# **Chromospheric Dynamics and Line Formation**

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# **BASIC HYDRODYNAMICS AND THERMODYNAMICS**

The dynamics of stellar chromospheres can be simulated by using the time-dependent equations of radiation hydrodynamics. To outline these equations we first describe the basic hydrodynamic and thermodynamic equations and then discuss the elements of radiation theory.

## **Continuity equation, Euler frame**



**FIGURE 1.** Euler frame  $\Sigma_E$  (left): observe the mass flow out of a fixed volume V, Lagrange frame  $\Sigma_L$  (right): follow a mass element M

In the *Euler frame*  $\Sigma_E$  we have a cartesian coordinate system with origin O and the unit vectors  $\mathbf{e}_x$ ,  $\mathbf{e}_y$ ,  $\mathbf{e}_z$  in x, y, z-directions and observe the density  $\rho(\mathbf{r},t)$   $(g/cm^3)$ , gaspressure  $p(\mathbf{r},t)$   $(dyn/cm^2)$  and temperature  $T(\mathbf{r},t)$  (K) flowing out of a fixed volume V with velocity  $\mathbf{v}(\mathbf{r},t)$  (cm/s) (Fig. 1.1).  $\mathbf{r}$  is the radius vector with components x,y,z and t is the time. Let dA be a directed surface element of V. Then the amount of matter which flows out of V across its surface per sec (Fig. 1.1) is given by

$$\int \boldsymbol{\rho} \mathbf{v} \cdot d\mathbf{A} = \int \nabla \cdot \boldsymbol{\rho} \mathbf{v} \, dV \quad , \tag{1}$$

where the equal sign is due to Gauss's theorem. The amount of matter missing per sec from volume V is also given by

$$-\frac{\partial}{\partial t}\int \rho dV = -\int \frac{\partial \rho}{\partial t} dV \quad , \tag{2}$$

where the equal sign is due to the fact that *V* does not depend on time. Equating Eqs. (1), (2) and because *V* is arbitrary one finds the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \quad . \tag{3}$$

# Lagrange frame

In a Lagrange frame  $\Sigma_L$  we monitor the mass element M (Fig. 1.1) initially contained in V (at time t = 0 and position  $\mathbf{r} = \mathbf{a}$ ), which at later time t = t is in volume V' at radius vector  $\mathbf{r}(\mathbf{a},t) = (x(\mathbf{a},t), y(\mathbf{a},t), z(\mathbf{a},t))$ . Vector  $\mathbf{a}$  uniquely identifies a mass element at the initial time t = 0. We consider physical variables such as  $\rho$  on an x, t plane on which the position of the mass element M describes a path x(t) where x(t=0)=a and gives  $\rho(x(t),t)$ . The rate of change of  $\rho$  along that path can be pictured as a sum of the indicated incremental variations of  $\rho$  in t- and x-directions

$$\frac{d\rho}{dt} \equiv \left(\frac{\partial\rho}{\partial t}\right)_{a} = \left(\frac{\partial\rho}{\partial t}\right)_{x} + \left(\frac{\partial\rho}{\partial x}\right)_{t} \frac{dx(t)}{dt} = \left(\frac{\partial\rho}{\partial t}\right)_{x} + v\left(\frac{\partial\rho}{\partial x}\right)_{t} \quad , \qquad (4)$$

where the sign  $(\equiv)$  means equal by definition and

$$v = \frac{dx(t)}{dt} \quad , \tag{5}$$

is the velocity of the mass element. The time derivative  $d/dt \equiv (\partial/\partial t)_a$  in the Lagrange frame is the so called *substantial derivative*. It describes the time-dependence of a physical quantity in a moving mass element. In three dimensions, and for a physical function *h* (such as  $\rho$ , *p*, *T*, etc.), the substantial derivative similarly as in (4) is given by

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \mathbf{v} \cdot \nabla h \quad . \tag{6}$$

### **Equation of motion**

In the Lagrange frame  $\Sigma_L$ , Newton's third law (that the inertial force, mass dM times acceleration  $d\mathbf{v}/dt$ , is balanced by the forces acting on this mass element) can be written

$$\int \frac{d\mathbf{v}}{dt} \, dM = \int \rho \frac{d\mathbf{v}}{dt} \, dV = \int \mathbf{f} \, dV - \int \nabla p \, dV \quad . \tag{7}$$

