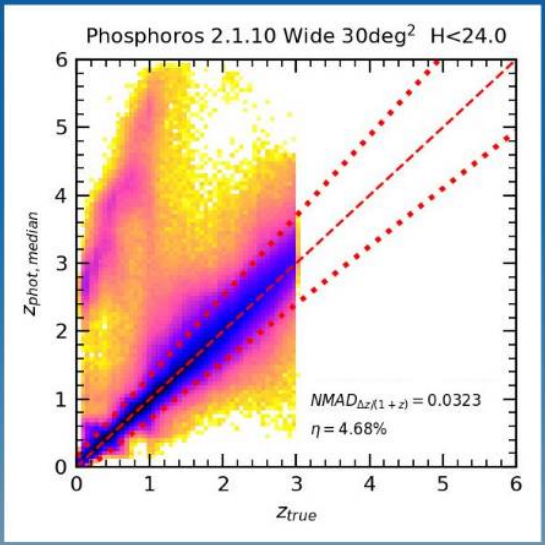
Slide from Micol Bolzanella

Why do we need other surveys??

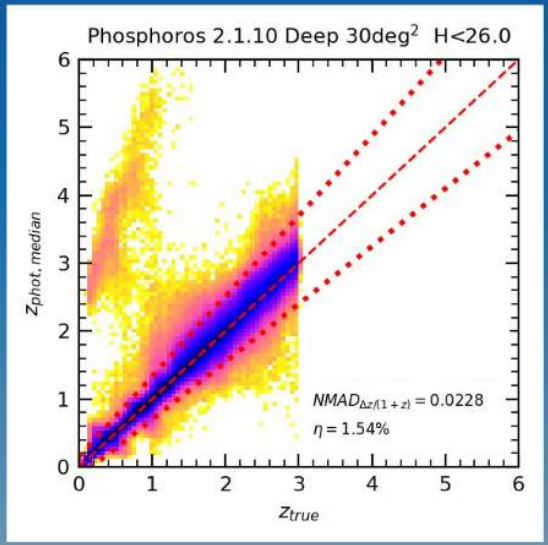
OU-PHZ: Paltani

Simulations for Galaxy Environment: photoz

Wide Survey



Deep Fields

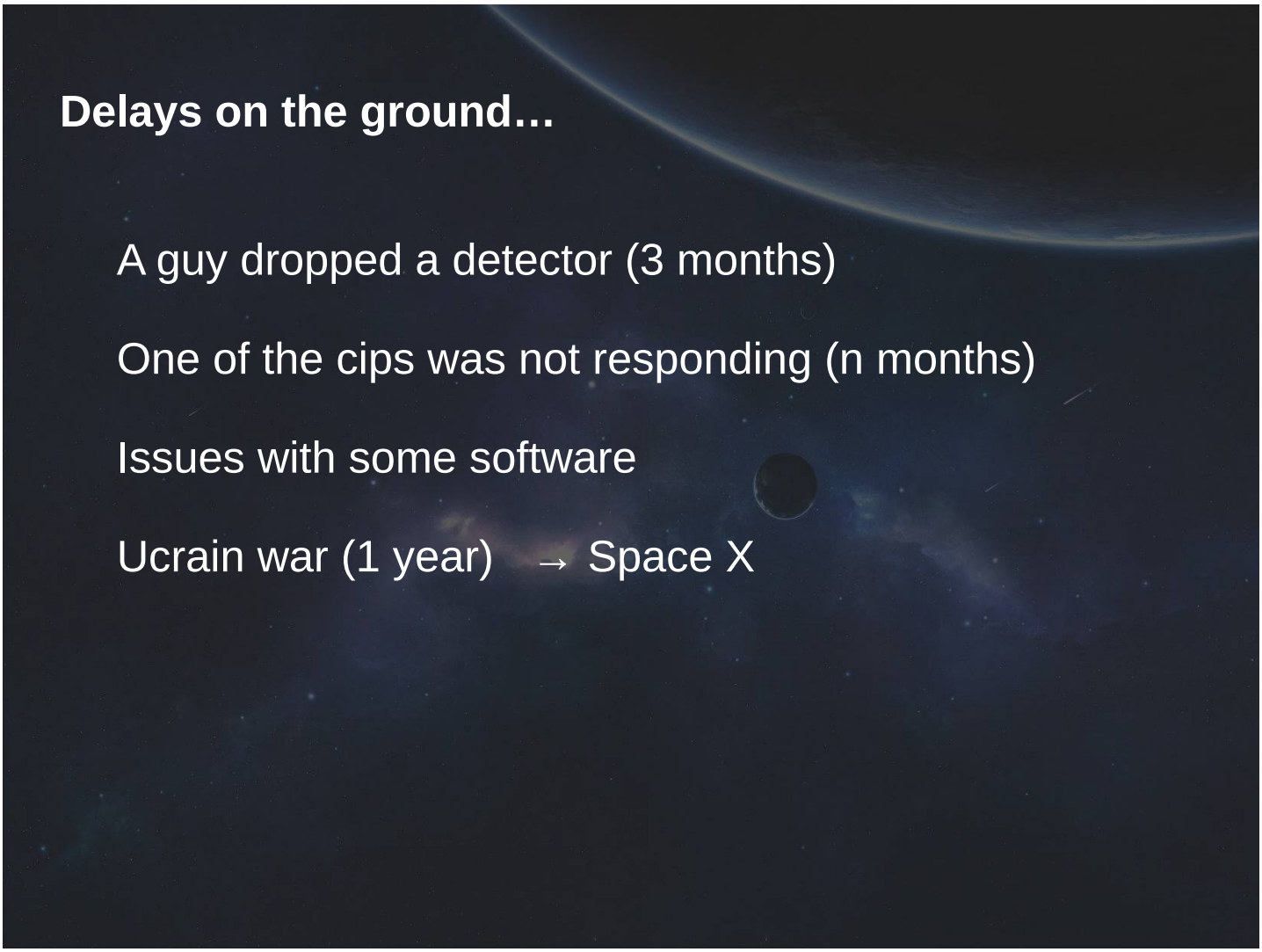


Porto Ercole, 29/09/2023 ✧ *A journey through galactic environments*

Slide from Micol Bolzanella

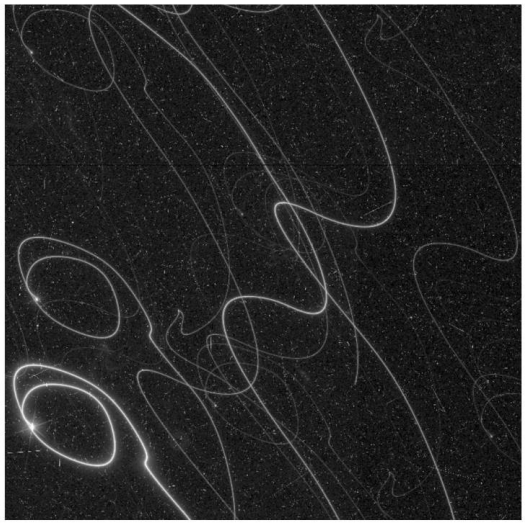
The adventure….



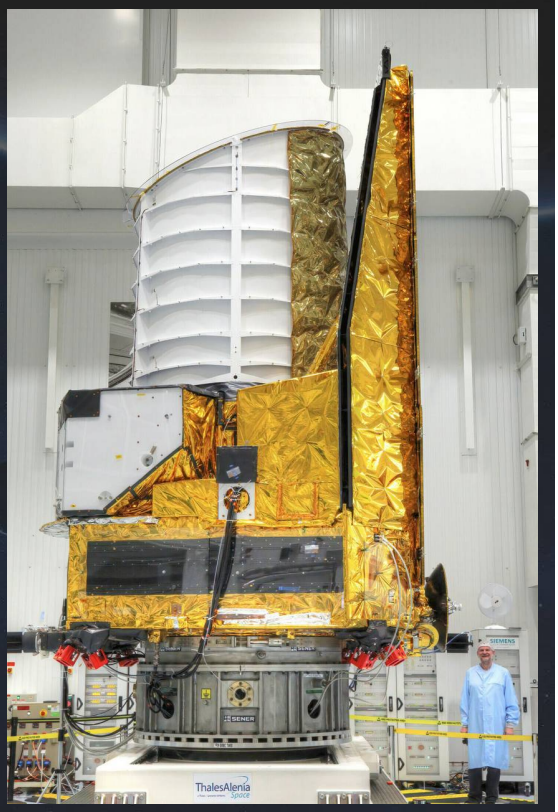

Delays on the ground…
A guy dropped a detector (3 months)
One of the cips was not responding (n months)
Issues with some software
Ucrain war (1 year) → Space X

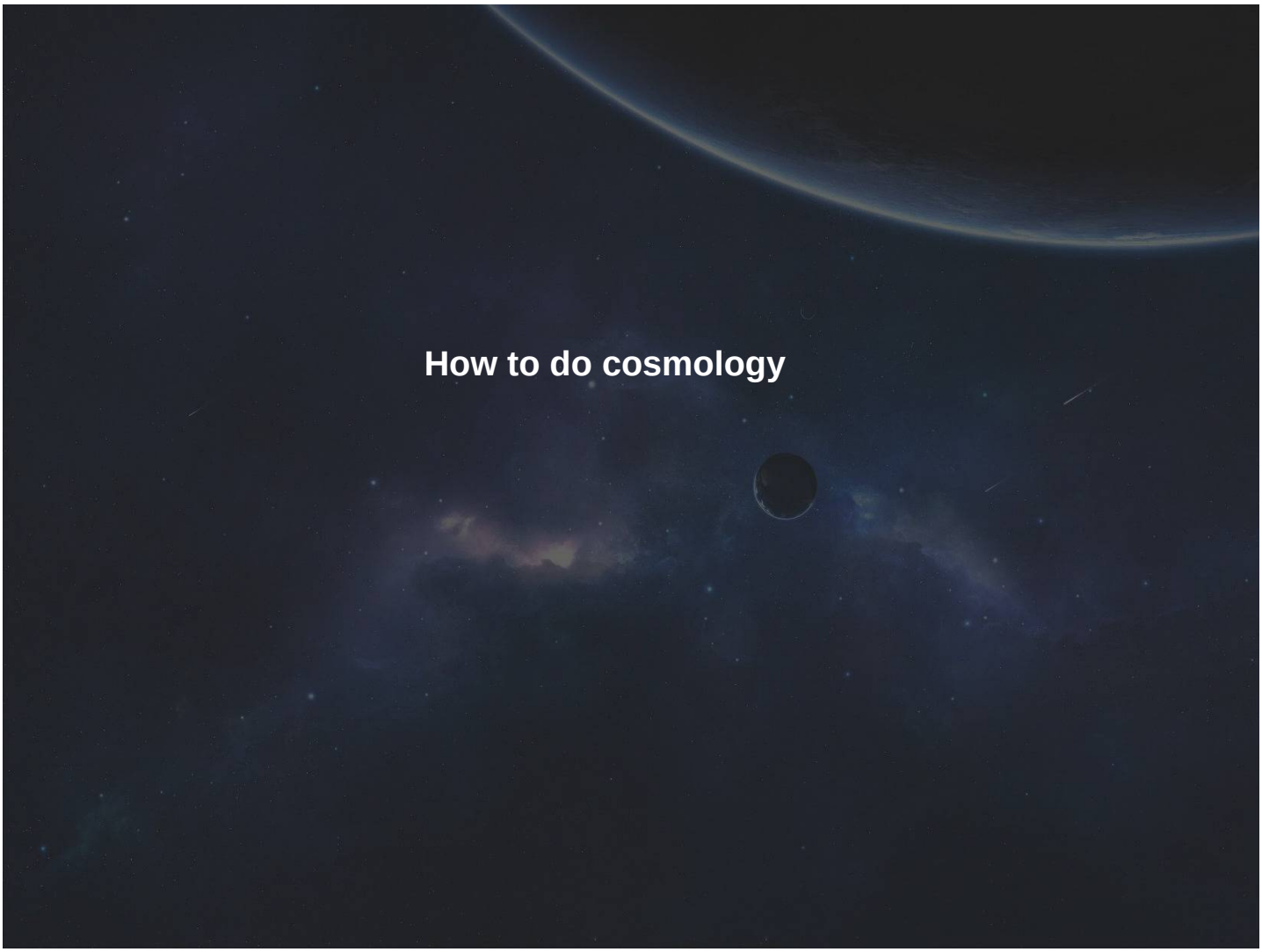
Delays in space….

- Stray light : we had to redesign the survey strategy
- Fine guide issues. Solved: only 3% slower
- Sun flares: X-rays in VIS



— Loopy star trails show the effect of Euclid's Fine Guidance Sensor intermittently losing its guide stars



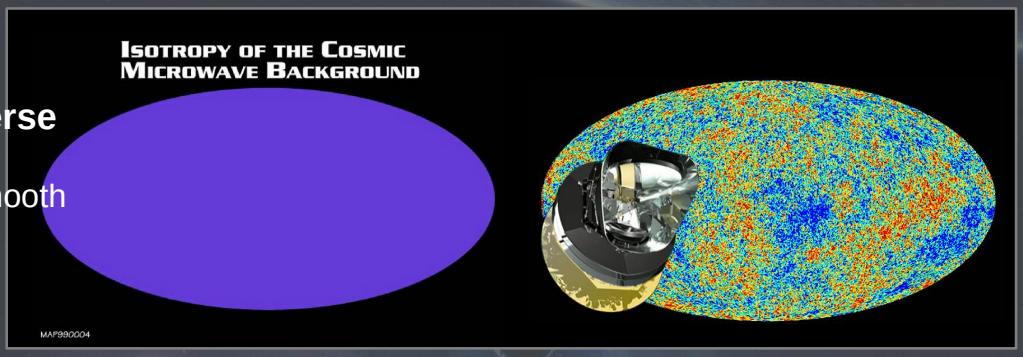

How to do cosmology

How do we approach the universe?
Homogeneity and isotropy

The young universe

Very uniform and smooth

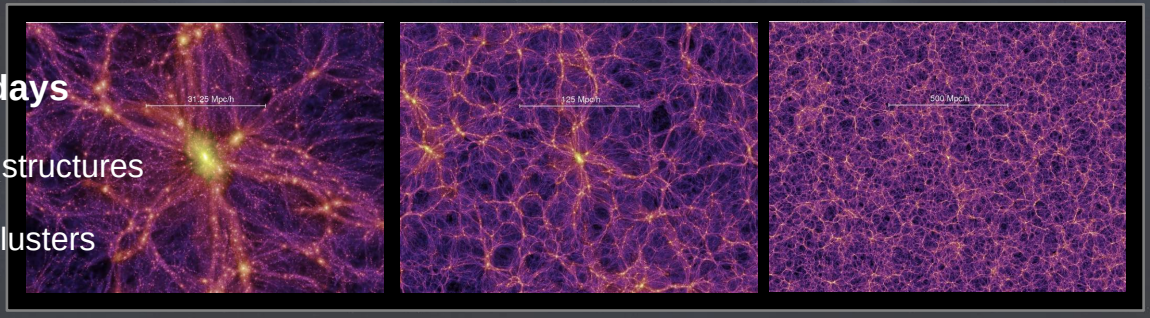
DT/T = 10^{-5}



Now days

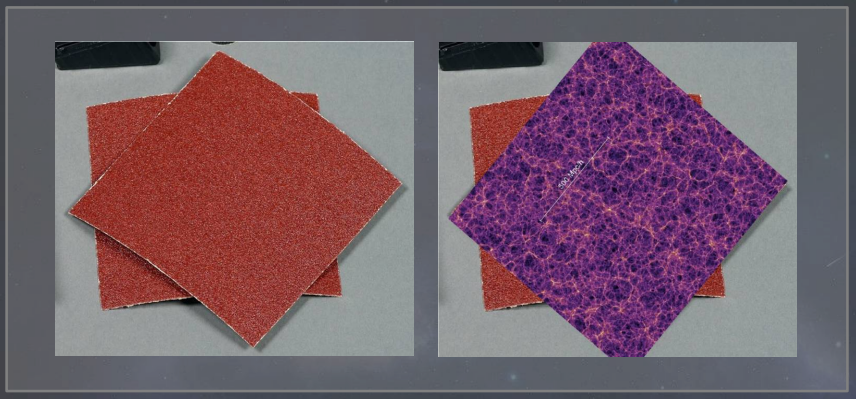
Large scale structures

Galaxy clusters



Let's build a model of the universe

The universe is like sandpaper: rough but homogeneous and isotropic



To describe sand paper we do not need to describe each single grain!