Here the acting forces consist of a volume force density  $\mathbf{f} (dyn/cm^3)$  and a pressure force. In stellar atmospheres the volume force density is due to gravity

$$\mathbf{f} = -\frac{G M_{\star} \rho}{r^2} \mathbf{e}_r = -\rho g \mathbf{e}_r \quad , \tag{8}$$

where  $\mathbf{e}_r$  is the unit vector in radial direction,  $M_{\star}$  the stellar mass,  $G = 6.672 \times 10^{-8}$  (cm<sup>3</sup> g<sup>-1</sup> s<sup>-2</sup>) the gravitational constant, and *r* the radial distance from the center of the star. As the extent  $\Delta r$  of stellar atmospheres is often small ( $\Delta r \ll r$ ), we can replace  $GM_{\star}/r^2$  by a constant gravitational acceleration *g*, with e.g.  $g = 2.74 \times 10^4$  (cm s<sup>-2</sup>) for the Sun. Using Eqs. (6), (7), (8) and noting that the volume of the mass element is arbitrary, we obtain the *equation of motion* 

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p - \rho g \,\mathbf{e}_r \quad . \tag{9}$$

In this equation we have neglected *viscosity* and *radiation pressure*. It can be shown that in typical late-type stellar atmospheres both are unimportant.

# **Energy equation**

It is convenient to picture the moving mass element as enclosed by a wall and to carefully monitor the energy which flows through that wall. A powerful book-keeping quantity for this is the entropy S (erg g<sup>-1</sup>K<sup>-1</sup>). If the walls are impermeable, energy conservation just means S = constant in  $\Sigma_L$ . This can be written with Eq. (6)

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = 0 \quad . \tag{10}$$

In atmospheric regions close to the star (in photospheres and most of the chromospheres) only two processes by which mass elements gain energy are important, *radiation* and *Joule heating*. In coronae and some chromospheric regions one has steep temperature gradients and sometimes large velocity gradients. Here *thermal conduction* and *viscosity* are other important heating (or loss) mechanisms. One thus has the relation

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = \frac{dS}{dt} \bigg|_{ext} = \frac{\Phi_R}{\rho T} + \frac{\Phi_J}{\rho T} + \frac{\Phi_C}{\rho T} + \frac{\Phi_V}{\rho T} \quad , \tag{11}$$

called *entropy conservation equation*. Here  $dS/dt|_{ext}$  (erg g<sup>-1</sup>K<sup>-1</sup>s<sup>-1</sup>) is the heating function resulting from external heating by radiative, Joule, thermal conductive and viscous heating, where  $\Phi_R$ ,  $\Phi_C$  and  $\Phi_V$  (erg cm<sup>-3</sup> s<sup>-1</sup>) are the net radiative, thermal conductive and viscous heating rates, respectively. These functions will be discussed below. Joule heating,  $\Phi_J$ , will not be considered here although magnetic fields and heating (magnetohydrodynamics, MHD) are very important in chromospheres.

### The atmospheric gas composition

Stellar atmospheres consist of ideal gases which are described by the universal gas law

$$p = \sum_{i} n_{i} kT = \rho \frac{\Re T}{\mu} \quad , \tag{12}$$

where  $k = 1.3807 \times 10^{-16}$  (erg K<sup>-1</sup>) is the Boltzmann constant,  $\mu$  (g mol<sup>-1</sup>) the mean molecular weight,  $\Re = 8.3144 \times 10^7$  (erg K<sup>-1</sup> mol<sup>-1</sup>) the universal gas constant.  $n_i$ are the number densities (cm<sup>-3</sup>) of the different types of particles *i* (atoms, ions and electrons). With a given mixture of chemical elements in the stellar gas the mean molecular weight can be computed as a function  $\mu = \mu(T, p)$ . For solar type neutral gas one has  $\mu \approx 1.24$ , while for fully ionized gas one has  $\mu \approx 0.60$ .

## The basic elements of thermodynamics.

In many applications in stellar photospheres, chromospheres and coronae it is sufficient to consider atmospheric gases which are either neutral or fully ionised. Under these conditions the thermodynamic relations are particularly simple. The internal energy E (erg g<sup>-1</sup>) is given by

$$E = c_v T \quad , \tag{13}$$

where  $c_v$  (erg g<sup>-1</sup> K<sup>-1</sup>) is the specific heat for constant volume per gram given by

$$c_{\nu} = \frac{1}{\gamma - 1} \frac{\Re}{\mu} \quad , \tag{14}$$

and  $\gamma = c_p/c_v = 5/3$  is the ratio of specific heats. With the specific volume V (cm<sup>3</sup> g<sup>-1</sup>)

$$V = \frac{1}{\rho} \quad , \tag{15}$$

the combined first and second law of thermodynamics is written

$$T dS = dE + p dV = dE - \frac{p}{\rho^2} d\rho \quad . \tag{16}$$

>From these equations the relationships between the thermodynamical variables S, E, T, p,  $\rho$  can be computed. Specification of two of these allows to determine the remaining variables.