3 parameters are enough:
*material (**M**), grains dimension (**D**), grains number density (**N**)*

The same for the universe...

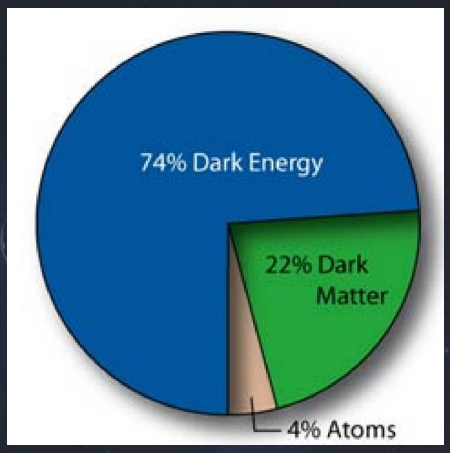
The ingredients of the cake

Cosmological parameters

Dynamics:
- H_0: Hubble parameter

Ingredients:
- Ω_r : radiation density
- Ω_b : barionic density
- Ω_d : dark matter density
- Ω_Λ: dark energy density

"Raffness":
- σ_8



And the roles of the game?

How do the ingredients interact?

The rules mastering the universe

Theory	Interaction	Behavior	Typical scale
QCD	Strong	$1/r^7$	1.4×10^{-15} m
Electroweak	Weak	$1/r^5$ to $1/r^7$	10^{-18} m
QED	Electromagnetic	$1/r^2$	∞
GR	Gravity	$1/r^2$	∞

We need a long distance force acting on a neutral universe
Gravity!!

The “homogeneous and isotropic universe”...

Einstein's field equation

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \,,$$

$$ds^2 = (cdt)^2 - a^2(t) \left[\frac{dr^2}{(1 - kr^2)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi) \right]$$

Friedmann–Lemaître–Robertson–Walker metric

$$T_{\alpha\beta} = -p g_{\alpha\beta} + \left(p + \rho c^2 \right) \mu_\alpha \mu_\beta - \frac{c^4 \Lambda}{8\pi G} g_{\alpha\beta} \,.$$

Universe ingredients

Friedmann's equations

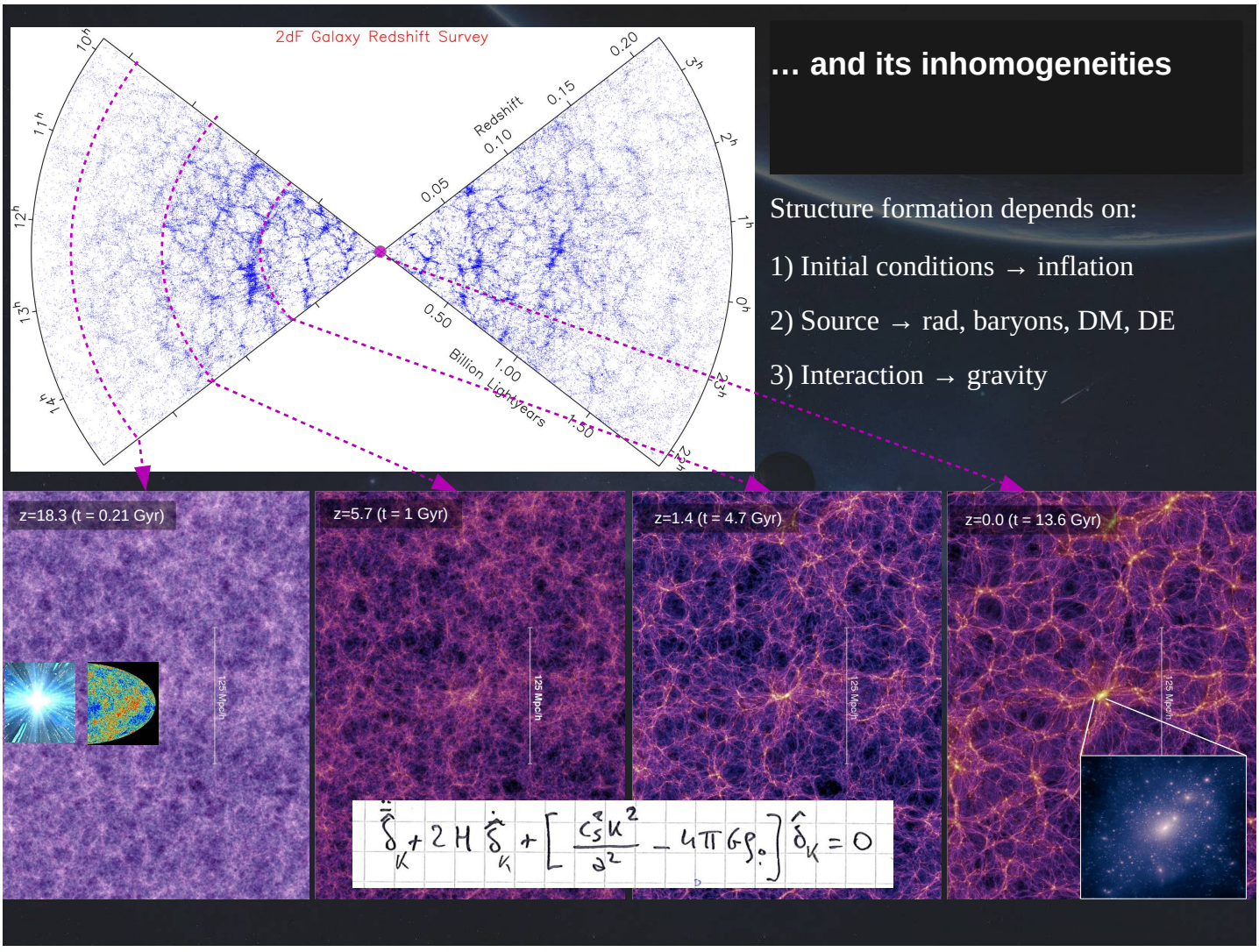
$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \frac{\Lambda}{3}$$

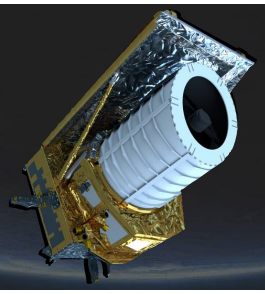
Newtonian

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho)$$

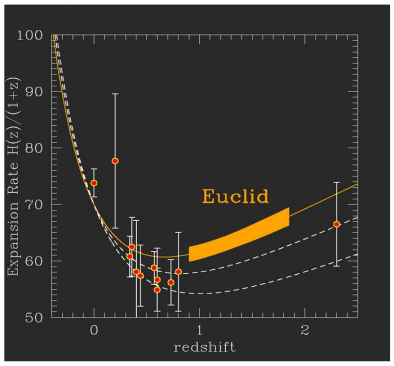

2dF Galaxy Redshift Survey
Redshift
0.05
0.10
0.15
0.20
0.50
1.00
1.50
Billion Lightyears
10h
11h
12h
13h
14h
1h
2h
3h
0h
23h
22h
... and its inhomogeneities
Structure formation depends on:
1) Initial conditions → inflation
2) Source → rad, baryons, DM, DE
3) Interaction → gravity
z=18.3 (t = 0.21 Gyr)
z=5.7 (t = 1 Gyr)
z=1.4 (t = 4.7 Gyr)
z=0.0 (t = 13.6 Gyr)
125 Mpc/h
$\ddot{\delta}_k + 2H\dot{\delta}_k + \left[\frac{c_s^2 k^2}{a^2} - 4\pi G\rho_0\right]\hat{\delta}_k = 0$

The key point: pushing the redshift range



All this is vital, if we want to get this...

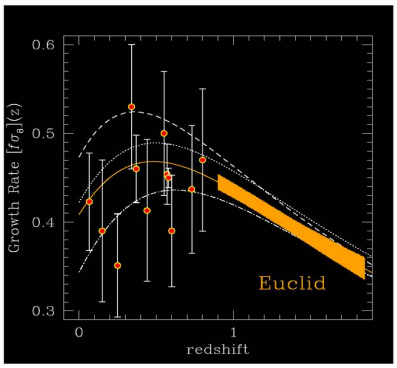
Euclid Consortium



EC General Meeting Lisbon 1 Jun 2016 18

... and this

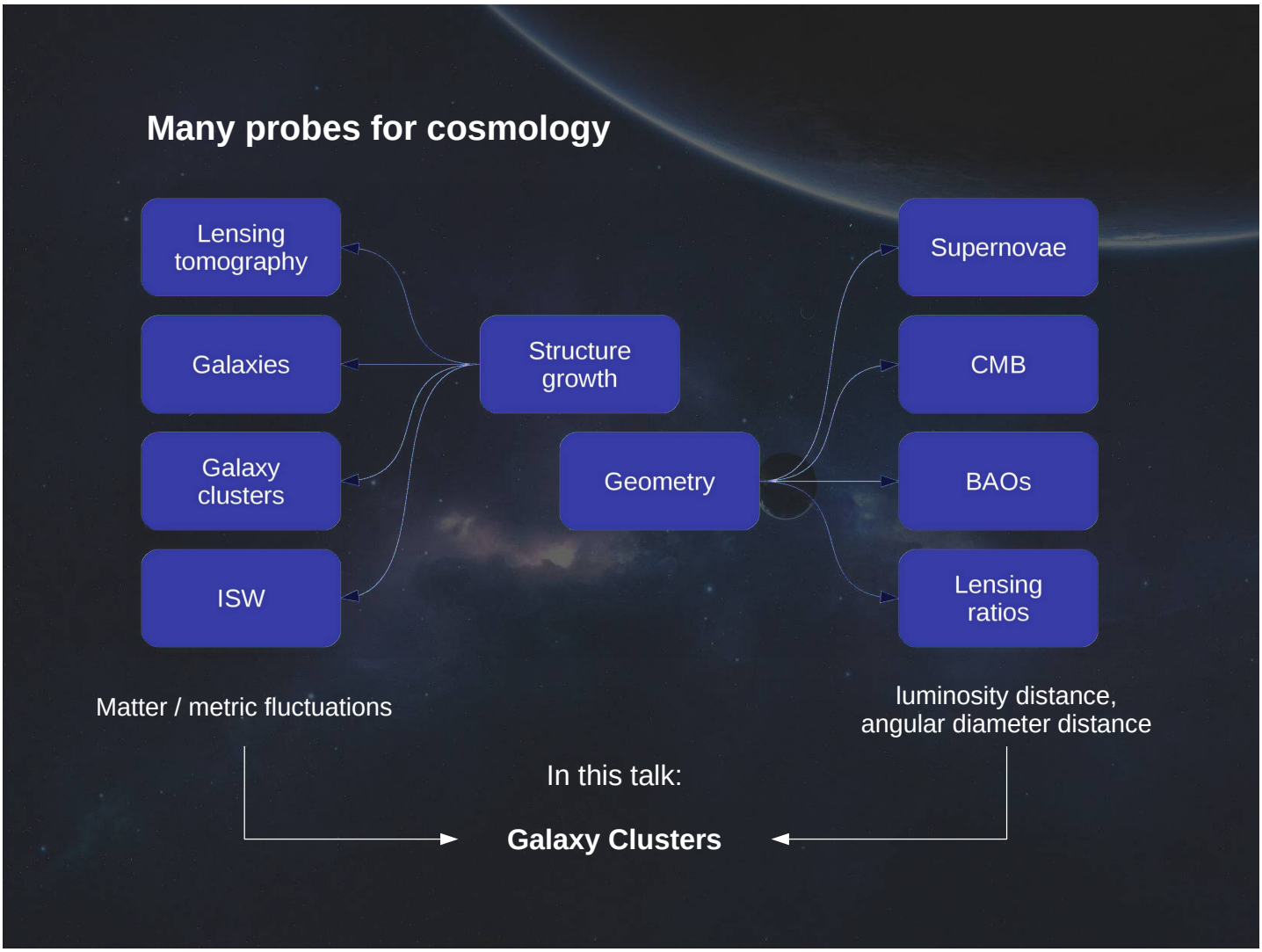
Euclid Consortium



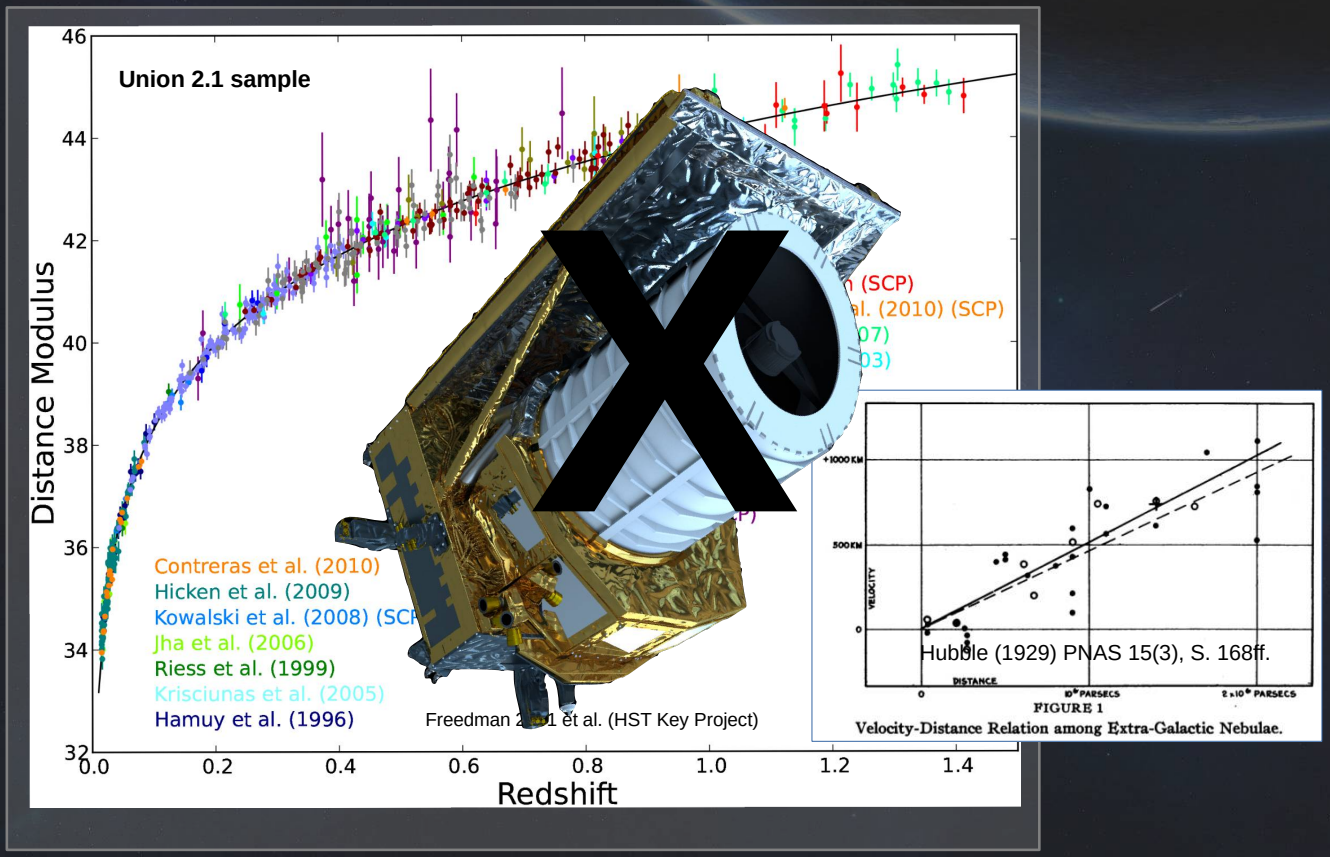
EC General Meeting Lisbon 1 Jun 2016 19

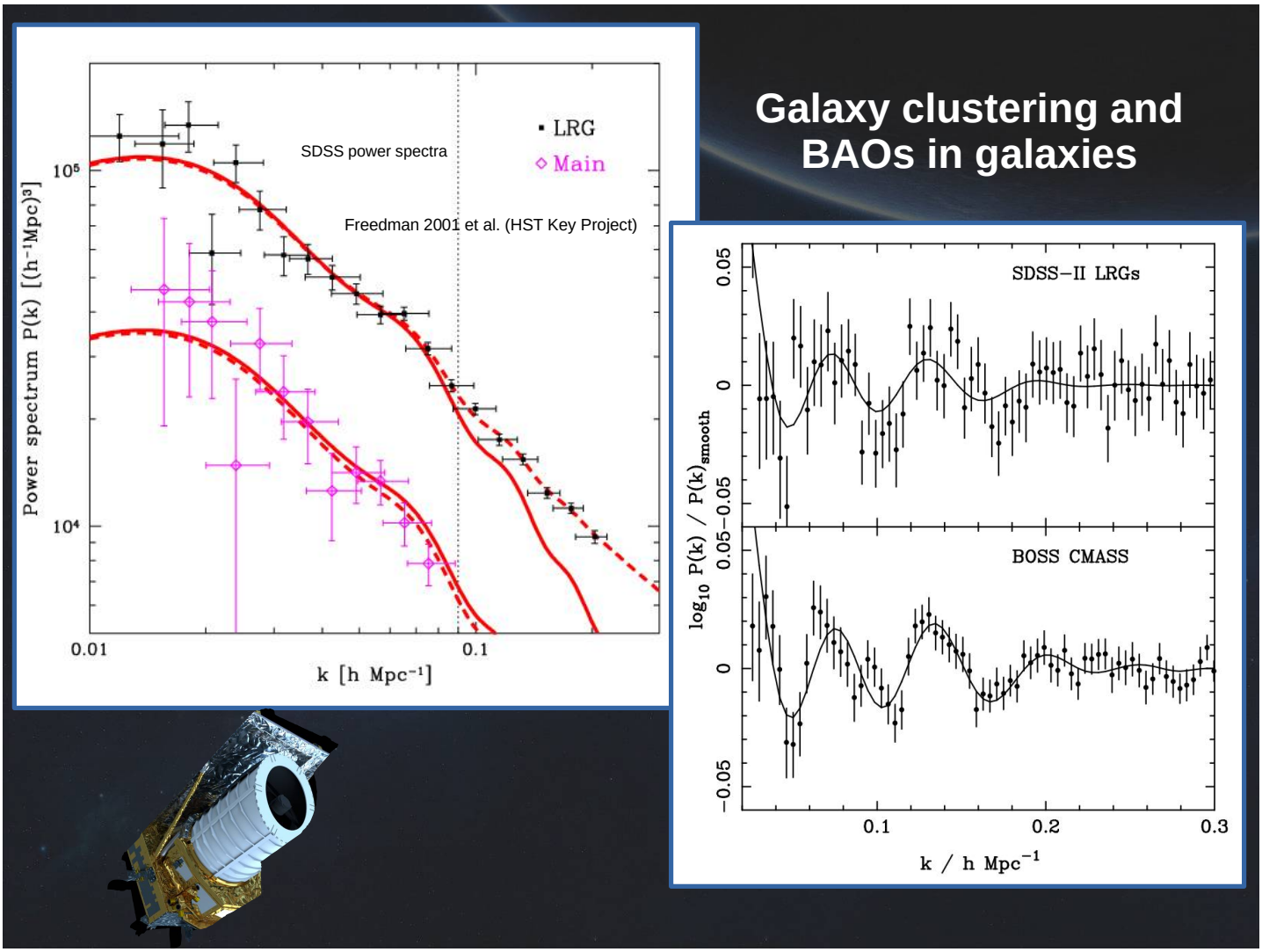
Slides from J. Mellier

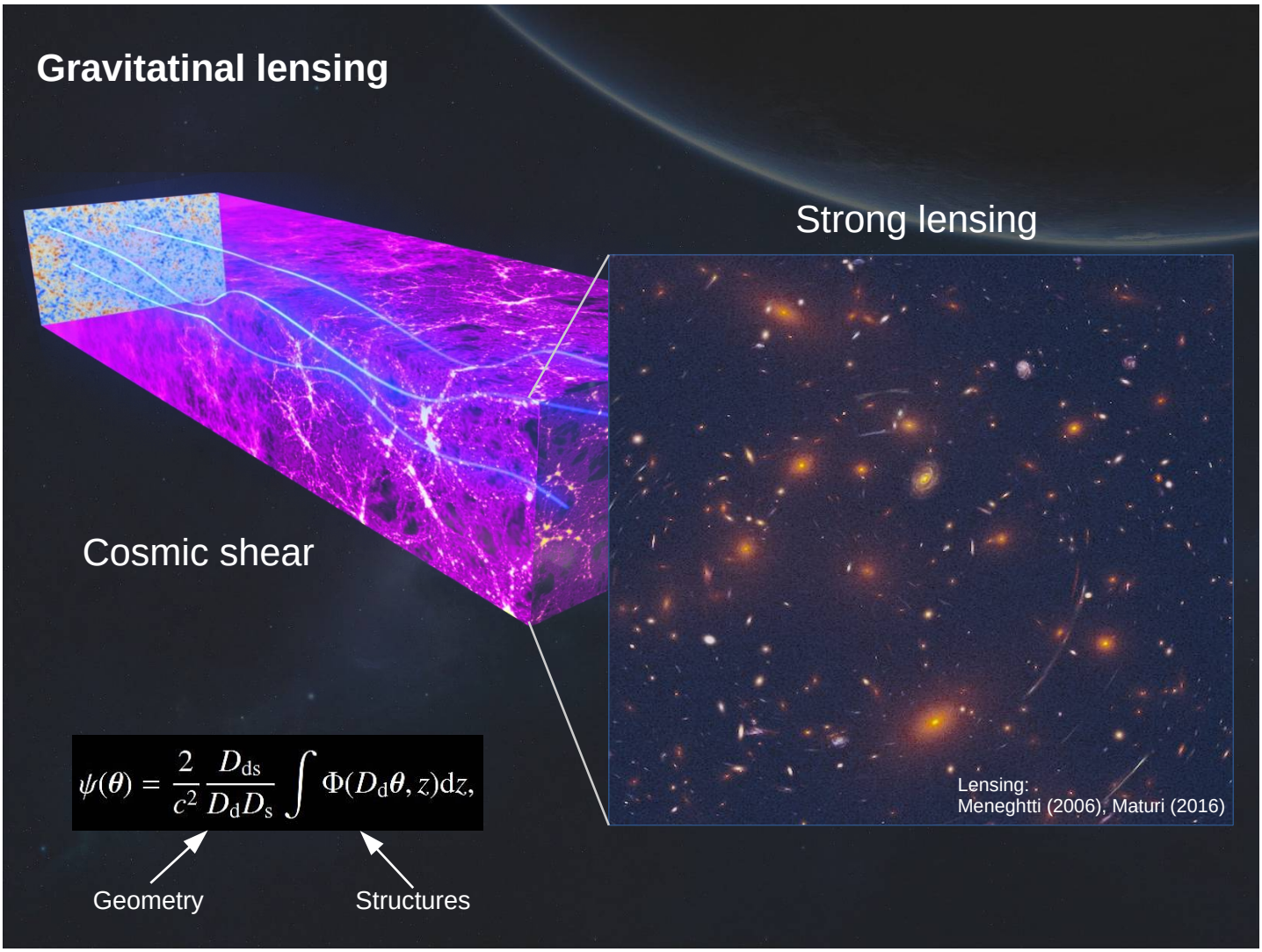
Increase leverage for the possible evolution of DE

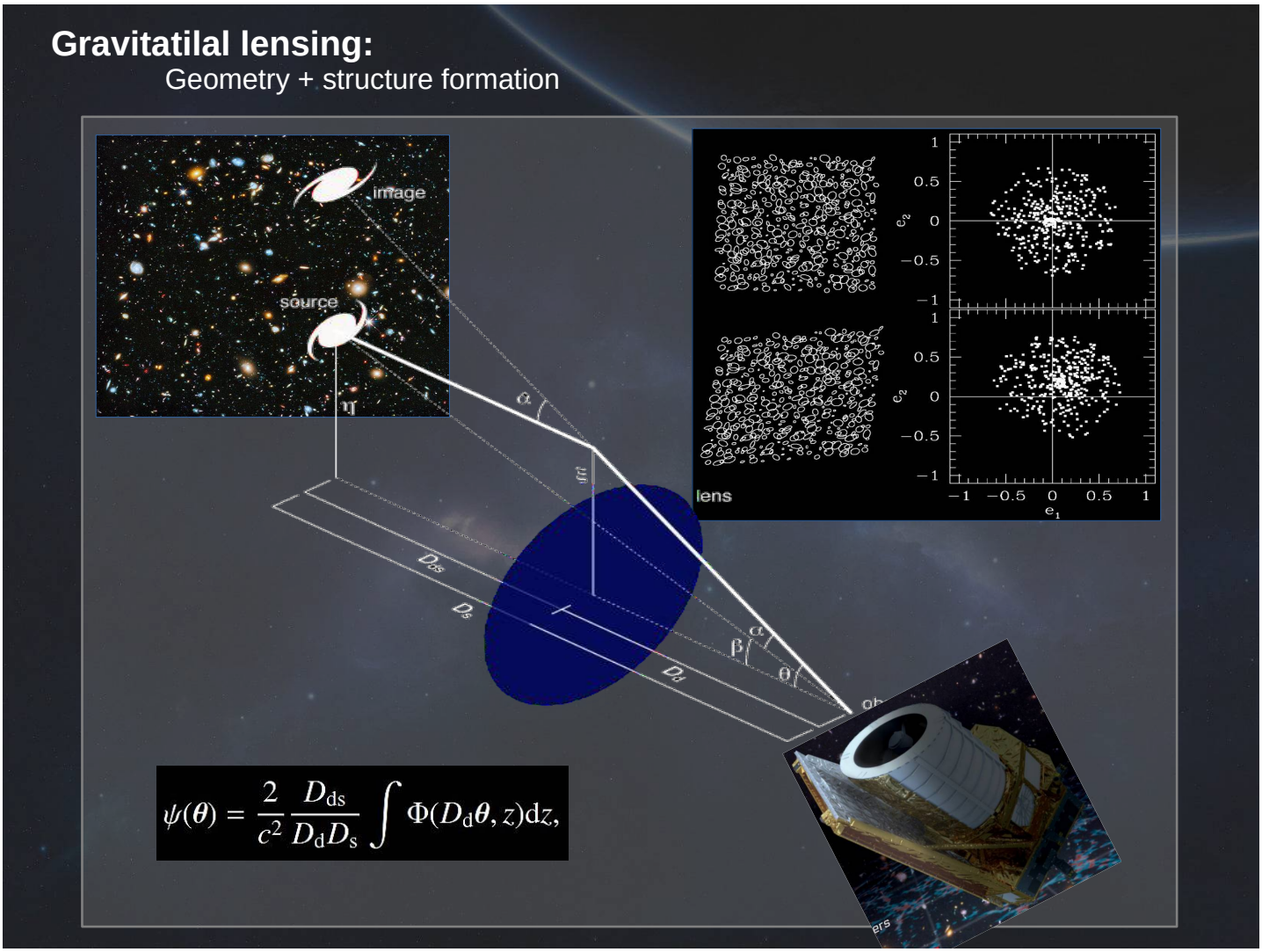

Many probes for cosmology
Lensing tomography
Galaxies
Galaxy clusters
ISW
Structure growth
Geometry
Supernovae
CMB
BAOs
Lensing ratios
Matter / metric fluctuations
luminosity distance, angular diameter distance
In this talk:
Galaxy Clusters

Example: supernovae (luminosoty distance)