# **Conservation equations**

It is often convenient to write the hydrodynamic equations in conservation form. If one has conserved quantities f (mass, momentum, energy) and fluxes **F** (mass flux, momentum flux, energy flux) the equations can be written in the form:

$$\frac{\partial f}{\partial t} = -\nabla \cdot \mathbf{F} + C \quad , \tag{17}$$

where C is a source term.

### a. Conservation of mass

The continuity equation is already in conservation form:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v} \quad . \tag{18}$$

### **b.** Conservation of momentum

From Eqs. (3) and (9) one obtains

$$\frac{\partial \rho \mathbf{v}}{\partial t} = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t} = -\rho \mathbf{v} \cdot \nabla \mathbf{v} - \mathbf{v} \nabla \cdot \rho \mathbf{v} - \nabla p - \rho g \mathbf{e}_r \quad , \tag{19}$$

which can be written

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot \left( \rho \mathbf{v} \mathbf{v} + p \overline{\overline{U}} \right) - \rho g \mathbf{e}_r \quad . \tag{20}$$

Here vv is a dyad, and

$$\overline{\overline{U}} = \mathbf{e}_x \mathbf{e}_x + \mathbf{e}_y \mathbf{e}_y + \mathbf{e}_z \mathbf{e}_z \quad , \tag{21}$$

is a unit dyad.

# c. Conservation of energy

There are three types of energy densities (erg  $cm^{-3}$ ) for gas elements in stellar atmospheres:

 $\rho E$  internal energy (energy of microscopic undirected motion of atoms and ions),  $\frac{1}{2}\rho v^2$  kinetic energy (directed motion of the entire gas element),

 $\rho \phi$  potential energy (gas element in the gravitational field).

The gravitational potential is given by

$$\phi \equiv -\frac{GM_{\star}}{r} \quad . \tag{22}$$

With this and Eq. (8) the volume force density can be written

$$\mathbf{f} = -\rho g \mathbf{e}_r = -\rho \frac{GM_\star}{r^2} \mathbf{e}_r = -\rho \nabla \phi \quad . \tag{23}$$

Consider the time derivative of the total energy density, which can be written as a sum of four terms

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho E + \rho \phi \right) = \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \rho E}{\partial t} + \frac{\partial \rho \phi}{\partial t} \quad . \tag{24}$$

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Modifying the terms on the RHS using Eqs. (3), (9), (11), (16), (23) one obtains the *energy conservation equation*:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho E + \rho \phi \right) = -\nabla \cdot \rho \mathbf{v} \left( \frac{1}{2} v^2 + E + \frac{p}{\rho} + \phi \right) + \rho T \left. \frac{dS}{dt} \right|_{ext}$$
$$= -\nabla \cdot \rho \mathbf{v} \left( \frac{1}{2} v^2 + E + \frac{p}{\rho} + \phi \right) + \Phi_R + \Phi_J + \Phi_C + \Phi_V \quad . \tag{25}$$

It is seen that there are three energy flux components (erg cm<sup>-2</sup> s<sup>-1</sup>):  $\rho \mathbf{v}_2^1 v^2$  kinetic energy flux,  $\rho \mathbf{v} (E + p/\rho)$  enthalpy flux,

 $\rho v \phi$  potential energy flux.

# Heating by viscosity and thermal conductivity

The particle transport processes which occur in the presence of velocity and temperature gradients contribute to the local heating.

The thermal conductive heating rate  $\Phi_C$  (erg cm<sup>-3</sup> s<sup>-1</sup>) is given by

$$\Phi_C = \frac{d}{dx} \kappa_{th} \frac{dT}{dx} \quad . \tag{26}$$

The viscous heating rate  $\Phi_V$  (erg cm<sup>-3</sup>s<sup>-1</sup>) by

$$\Phi_V = \eta_{vis} \left(\frac{dv}{dx}\right)^2 \quad . \tag{27}$$

The coefficients of thermal conductivity  $\kappa_{th}$  and of viscosity  $\eta_{vis}$  are functions of T and p.

# **ELEMENTARY RADIATION THEORY**

Using the three time-dependent hydrodynamic equations Eqs. (18), (20), (25) together with the ideal gas law Eq. (12), the thermodynamic relations and functions such as  $\mu(T,p)$ ,  $\kappa_{th}(T,p)$  and  $\eta_{vis}(T,p)$  one would be able to compute the dynamics of stellar chromospheres except that there radiation is critically important. For this reason we now give a short review of the basic equations of the radiation theory and derive radiative heating rates.

# **Basic radiation quantities**

#### a. Solid angle



FIGURE 2. Left: Solid angle. Right: Intensity

Consider an orthogonal cartesian coordinate system x, y, z. Let the latitude angle  $\vartheta$ , and azimuth angle  $\varphi$  be defined as seen in Fig. 2.1, left. The *solid angle* is a surface element on a sphere with unit radius

$$d\Omega \equiv \sin\vartheta \, d\vartheta \, d\varphi \quad , \tag{28}$$

and has the dimension sr = steradian.