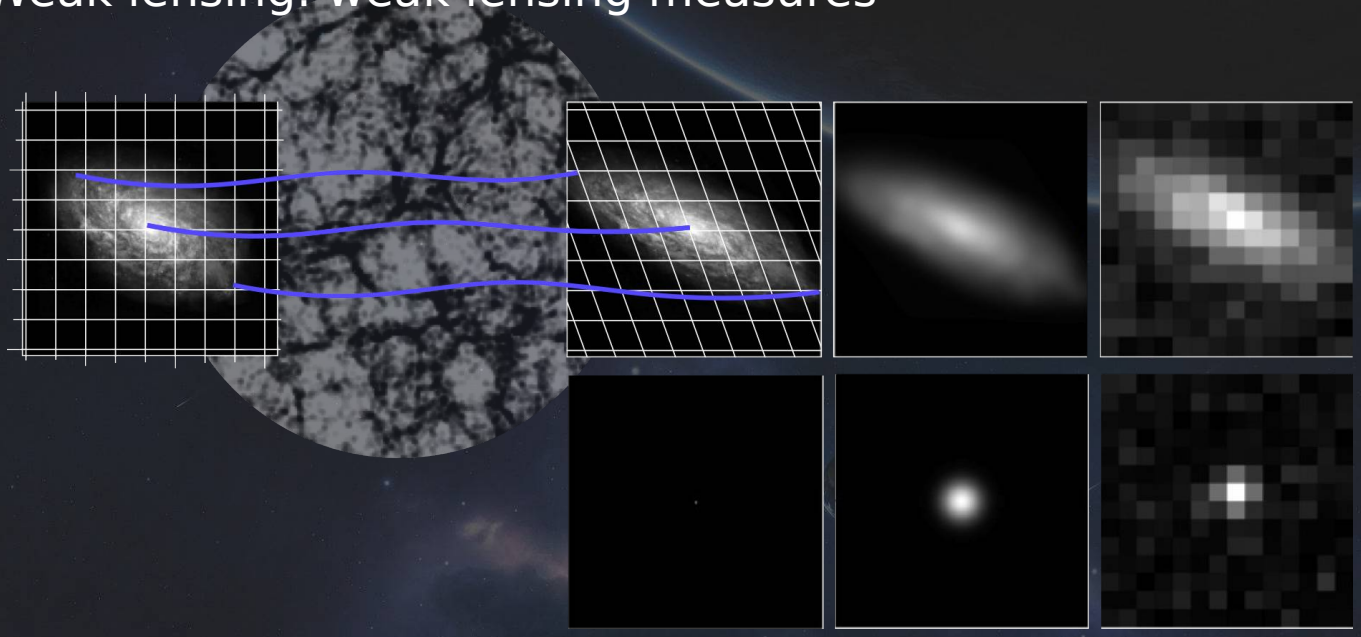



Galaxy clustering and
BAOs in galaxies
SDSS power spectra
LRG
Main
Freedman 2001 et al. (HST Key Project)
Power spectrum P(k) [(h⁻¹Mpc)³]
10⁵
10⁴
0.01
0.1
k [h Mpc⁻¹]
SDSS−II LRGs
BOSS CMASS
log₁₀ P(k) / P(k)smooth
0.05
0
−0.05
0.1
0.2
0.3
k / h Mpc⁻¹

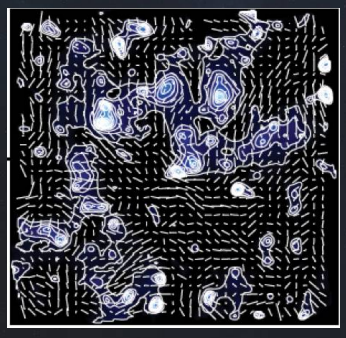

Gravitatinal lensing
Strong lensing
Cosmic shear
$\psi(\theta) = \frac{2}{c^2}\frac{D_{ds}}{D_d D_s}\int \Phi(D_d\theta, z)dz,$
Geometry
Structures
Lensing:
Meneghtti (2006), Maturi (2016)

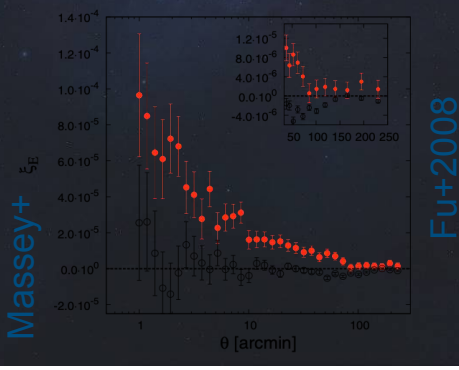

Gravitatilal lensing:
Geometry + structure formation
image
source
lens
$\psi(\theta) = \frac{2}{c^2}\frac{D_{ds}}{D_d D_s}\int \Phi(D_d\theta, z)dz,$

Weak lensing: weak lensing measures



- Correlate shapes of millions of galaxies to measure the cosmological signal at 10^{-3} in ellipticity





Galaxy Clusters

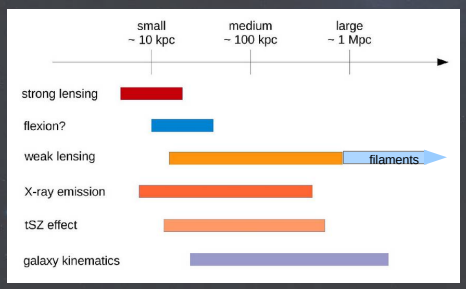
What they are, how to use them

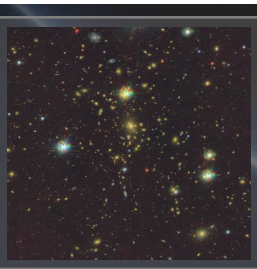
Properties:

Mass $M_v \sim 10^{14}-10^{15}\ M_\odot$
Virial radius $r_v \sim 1-3$ Mpc

+ Dark-matter halo
+ gas $T\sim10^7-10^8$ K
+ stars & galaxies

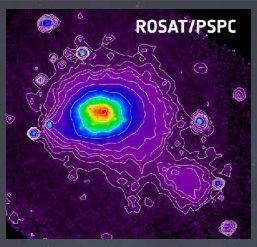
Different regions
Different response to distance





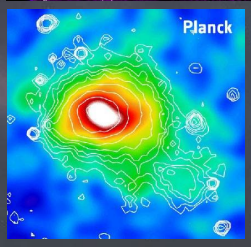
(1) Stars / galaxies

Optical emission from galaxies

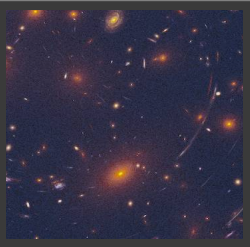


(2) ICM Temperature $T\sim10^7-10^8$ K

Bremsstrahlung emission
- X-ray luminosity $L_x \simeq 10^{45}$ erg/s



Inverse Compton scattering of CMB photons
- tSZ effect (CMB) ΔT tSZ $\sim 100\ \mu K$
- kSZ effect (CMB) ΔT kSZ $\sim 10 - 30\ \mu K$



(3) Dark matter + baryons

Gravitational Lensing (background galaxies & CMB)
- strong $R_E \approx 5$"
- weak

X-rays: baryons - bremsstrahlung

ICM → Plasma T∼10^7−10^8 K

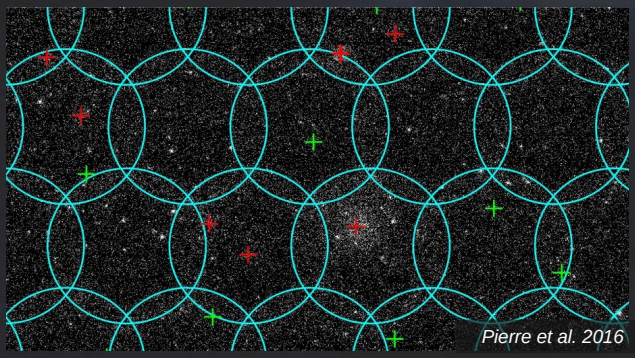
$$\epsilon_{\rm ff} \approx 1.2 \times 10^{-27} T^{1/2} n_{\rm P}^2 g(T)$$

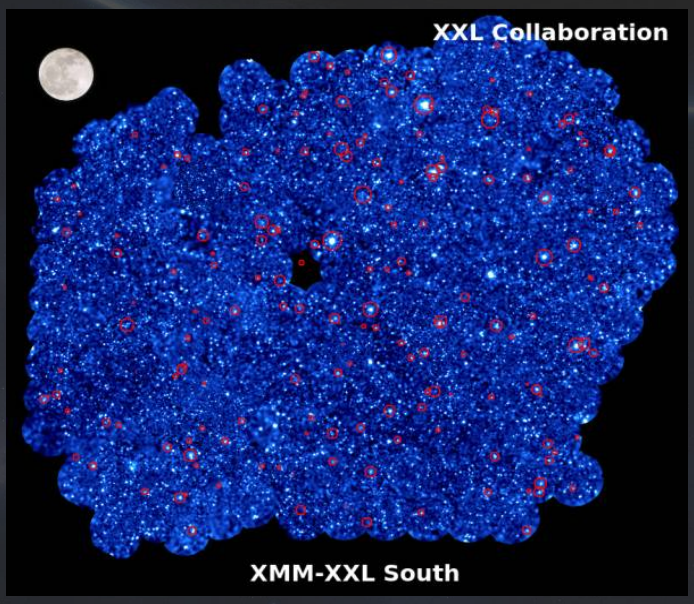
Bremsstrahlung bolometric emissivity

$$I_{\rm X,i} = \frac{L_{\rm X,i}}{4\pi d_{\rm L}(z)^2}\ {\rm erg/s/cm^2},$$

X-ray flux

Drop with distance / cosmic-time





$L_x \simeq 10^{45}$ erg/s

Sunyaev Zel'dovich (SZ): baryons – inverse compton scattering

- CMB photons - Intra cluster medium
  $T \sim 10^7 - 10^8$ K
- Inverse compton scattering

Thermal: tSZ

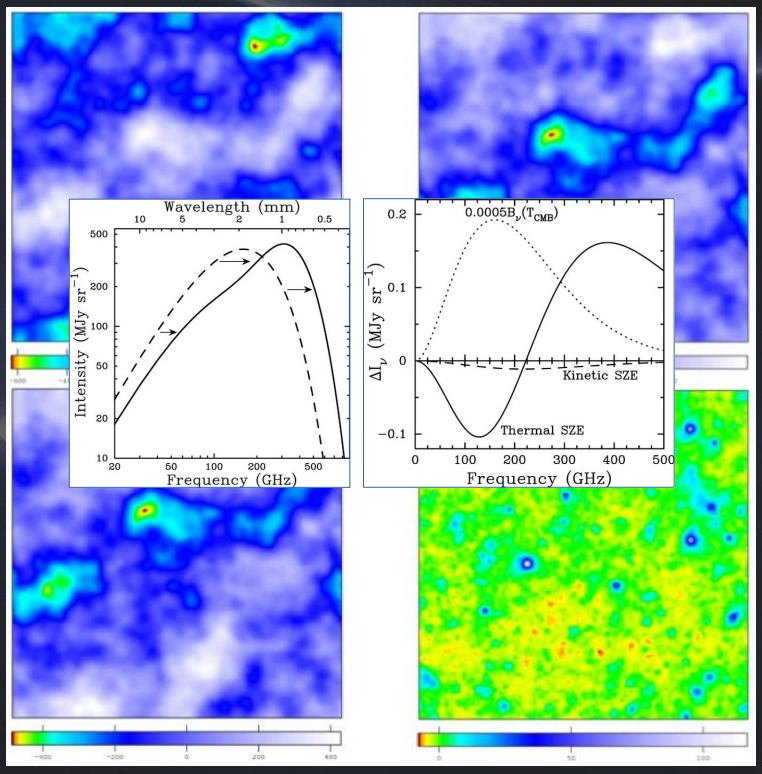
$$\frac{\Delta T}{T} = y g_\nu(x), \qquad y = \frac{k_B \sigma_T}{m_e c^2} \int dl\, n_e T_e,$$

$$g_\nu(x) = \frac{x}{\tanh(x/2)} - 4$$

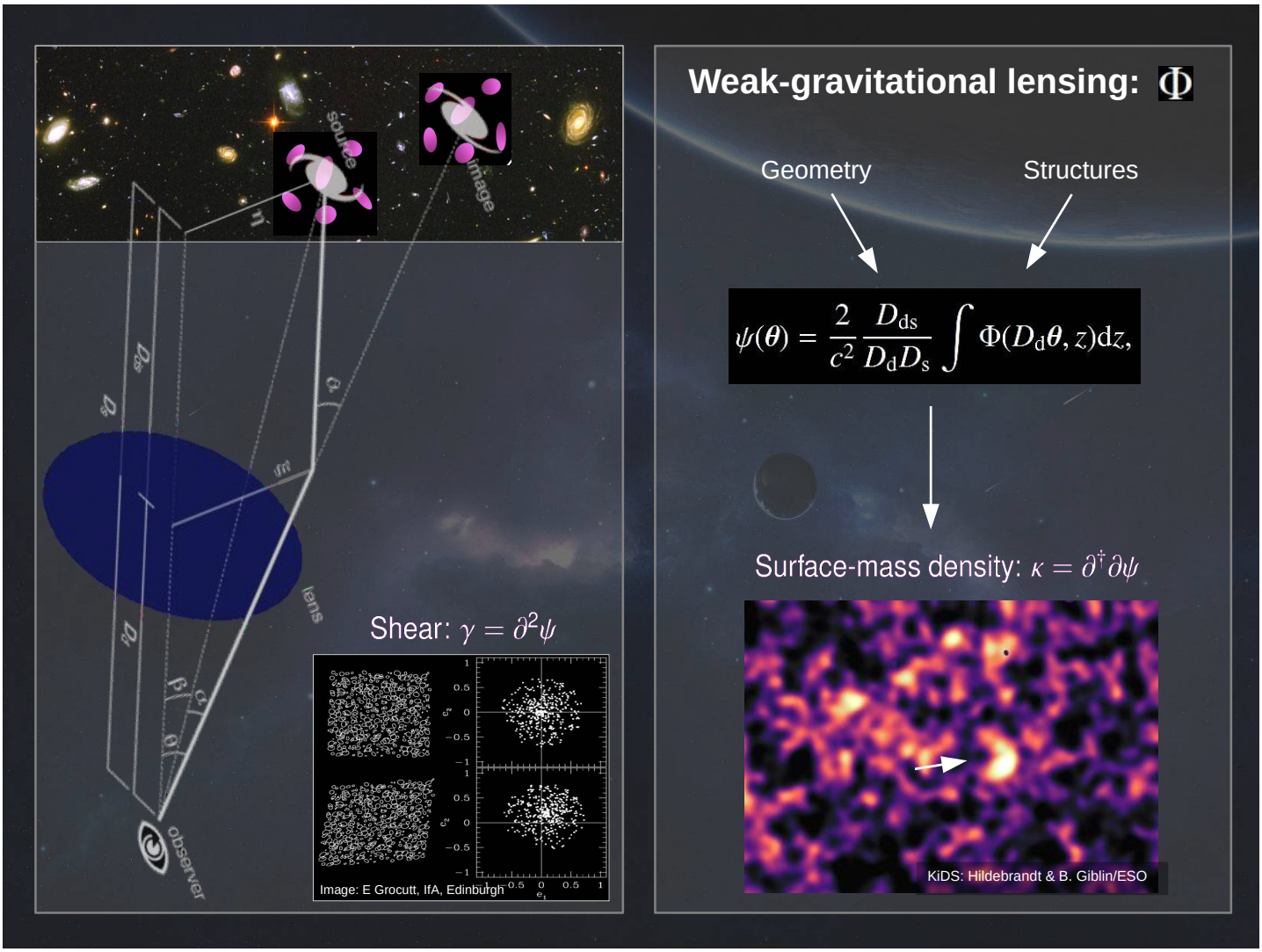
Kinetic: kSZ

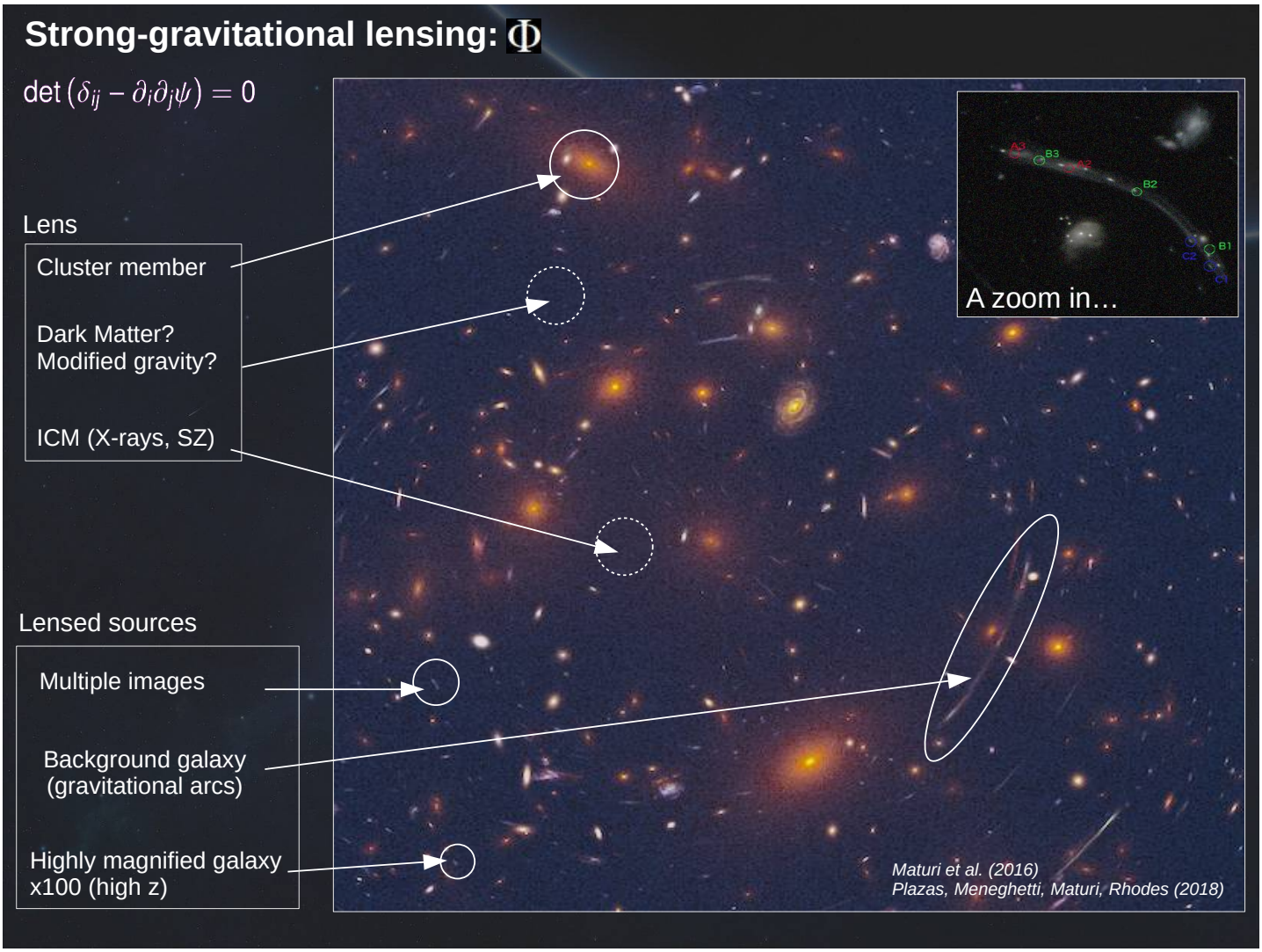
$$\frac{\Delta T}{T} = -b = -\frac{\sigma_T}{c} \int dl\, n_e v_r,$$

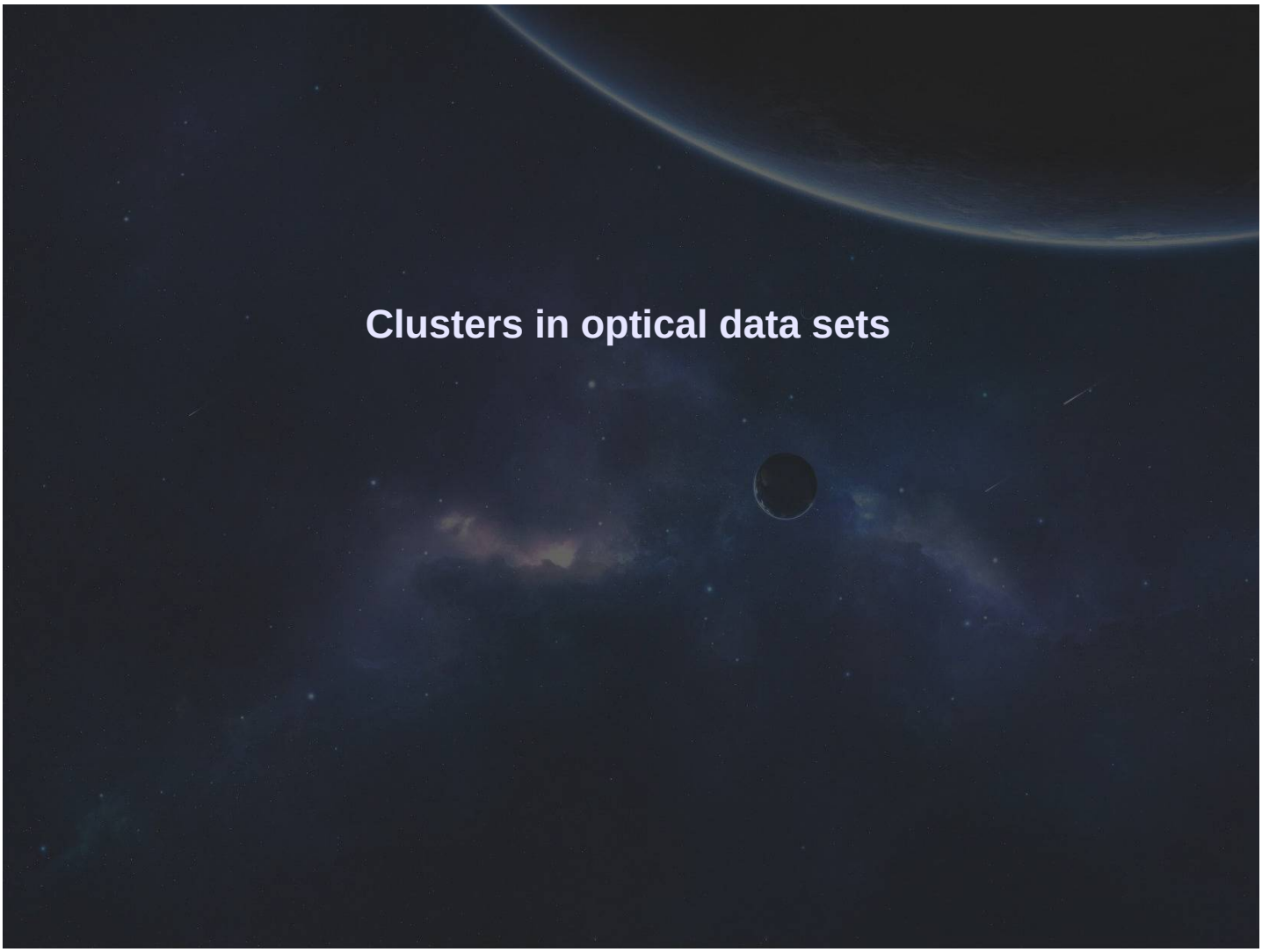
- What for?
  - Detection
  - Mass estimates
- Almost z independent



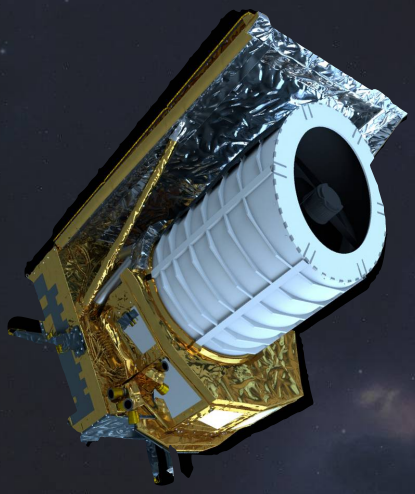
Pace et al. (2008)


Weak-gravitational lensing: Φ
Geometry
Structures
$\psi(\boldsymbol{\theta}) = \frac{2}{c^2}\frac{D_{\rm ds}}{D_{\rm d}D_{\rm s}}\int \Phi(D_{\rm d}\boldsymbol{\theta}, z)\mathrm{d}z,$
Surface-mass density: $\kappa = \partial^\dagger\partial\psi$
Shear: $\gamma = \partial^2\psi$
source
image
lens
observer
Image: E Grocutt, IfA, Edinburgh
KiDS: Hildebrandt & B. Giblin/ESO


Strong-gravitational lensing: Φ
det (δij – ∂i∂jψ) = 0
Lens
Cluster member
Dark Matter?
Modified gravity?
ICM (X-rays, SZ)
Lensed sources
Multiple images
Background galaxy
(gravitational arcs)
Highly magnified galaxy
x100 (high z)
A zoom in…
Maturi et al. (2016)
Plazas, Meneghetti, Maturi, Rhodes (2018)

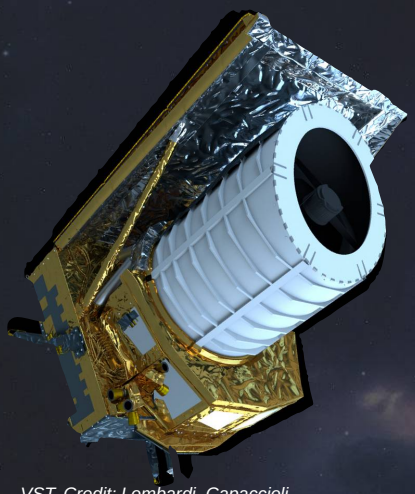

Clusters in optical data sets

Clusters in optical data sets

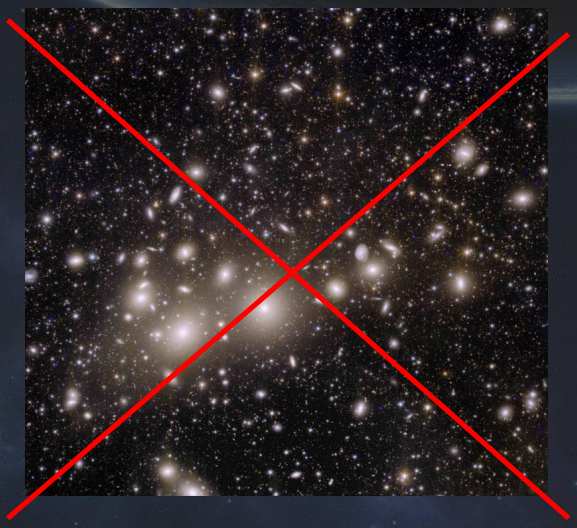




Clusters in optical data sets



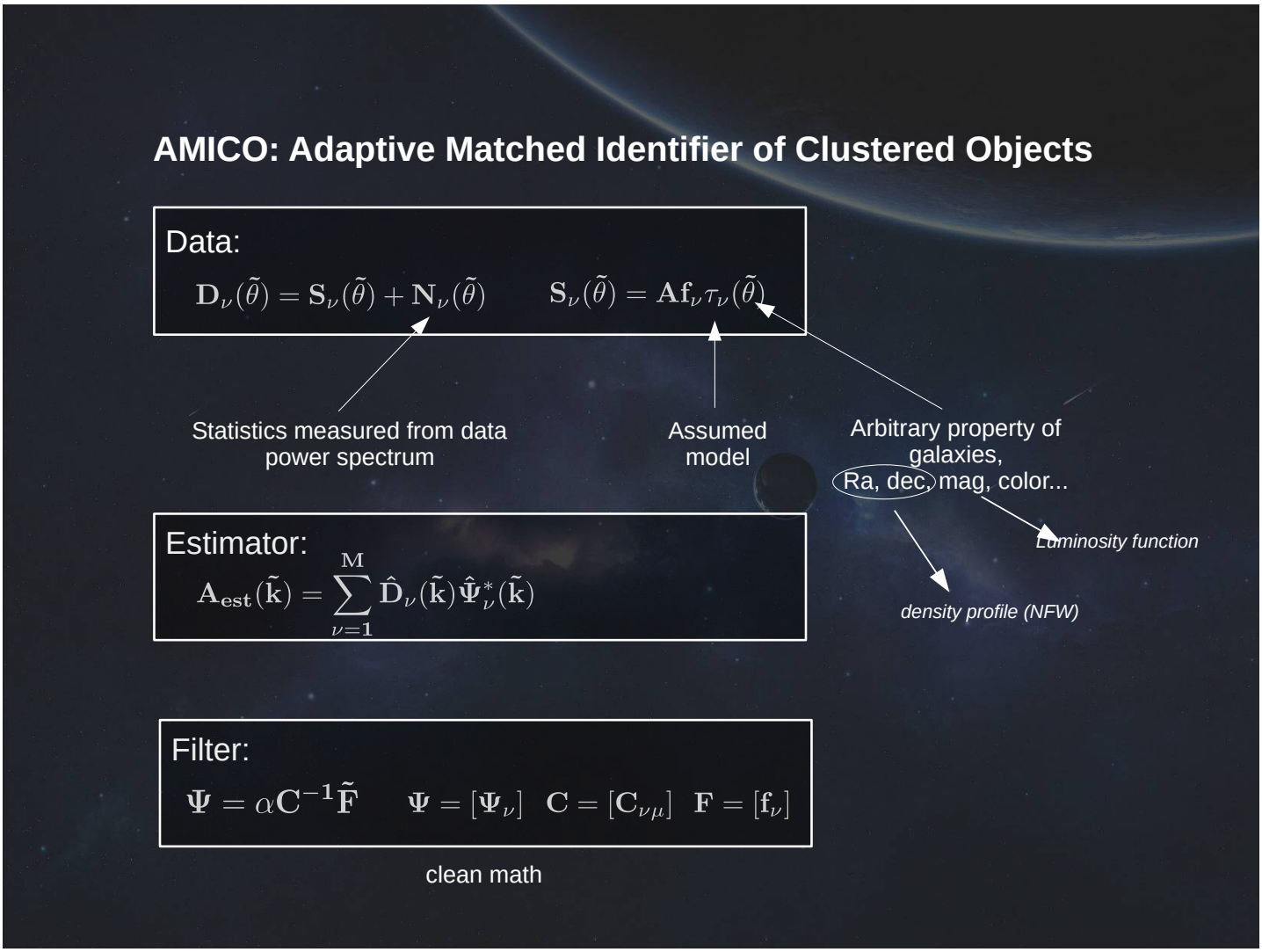
VST, Credit: Lombardi, Capaccioli



3706	7463.61	157.232	2	1.25	-0.018673	0.0329224	6.87081e-06	10.2722	-0.00933983	0.→
4079	785.072	160.885	2	1.25	-0.00677133	-0.0191166	1.33513e-05	10.2722	-0.00338601	-0→
4126	4863.13	204.884	2	1.25	-0.0457396	0.0266199	1.86214e-05	10.2722	-0.0228858	0.→
4248	2885.11	215.226	2	1.25	-0.00806341	0.0387643	1.42859e-05	10.2722	-0.00403329	0.→
4792	6984.93	237.745	2	1.25	-0.0398985	0.0350447	4.19831e-06	10.2722	-0.0199633	0.→
5372	2113.37	282.304	2	1.25	-0.0208591	0.0487989	3.15905e-06	10.2722	-0.0104369	0.→
5475	7424.76	301.021	2	1.25	-0.0436674	0.0340955	9.44324e-06	10.2722	-0.0218505	0.→
5956	9548.46	321.836	2	1.25	-0.0436195	-0.0446261	4.49729e-06	10.2722	-0.021831	-0→