#### **b.** Intensity

The intensity is the basic quantity of radiation theory. Consider the energy  $\Delta E$  of all photons which flow through a window of area  $\Delta A$  in normal direction  $\mathbf{e}_n$  into the solid angle  $\Delta \Omega$  per time interval  $\Delta t$  and frequency interval  $\Delta v$  (Fig. 2.1, right). Assume that **r** is the radius vector that points from the origin *O* of a coordinate system (with unit vectors  $\mathbf{e}_x$ ,  $\mathbf{e}_y$ ,  $\mathbf{e}_z$ ) to the location of the area  $\Delta A$ . Then the *monochromatic intensity*  $I_v$ , also called *specific intensity*, is given by

$$I_{\nu}(\mathbf{r}, \mathbf{e}_{n}, t) \equiv \lim \frac{\Delta E}{\Delta A \,\Delta t \,\Delta \Omega \,\Delta \nu} \quad . \tag{29}$$

The limit is taken for  $\Delta A, \Delta t, \Delta \Omega, \Delta v \rightarrow 0$ .  $I_v$  has the dimension erg cm<sup>-2</sup> s<sup>-1</sup> sr<sup>-1</sup> Hz<sup>-1</sup> = 10<sup>-3</sup> W m<sup>-2</sup> sr<sup>-1</sup> Hz<sup>-1</sup>. Recall that 1 J = 1 Ws = 1 Nm = 1 kg m<sup>2</sup>s<sup>-2</sup> = 10<sup>7</sup> erg.

#### c. Mean intensity

The *mean intensity* is the specific intensity averaged over all angles

$$J_{\nu}(\mathbf{r},t) = \frac{1}{4\pi} \int I_{\nu}(\mathbf{r},\mathbf{e}_{n},t) \ d\Omega = \frac{1}{4\pi} \int_{0}^{\pi} \int_{0}^{2\pi} I_{\nu} \sin\vartheta \ d\vartheta \ d\varphi \quad . \tag{30}$$

In most cases there is no dependence of  $I_v$  on  $\varphi$ , and with the *angle cosine* 

$$\mu \equiv \cos \vartheta \quad , \tag{31}$$

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the mean intensity can be written

$$J_{\mathcal{V}}(\mathbf{r},t) = \frac{1}{2} \int_{0}^{\pi} I_{\mathcal{V}} \underbrace{\sin\vartheta \, d\vartheta}_{-d\mu} = \frac{1}{2} \int_{-1}^{+1} I_{\mathcal{V}} \, d\mu \quad .$$
(32)

The dimension of  $J_{\nu}$ , erg cm<sup>-2</sup> s<sup>-1</sup> sr<sup>-1</sup> Hz<sup>-1</sup>, is the same as that of  $I_{\nu}$ .

### d. Radiative flux

Consider a window with area  $\Delta A$  and normal  $e_n$  (see Fig. 3.3). Assume that from both sides photons flow in arbitrary directions  $\mathbf{e}'_n$  through  $\Delta \mathbf{A}$ . The net energy  $\Delta E$  transported in  $\mathbf{e}_n$  direction by all the photons flowing through  $\Delta A$  per time interval  $\Delta t$  and frequency band  $\Delta v$  is the it radiative flux given by

$$F_{\mathbf{v}}(\mathbf{r}, \mathbf{e}_n, t) = \lim \frac{\Delta E}{\Delta A \,\Delta t \,\Delta v} \quad , \tag{33}$$

where the limit is taken for  $\Delta A, \Delta t, \Delta v \rightarrow 0$ .  $F_v$  has the dimension erg cm<sup>-2</sup> s<sup>-1</sup> Hz<sup>-1</sup> and can be derived from  $I_{v}$  by summing over all projected contributions from the individual light rays in  $\mathbf{e}'_n$  direction,

$$F_{\mathbf{v}}(\mathbf{r},\mathbf{e}_{n},t) = \int I_{\mathbf{v}}(\mathbf{r},\mathbf{e}_{n}',t) \,\mathbf{e}_{n}' \cdot \mathbf{e}_{n} \,d\Omega' = \int_{0}^{\pi} \int_{0}^{2\pi} I_{\mathbf{v}} \underbrace{\cos\vartheta}_{\mu} \underbrace{\sin\vartheta}_{-d\mu} \,d\vartheta \,\,, \qquad (34)$$

where  $\vartheta$  is measured from the  $\mathbf{e}_n$  direction. Here one takes into account that photons from  $\mathbf{e}'_n$  see only a projected area  $\Delta A \mathbf{e}'_n \cdot \mathbf{e}_n$ . In most cases there is no  $\varphi$  dependence and one has

$$F_{\mathbf{v}}(\mathbf{r}, \mathbf{e}_n, t) = 2\pi \int_{-1}^{+1} I_{\mathbf{v}} \ \mu \ d\mu \quad .$$
(35)

Note that  $\mathbf{F}_{\nu}$  is a vector which points in the direction  $\mu = 1$ , or  $\vartheta = 0$ . In an isotropic radiation field one has  $F_{y} = 0$ . A note of caution: in some texts on radiation theory the radiative flux is defined differently and given by  $\pi F_{v}$ .