Galaxies = vectors

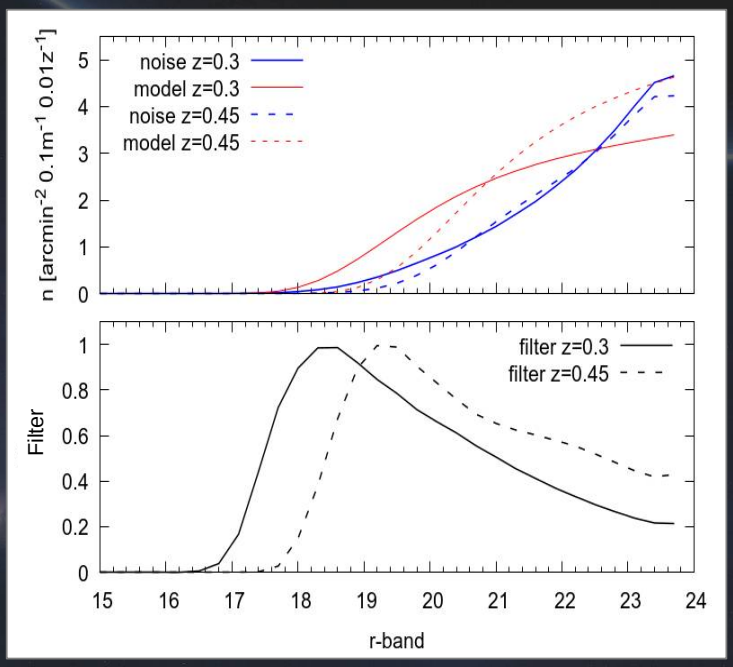
Galaxies Stars


AMICO: Adaptive Matched Identifier of Clustered Objects
Data:
$\mathbf{D}_\nu(\tilde{\theta}) = \mathbf{S}_\nu(\tilde{\theta}) + \mathbf{N}_\nu(\tilde{\theta})$
$\mathbf{S}_\nu(\tilde{\theta}) = \mathbf{A}\mathbf{f}_\nu\tau_\nu(\tilde{\theta})$
Statistics measured from data power spectrum
Assumed model
Arbitrary property of galaxies,
Ra, dec, mag, color...
Luminosity function
density profile (NFW)
Estimator:
$\mathbf{A}_{\mathbf{est}}(\tilde{\mathbf{k}}) = \sum_{\nu=1}^{\mathbf{M}} \hat{\mathbf{D}}_\nu(\tilde{\mathbf{k}})\hat{\mathbf{\Psi}}^*_\nu(\tilde{\mathbf{k}})$
Filter:
$\mathbf{\Psi} = \alpha\mathbf{C}^{-1}\tilde{\mathbf{F}}$ $\mathbf{\Psi} = [\Psi_\nu]$ $\mathbf{C} = [\mathrm{C}_{\nu\mu}]$ $\mathbf{F} = [\mathrm{f}_\nu]$
clean math

Filter (magnitude sub-space)

$$\Psi = \alpha \mathrm{C}^{-1} \tilde{\mathrm{F}}$$

Noise estimate Cluster model



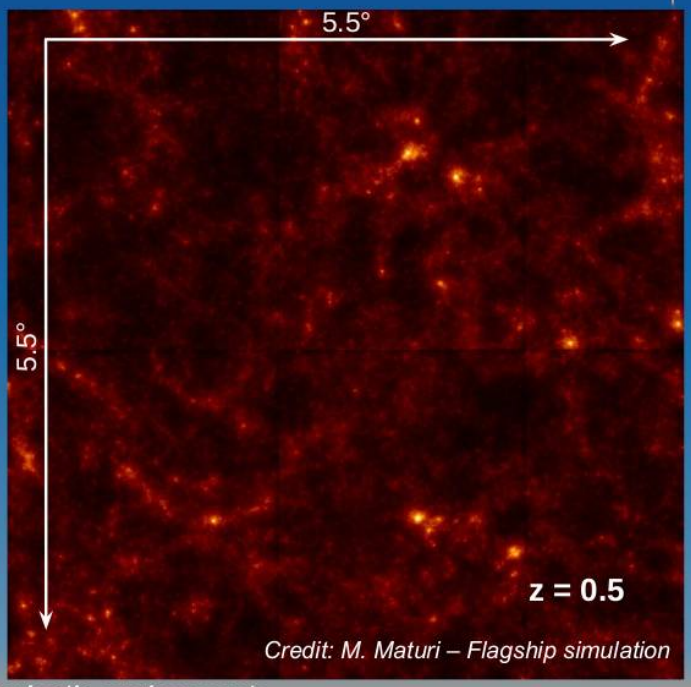
Filter (redshift sub-space)

No redshift slicing, photo-z P(z) of each individual galaxy

Galaxy Clusters

AMICO (Adaptive Matched Identifier of Clustered Objects, Bellagamba+18) based on the Optimal Filtering technique

Amplitude map response of the filter in the Flagship simulation with the depth of the Deep Fields



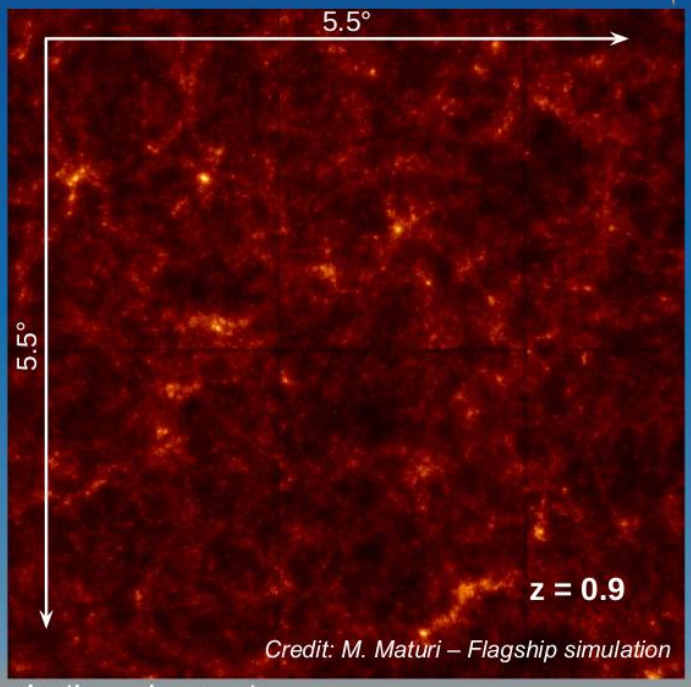
Porto Ercole, 29/09/2023 ✧ *A journey through galactic environments*

Slide from Micol Bolzanella

Galaxy Clusters

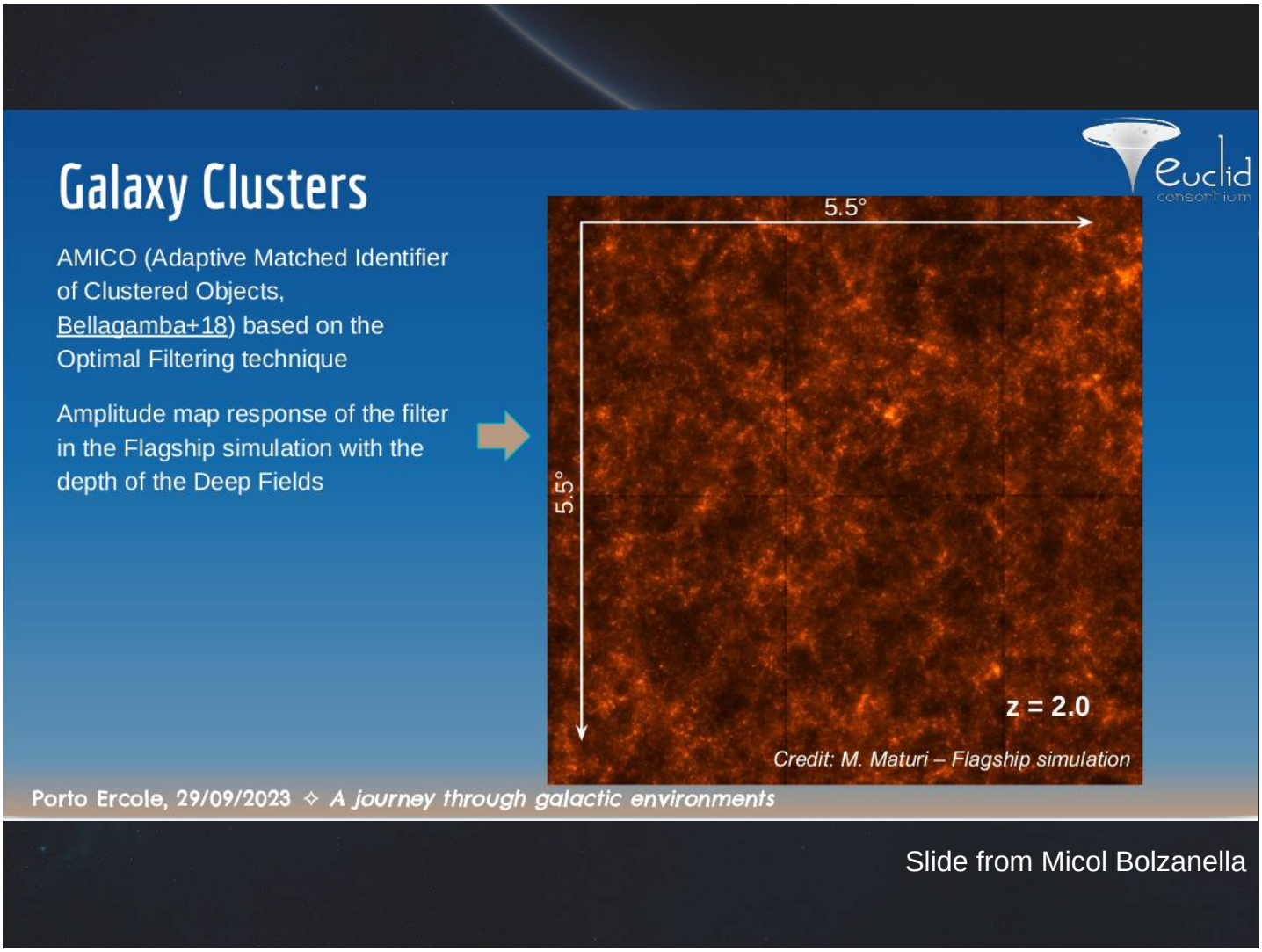
AMICO (Adaptive Matched Identifier of Clustered Objects, Bellagamba+18) based on the Optimal Filtering technique

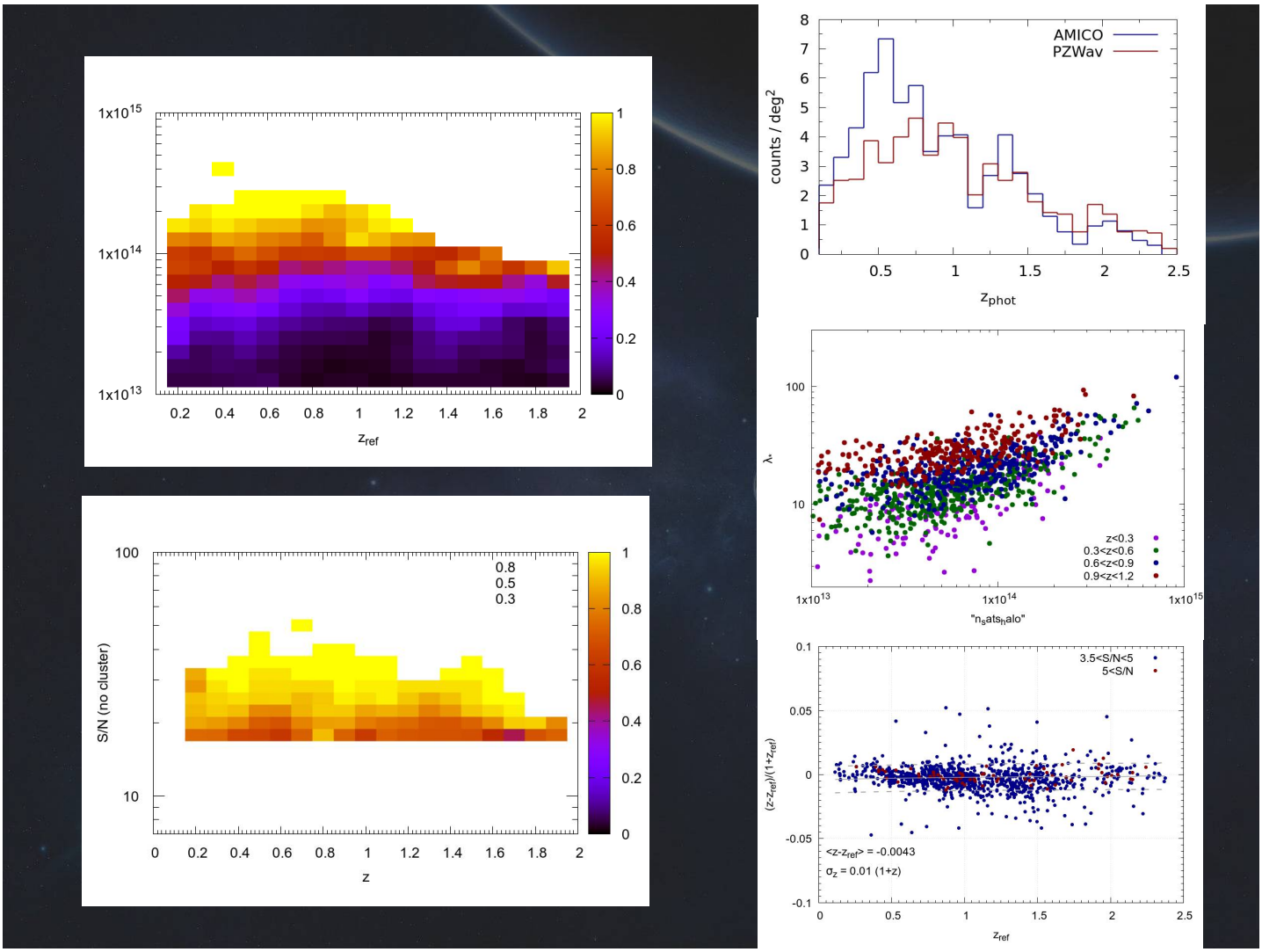
Amplitude map response of the filter in the Flagship simulation with the depth of the Deep Fields

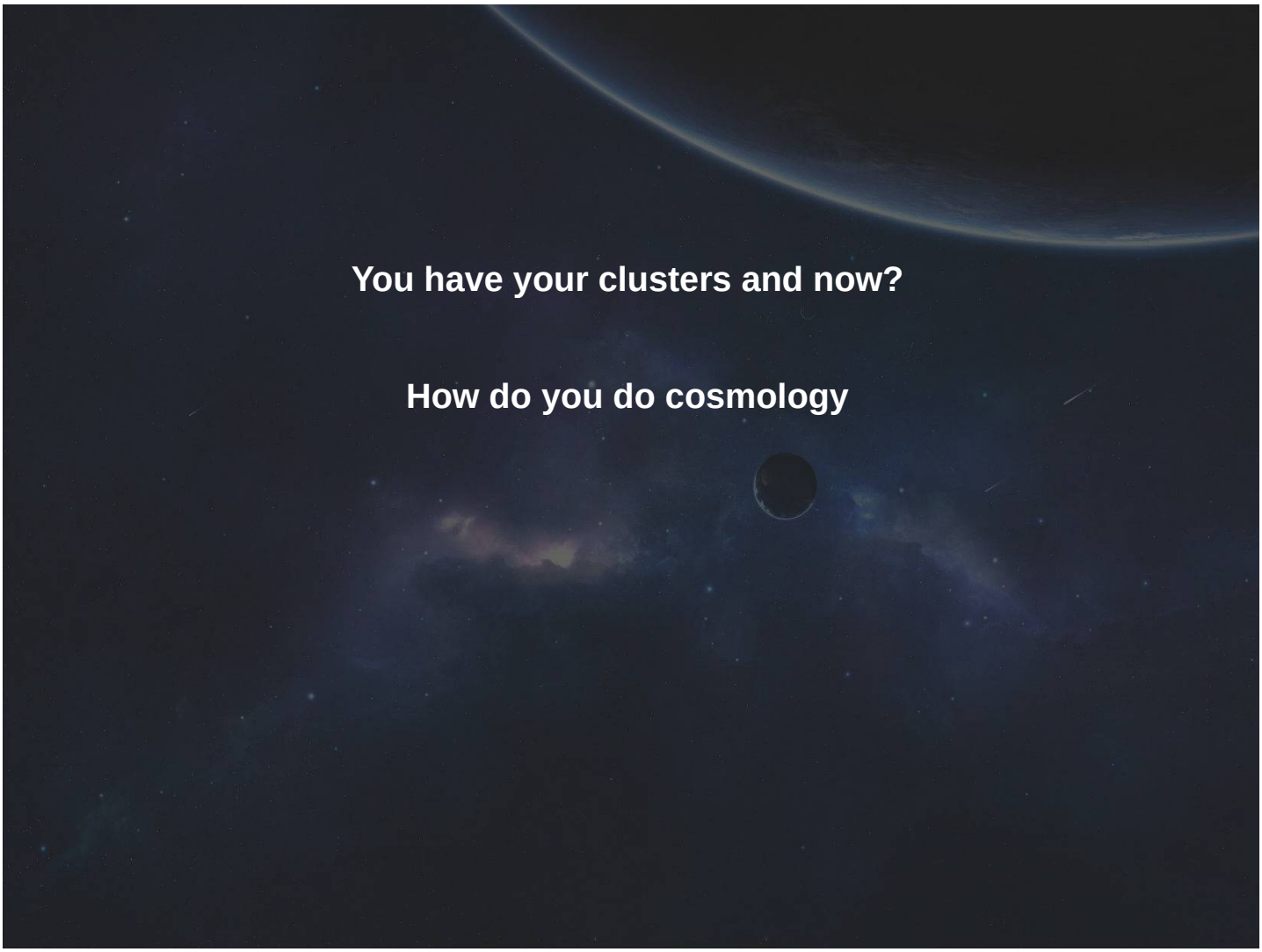


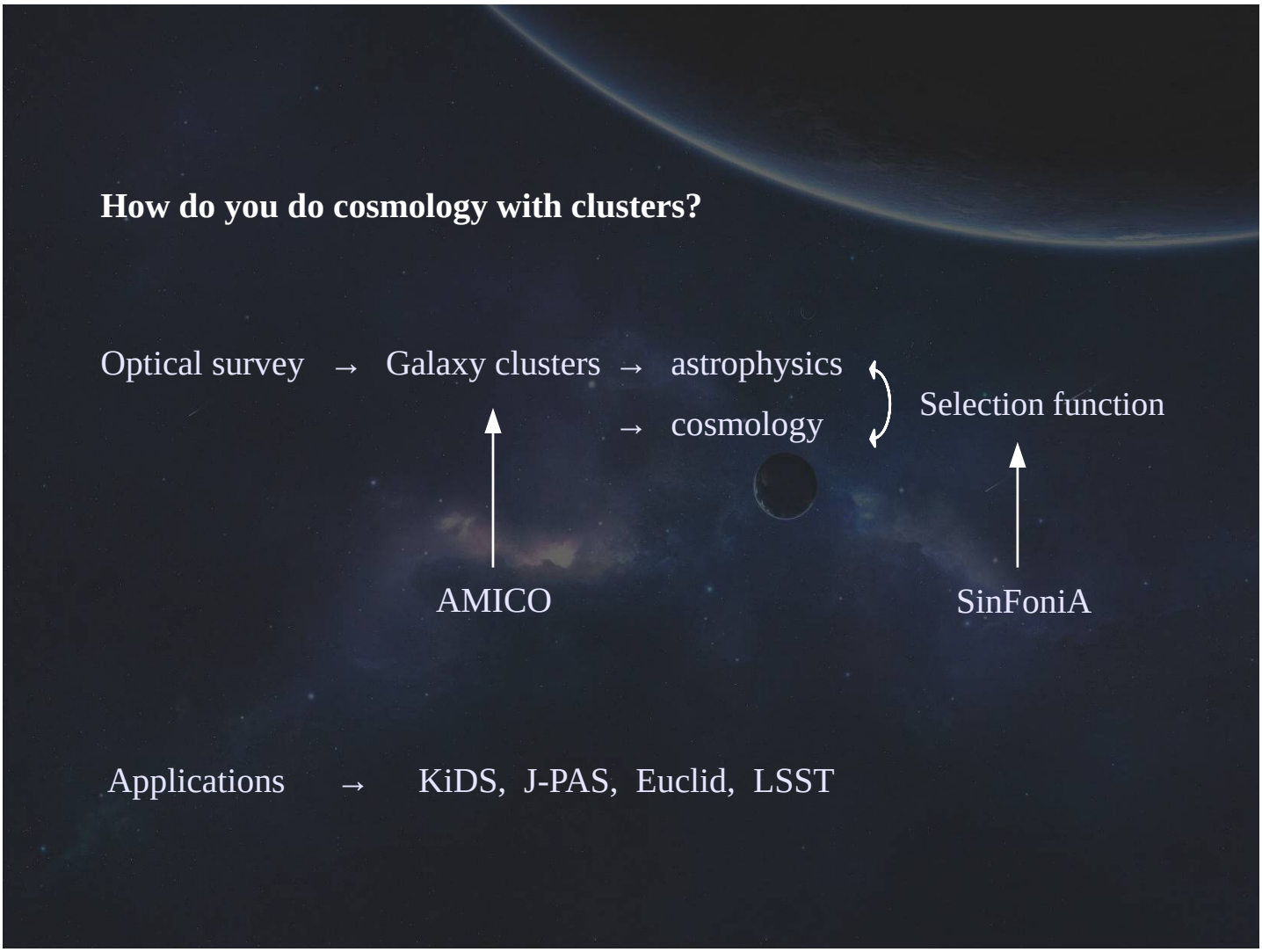
Porto Ercole, 29/09/2023 ✧ *A journey through galactic environments*

Slide from Micol Bolzanella


Galaxy Clusters
AMICO (Adaptive Matched Identifier of Clustered Objects, Bellagamba+18) based on the Optimal Filtering technique
Amplitude map response of the filter in the Flagship simulation with the depth of the Deep Fields
5.5°
5.5°
z = 2.0
Credit: M. Maturi – Flagship simulation
Porto Ercole, 29/09/2023 ✧ A journey through galactic environments
Slide from Micol Bolzanella


1x10^15
1x10^14
1x10^13
0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
z_ref
1 0.8 0.6 0.4 0.2 0
S/N (no cluster)
100
10
0.8
0.5
0.3
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
z
counts / deg^2
AMICO
PZWav
z_phot
λ
"n_satsh_alo"
z<0.3
0.3<z<0.6
0.6<z<0.9
0.9<z<1.2
(z-z_ref)/(1+z_ref)
3.5<S/N<5
5<S/N
<z-z_ref> = -0.0043
σ_z = 0.01 (1+z)

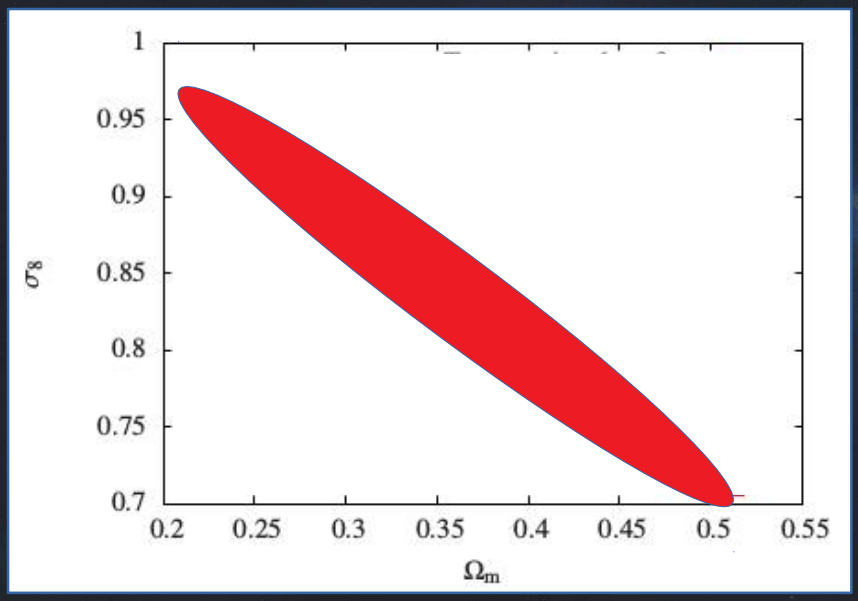

You have your clusters and now?
How do you do cosmology


How do you do cosmology with clusters?
Optical survey → Galaxy clusters → astrophysics
→ cosmology
Selection function
AMICO
SinFoniA
Applications → KiDS, J-PAS, Euclid, LSST

Cosmology with galaxy clusters

- We look for large samples: large volumes, small masses, high z

- But what you want in the end is:

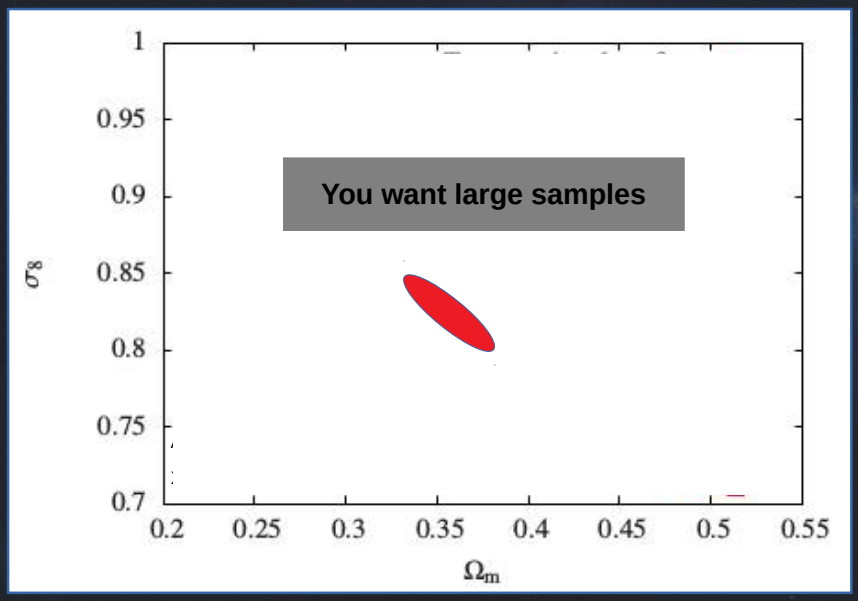


$$w(a) = w_0 + (1-a)\,w_a$$

Cosmology with galaxy clusters

- We look for large samples: large volumes, small masses, high z

- But what you want in the end is:

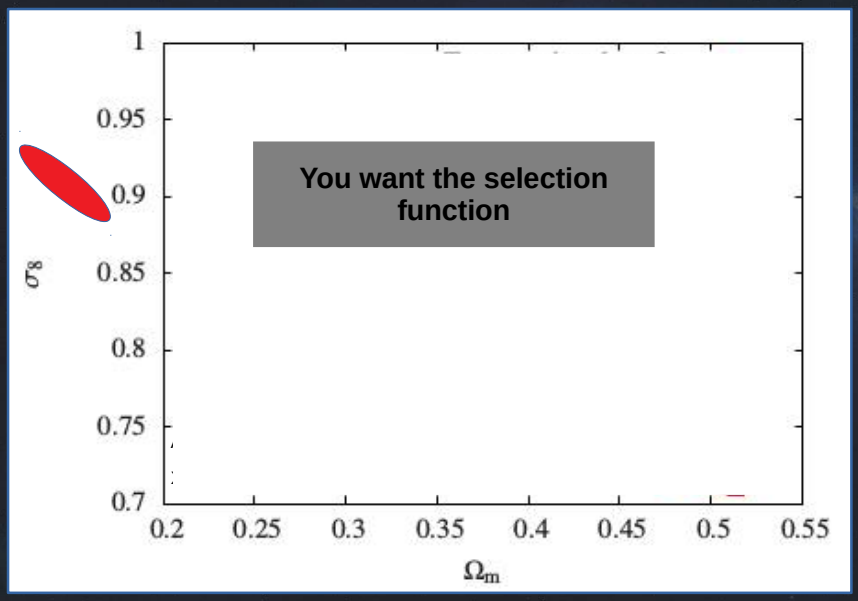


$$w(a) = w_0 + (1-a)\,w_a$$

Cosmology with galaxy clusters

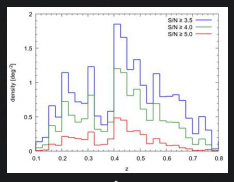
- We look for large samples: large volumes, small masses, high z

- But what you want in the end is:

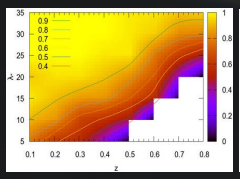


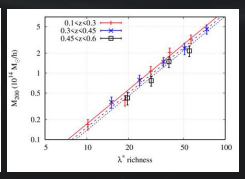
$$w(a) = w_0 + (1-a)\,w_a$$

Number counts, mass function & selection function

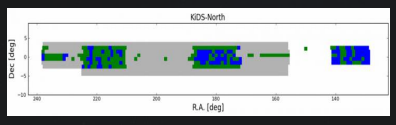


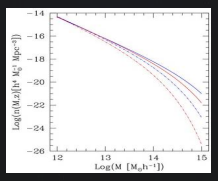
$D_a^2 dD_p$





$$N_{\ell,m} = \frac{\Omega_{\rm sky}}{4\pi} \int_{z_\ell}^{z_{\ell+1}} dz \frac{dV}{dz} \int_{M^{\rm ob}_{\ell,m}}^{M^{\rm ob}_{\ell,m+1}} \frac{dM^{\rm ob}}{M^{\rm ob}} \int_0^{+\infty} dM\, n(M,z)\, p(M^{\rm ob}|M)\ ,$$