# Radiation field, level populations, LTE and NLTE

Consider the interior of a cavity which has been submerged for a long time in a gas of constant temperature T. All time-dependent processes have come to an end and a final state has been reached, called *thermal equilibrium* (TE). As derived in Appendix B one then has four important relations: the *Planck function*, the *Boltzmann distribution*, the Saha equation and the Maxwell velocity distribution.

#### a. Planck function



FIGURE 3. Radiative flux



**FIGURE 4. a.** Measurement of the flux. **b.** Measurement of the intensity. **c.** dependence of  $F_v$  and  $I_v$  on the distance from the star

In TE the intensity is given by the *Planck function* (see Appendix B, Eq. (B.52)):

$$I_{\nu} = B_{\nu} \equiv \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad , \tag{36}$$

which upon frequency integration gives

$$I = \int I_{\nu} d\nu = B \equiv \int B_{\nu} d\nu = \frac{\sigma}{\pi} T^4 \quad . \tag{37}$$

*B* is called the *integrated Planck function*. Here  $k = 1.3807 \times 10^{-16}$  erg K<sup>-1</sup> is the Boltzmann constant and  $\sigma = 5.670 \times 10^{-5}$  erg cm<sup>-2</sup> s<sup>-1</sup> K<sup>-4</sup> the Stefan-Boltzmann constant.

#### b. Boltzmann and Saha equations

Consider the energy levels of the atoms or ions of a gas (see Fig. 3.5). The number  $n_l$  of atoms or ions per cm<sup>3</sup> which have their electrons in the bound energy level *l* is called the *population of level l. n<sub>k</sub>* is the population of the continuum, that is, the number of atoms or ions per cm<sup>3</sup> which have their electron removed by ionization. For bound levels in TE



FIGURE 5. Energy levels

one has the Boltzmann distribution (see Appendix B, Eq. (B.34)):

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-E_{lu}/kT} = \frac{g_u}{g_l} e^{-h\mathbf{v}_{lu}/kT}$$
(38)

where the  $g_i$  are *statistical weights* (for hydrogen e.g.  $g_i = 2i^2$ ) and  $E_{lu}$  is the *energy difference* between the levels (for tabulated values of these quantities see Allen 1973). For continuum levels in TE one has the *Saha equation* (see Appendix B, Eq. (B.47)):

$$\frac{n_k n_e}{n_l} = \frac{2u_k}{g_l} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-E_l/kT}$$
(39)

where (see Allen 1973)  $u_k$  is the *partition function* (see Eq. (B.36)),  $E_l$  the *ionization energy* from level l,  $m_e = 9.1095 \times 10^{-28}$  g the mass of the electron and  $n_e$  the number density of electrons.

#### c. Maxwell distribution

In addition in TE, atoms, ions and electrons obey the Maxwell velocity distribution:

$$\frac{dn(v)}{n} = 4\pi v^2 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} \, dv \tag{40}$$

Here dn(v) is the number of particles per cm<sup>3</sup> with mass *m* and velocities in the interval *v* to v + dv, while *n* is total number of these particles per cm<sup>3</sup>, irrespective of velocity.

**TABLE 1.** Various physical situations in stellar atmospheres definedby the successive break-down of relations valid in thermodynamic equi-librium.TELTENLTEInterplanetary

$I_{\nu} = B_{\nu}$	$I_{\nu} \neq B_{\nu}$	$I_{\nu} \neq B_{\nu}$	$I_{\nu} \neq B_{\nu}$
Boltzmann	$\operatorname{Boltzmann}$	≠Boltzmann	≠Boltzmann
$\operatorname{Saha}$	$\operatorname{Saha}$	≠Saha	≠Saha
Maxwell	Maxwell	Maxwell	$\neq$ Max well

#### d. Departures from thermodynamic equilibrium: LTE, NLTE

Deep in a star one has TE. Rising towards the surface, the first thing which breaks down is the equality of the intensity and the Planck function. One then has a situation in which

 $I_v$  is not necessarily equal to  $B_v$ , but where the Maxwell and Boltzmann distributions as well as the Saha equation are still valid. This is called *local thermodynamic equilibrium* (LTE). LTE holds roughly up to the photosphere. Rising further into the chromosphere and inner corona, both the Boltzmann distribution and the Saha equation are no longer valid. This situation, where only the Maxwell distribution with a common temperature for all species still holds, is called *Non-LTE* (NLTE). Here the individual transition rates between the various energy levels must be considered in detail. In the outer corona the temperatures of the different particle species become unequal and eventually in the interplanetary medium even the Maxwell distribution breaks down.

## Absorption-, emission coefficient, source function

#### a. Absorption coefficient



FIGURE 6. Left: Absorption coefficient. Right: Emission coefficient

Consider in Fig. 3.7, left a ray of light with intensity  $I_v$ , penetrating a box with surface area  $\Delta A$  and thickness  $\Delta s$ . The absorbing or scattering atoms in the box have a number density  $n \ (cm^{-3})$  and cross section  $q_v \ (cm^2)$ . The sum of all cross sections is  $Q_v = \sum q_v$  then:

$$\frac{\Delta I_{\nu}}{I_{\nu}} = -\frac{Q_{\nu}}{\Delta A} = -\frac{\sum q_{\nu}}{\Delta A} = -\frac{q_{\nu} n \Delta A \Delta s}{\Delta A} = -\kappa_{\nu} \Delta s \quad , \tag{41}$$

where

$$\kappa_{\rm V} \equiv q_{\rm V} \ n \quad , \tag{42}$$

is the *absorption coefficient* or *opacity* and has the dimension  $\text{cm}^{-1}$ . The function  $(\kappa_v(T, p))$  is available as a program package. It is often sufficient to consider only a gray (= frequency independent) opacity:

$$\overline{\kappa} = \left(\frac{\int_{0}^{\infty} \frac{1}{\kappa_{v}} \frac{dB_{v}}{dT} dv}{\int_{0}^{\infty} \frac{dB_{v}}{dT} dv}\right)^{-1} \text{Rosseland opacity} \quad .$$
(43)

At photospheric and low chromospheric temperatures the gray opacity  $\overline{\kappa}$  can be approximated by the  $H^-$  contribution:

$$\frac{\overline{\kappa}}{\rho} = 1.376 \cdot 10^{-23} p^{0.738} T^5 \quad \left(\frac{cm^2}{g}\right) \quad . \tag{44}$$

#### b. Emission coefficient and source function

Let the volume  $\Delta V$  (see Fig. 3.8, right) emit photons of energy  $\Delta E$  in direction  $\mathbf{e}_n$  into the solid angle  $\Delta \Omega$  per frequency interval  $\Delta v$  and per time interval  $\Delta t$  then

$$\eta_{\nu}(\mathbf{r}, \mathbf{e}_{n}, t) \equiv \lim \frac{\Delta E}{\Delta V \,\Delta t \,\Delta \Omega \,\Delta \nu} \quad , \tag{45}$$

is the *emission coefficient* with the dimension  $erg \ cm^{-3}s^{-1}sr^{-1}Hz^{-1}$ . Here the limit is taken for  $\Delta V$ ,  $\Delta t$ ,  $\Delta \Omega$ ,  $\Delta v \rightarrow 0$ . The *source function* is defined as

$$S_{\nu}(\mathbf{r}, \mathbf{e}_{n}, t) \equiv \frac{\eta_{\nu}(\mathbf{r}, \mathbf{e}_{n}, t)}{\kappa_{\nu}(\mathbf{r}, t)} \quad , \tag{46}$$

which has the same dimension as the intensity. In TE the amount of energy absorbed,  $\Delta E_A$ , is exactly equal to the amount of energy emitted,  $\Delta E_E$ 

$$\Delta E_A = B_V \kappa_V \Delta s \Delta A \Delta t \Delta \Omega \Delta v \; ; \; \Delta E_E = \eta_V \Delta A \Delta s \Delta t \Delta \Omega \Delta v \; ,$$

which leads to Kirchhoff's law:

$$S_{\nu} = B_{\nu} \quad . \tag{47}$$

In LTE Kirchhoff's law still holds, because  $S_v$ , as we will see in Sec. 4.2, is essentially the population ratio,  $n_u/n_l$ . From our definition of LTE in Sec. 3.2d this ratio, same as in TE, obeys the Boltzmann distribution.