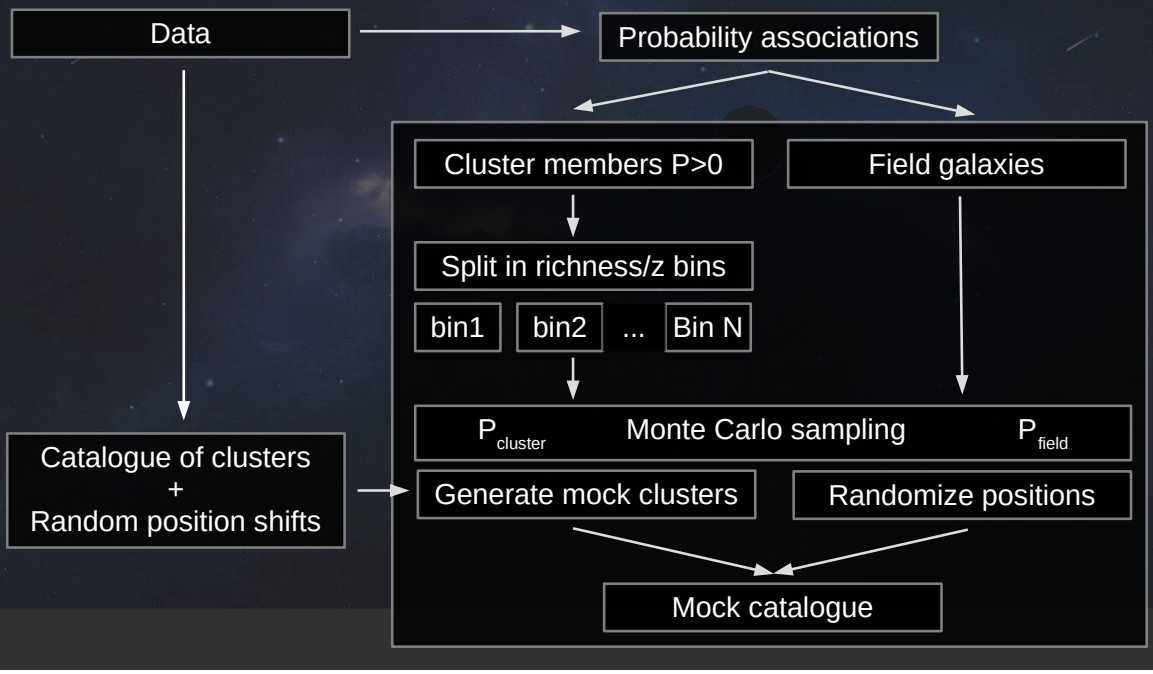


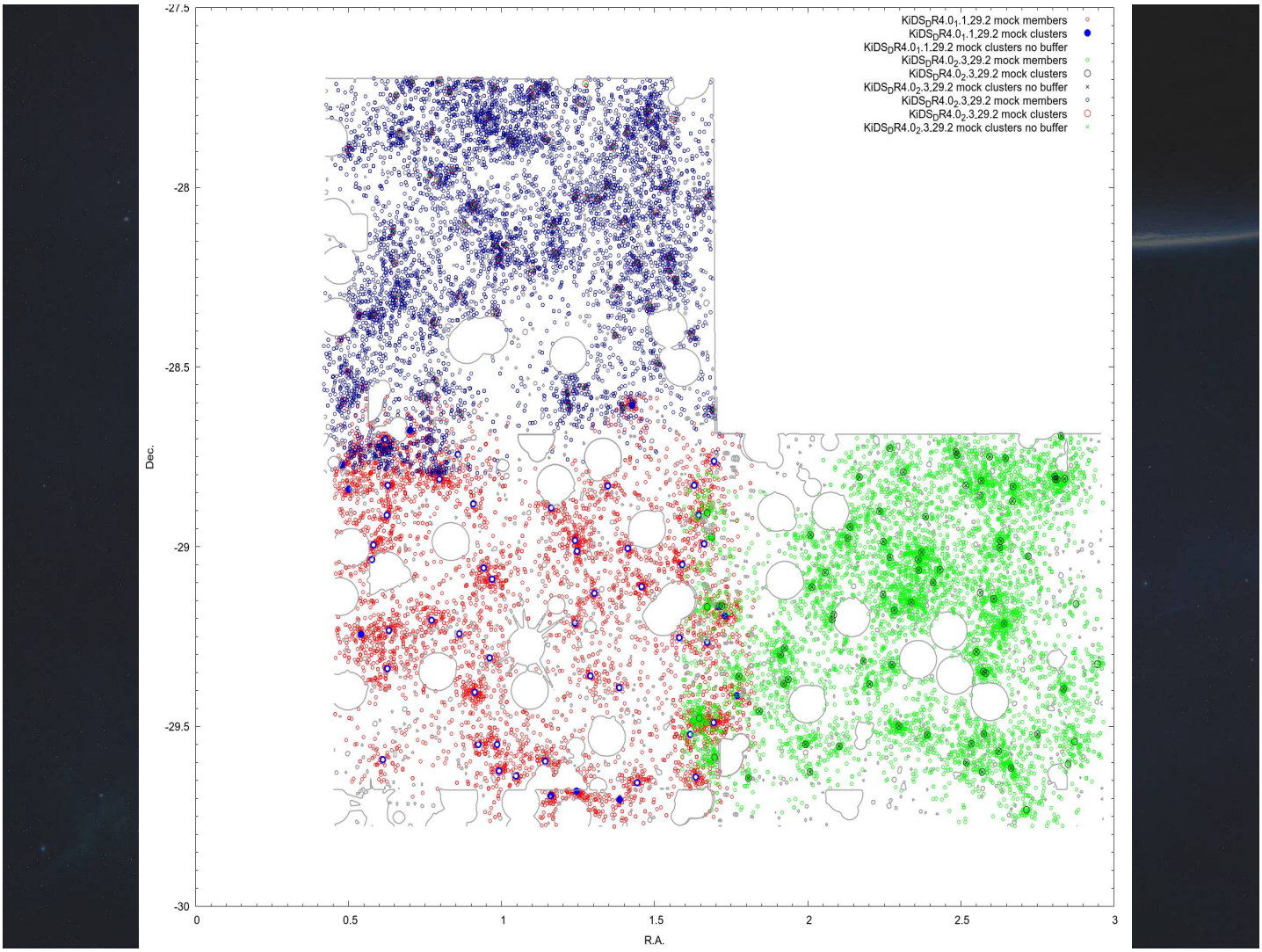


SinFoniA (Selection Function extrActor)

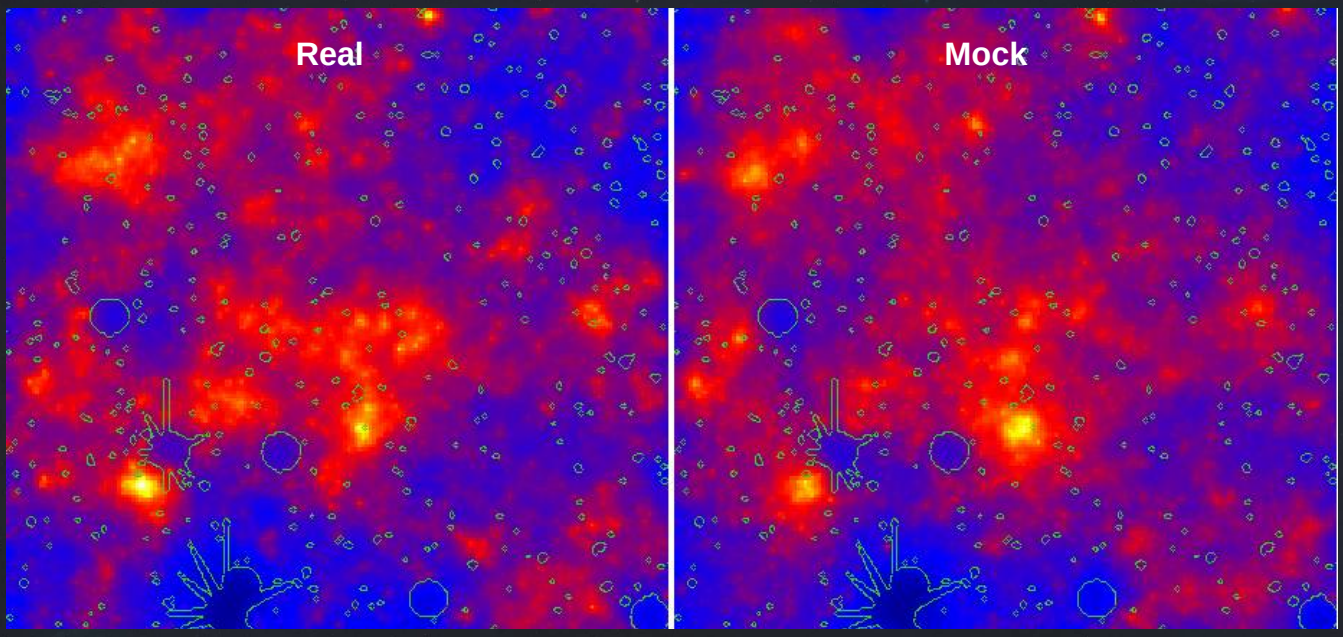
Mock catalogues from data

- Avoid numerical simulations
- Use the data as much as possible, stick to reality
- Produce mock catalogues with known field and cluster galaxies
- Run the detection algorithm and perform matching (similar to the CFCs)
- Purity, completeness, uncertainties on all quantities, …



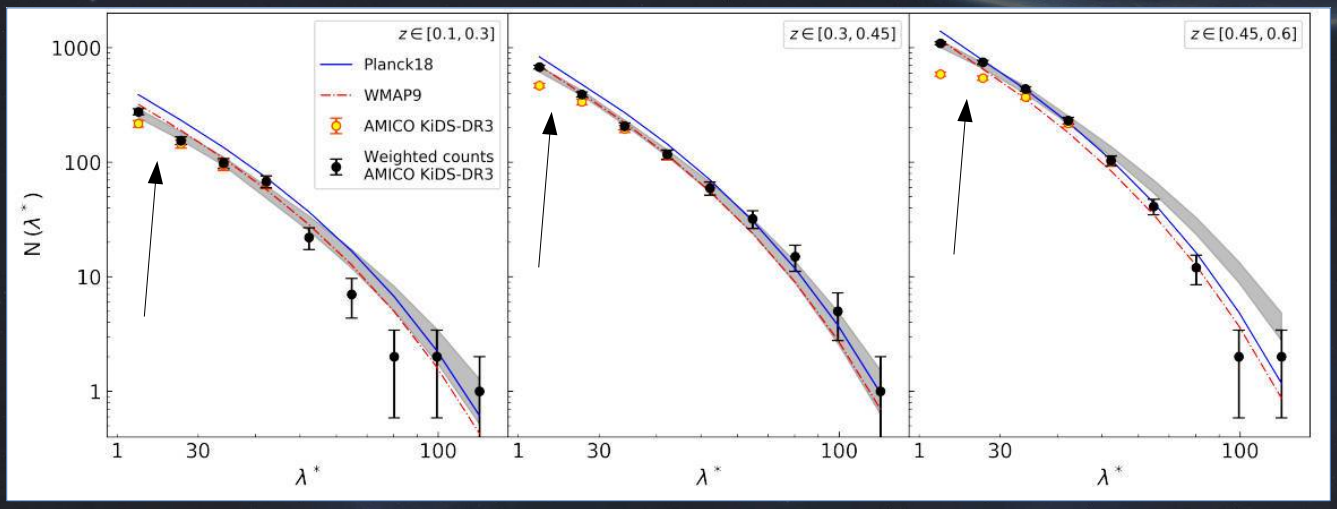

KiDS$_D$R4.0$_1$.1.29.2 mock members
KiDS$_D$R4.0$_1$.1.29.2 mock clusters
KiDS$_D$R4.0$_1$.1.29.2 mock clusters no buffer
KiDS$_D$R4.0$_2$.3.29.2 mock members
KiDS$_D$R4.0$_2$.3.29.2 mock clusters
KiDS$_D$R4.0$_2$.3.29.2 mock clusters no buffer
KiDS$_D$R4.0$_2$.3.29.2 mock members
KiDS$_D$R4.0$_2$.3.29.2 mock clusters
KiDS$_D$R4.0$_2$.3.29.2 mock clusters no buffer
-27.5
-28
-28.5
-29
-29.5
-30
Dec.
0
0.5
1
1.5
2
2.5
3
R.A.

Then you run AMICO….
A visual impression: amplitude map



An example: one of the KiDS tiles

Impact of selection function / purity on counts

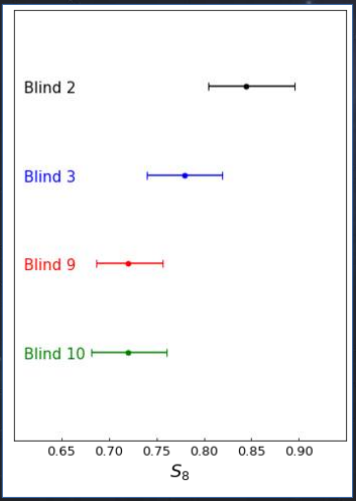


Lesci et al. 2020

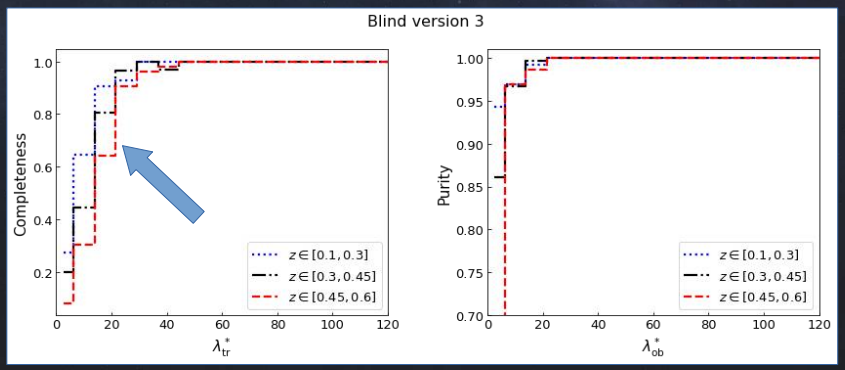
Ideally… blinding!

$$N_{\ell,m} = \frac{\Omega_{\rm sky}}{4\pi} \int_{z_\ell}^{z_{\ell+1}} dz \frac{dV}{dz} \int_{M^{\rm ob}_{\ell,m}}^{M^{\rm ob}_{\ell,m+1}} \frac{dM^{\rm ob}}{M^{\rm ob}} \int_0^{+\infty} dM\, n(M,z)\, p(M^{\rm ob}|M)\ ,$$

Matching of mocks → selection function
→ purity



Sadly not in Euclid for clusters , yes in KiDS




Cosmological forecast