### **Radiative transfer equation**

#### a. Transfer equation

Consider a gas layer in a stellar atmosphere (see Fig. 3.10). Let  $\mathbf{e}_n$  and the geometrical height x point in the outward vertical direction. Consider a light ray in an arbitrary direction  $\mathbf{e}'_n$ . The angle between  $\mathbf{e}_n$  and  $\mathbf{e}'_n$  is  $\vartheta$ . Let s be the geometrical distance along this light ray. From Eqs. (41), (45), (46) we have

$$dI_{\nu} = -I_{\nu} \kappa_{\nu} ds + \eta_{\nu} ds = -\kappa_{\nu} (I_{\nu} - S_{\nu}) ds \quad , \tag{48}$$

and

$$dx = -ds \cos(180 - \vartheta) = ds \cos \vartheta = \mu \, ds \quad . \tag{49}$$

With this we obtain the *radiative transfer equation*:

$$\mu \frac{dI_{\nu}}{dx} = -\kappa_{\nu} (I_{\nu} - S_{\nu}) \quad . \tag{50}$$

We define the optical depth  $\tau_v$ :

$$\tau_{\nu} \equiv -\int_{x}^{\infty} \kappa_{\nu} \, dx \quad \text{or} \quad d\tau_{\nu} = -\kappa_{\nu} \, dx \quad .$$
(51)

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FIGURE 7. Transfer equation

Using Eq. (51), the transfer equation can be written in terms of the optical depth:

$$\mu \frac{dI_{\nu}}{d\tau_{\nu}} = I_{\nu} - S_{\nu} \quad . \tag{52}$$

#### b. Formal solution of the transfer equation

Multiplying Eq. (52) with  $e^{-\tau_v/\mu}$  one finds

$$\frac{d\left(I_{\nu}e^{-\tau_{\nu}/\mu}\right)}{d\tau_{\nu}/\mu} = -S_{\nu}e^{-\tau_{\nu}/\mu} \quad , \tag{53}$$

which can be integrated between the limits *a* and *b*:

$$I_{\nu}e^{-\tau_{\nu}/\mu}\Big|_{a}^{b} = -\int_{a}^{b}S_{\nu} \ e^{-\tau_{\nu}'/\mu} \ d\tau_{\nu}'/\mu \quad .$$
 (54)

Boundary conditions:

i) outer boundary: No incoming radiation from outside the star

$$I_{\nu}(0,\mu) = 0 , \quad \mu < 0 ,$$
 (55)

ii) inner boundary: At the center of the star the outgoing intensity is finite

$$I_{\nu}(\tau_{\nu\infty},\mu) = \text{finite} , \quad \mu \ge 0 \quad . \tag{56}$$

Consider the case  $\mu < 0$ , take  $b = \tau_v$  and a = 0, multiply Eq. (54) with  $e^{\tau_v/\mu}$ :

$$I_{\nu}(\tau_{\nu},\mu) \ e^{-\tau_{\nu}/\mu} \ e^{\tau_{\nu}/\mu} - I_{\nu}(0,\mu) \ e^{\tau_{\nu}/\mu} =$$

$$I_{\nu}(\tau_{\nu},\mu) = -\int_{0}^{\tau_{\nu}} S_{\nu} \ (\tau_{\nu}') \ e^{-(\tau_{\nu}'-\tau_{\nu})/\mu} \ d\tau_{\nu}'/\mu \quad , \qquad (57)$$

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This is valid for *incoming radiation*, where  $\mu < 0$  and  $\vartheta > \pi/2$ .

Now consider the other case  $\mu \ge 0$ , take  $b = \tau_{v\infty}$  very large and  $a = \tau_v$ , multiply Eq. (54) with  $-e^{\tau_v/\mu}$ :

$$-I_{\nu}(\tau_{\nu\infty},\mu)\underbrace{e^{-\tau_{\nu\infty}/\mu}}_{=0} e^{\tau_{\nu}/\mu} + I_{\nu}(\tau_{\nu},\mu) e^{-\tau_{\nu}/\mu} e^{\tau_{\nu}/\mu} =$$

$$I_{\nu}(\tau_{\nu},\mu) = \int_{\tau_{\nu}}^{\infty} S_{\nu}(\tau_{\nu}') e^{-(\tau_{\nu}'-\tau_{\nu})/\mu} d\tau_{\nu}'/\mu \quad , \qquad (58)$$

This is valid for *outgoing radiation*, where  $\mu \ge 0$  and  $\vartheta < \pi/2$ .

#### **RADIATIVE EQUILIBRIUM, EDDINGTON APPROXIMATION**

#### a. Radiative equilibrium

Operate with  $2\pi \int_{-1}^{1} d\mu$  on Eq. (50) and use Eqs. (32), (35):

$$\frac{dF_{\nu}}{dx} = -4\pi \kappa_{\nu} \left(J_{\nu} - S_{\nu}\right) \quad , \tag{59}$$

where

$$2\pi \int_{-1}^{+1} S_{\nu} d\mu = 2\pi S_{\nu} \int_{-1}^{+1} d\mu = 4\pi S_{\nu} \quad , \tag{60}$$

if  $S_v$  is assumed independent of  $\mu$ .

Suppose that in a plane-parallel atmosphere the energy transport is by radiation only, then  $F = \int_0^\infty F_v \, dv = \text{const.}$ , that is, there is a constant energy flux through all layers. This condition is called *radiative equilibrium* and is given by

$$\frac{dF}{dx} = \int_0^\infty \frac{dF_\nu}{dx} d\nu = -4\pi \int_0^\infty \kappa_\nu \ (J_\nu - S_\nu) d\nu = 0 \quad , \tag{61}$$

which for gray opacity  $\kappa_v = \overline{\kappa}$  reduces to J = S as  $\overline{\kappa}$  can be taken out of the integration. **b. Eddington approximation** 

Consider the behavior of the intensity with depth (see Fig. 3.12). If in deeper layers the intensity is approximately isotropic we get for the *Eddington K-integral*:

$$K_{\nu} \equiv \frac{1}{2} \int_{-1}^{+1} I_{\nu} \mu^{2} d\mu \simeq \frac{1}{2} I_{\nu} \int_{-1}^{+1} \mu^{2} d\mu = \frac{1}{3} I_{\nu} \quad .$$
 (62)

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FIGURE 8. Isotropy of the radiation field as function of depth in the star

Similarly for the mean intensity at the same depth we have

$$J_{\nu} = \frac{1}{2} \int_{-1}^{+1} I_{\nu} d\mu \simeq \frac{1}{2} I_{\nu} \int_{-1}^{+1} d\mu = I_{\nu} \quad .$$
 (63)

We thus find the *Eddington approximation*:

$$K_{\nu} \simeq \frac{1}{3} J_{\nu} \quad , \tag{64}$$

which gives relatively good results even in situations where  $I_v$  is fairly anisotropic.

# Gray radiative equilibrium atmosphere

Assume LTE, the Eddington approximation, radiative equilibrium and gray opacity. Integrating all equations over frequency v and using frequency integrated quantities, we have:

$$S = B$$
,  $K = \frac{1}{3}J$ ,  $J = S$ , (65)

from which we find with Eq. (37):

$$J = S = B = \frac{\sigma}{\pi} T^4 \quad . \tag{66}$$

Operate with  $\frac{1}{2} \int_{-1}^{+1} \mu \, d\mu$  on the transfer eq. (52) and use Eq. (64):

$$\frac{dK}{d\overline{\tau}} = \frac{1}{3}\frac{dJ}{d\overline{\tau}} = \frac{1}{2}\int_{-1}^{+1} \mu I \, d\mu - \frac{1}{2}S\int_{-1}^{+1} \mu \, d\mu = \frac{F}{4\pi} = \text{const.}$$
(67)

Multiply with 3 and integrate over  $\overline{\tau}$ :

$$J = \frac{\sigma}{\pi} T^4 = \frac{3F}{4\pi} \overline{\tau} + J(0) \quad . \tag{68}$$

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As at the surface the ingoing intensity is zero:

$$J(0) = \frac{1}{2} \int_{0}^{1} B(0) d\mu = \frac{B(0)}{2} \quad , \tag{69}$$

$$F(0) = 2\pi \int_{0}^{1} B(0) \,\mu d\mu = \pi B(0) \quad , \tag{70}$$

we find

$$J(0) = \frac{F(0)}{2\pi} \quad . \tag{71}$$

Let us define:

$$F = F(0) \equiv \sigma T_{eff}^4 \quad , \tag{72}$$

where  $T_{eff}$  is the *effective temperature*. Then we have from Eqs. (68), (71) and (72) the so called *gray radiative equilibrium*  $T(\tau)$  *relation*:

$$T^{4} = \frac{3}{4} T_{eff}^{4} \left( \overline{\tau} + \frac{2}{3} \right) \quad .$$
 (73)

The relation between the optical and geometrical depths is obtained from Eq. (51):

$$d\overline{\tau} = -\overline{\kappa} \, dx \quad . \tag{74}$$

In addition from Eq. (1.9) for a plane, static (where the flow velocity  $\mathbf{v} = 0$ ) atmosphere one has the *equation of hydrostatic equilibrium*:

$$dp = -\rho \ g \ dx \quad . \tag{75}$$

Using for example the simple opacity law (44), it is seen that Eqs. (73), (74) and (75) can be integrated as function of x and give a *static*, *plane*, *gray*, *radiative equilibrium atmosphere*, if the two quantities  $T_{eff}$  and g, as well as suitable boundary conditions are given. Together with Eq. (1.12),  $p = \rho \Re T/\mu$ , we have four equations for the four unknowns  $T, p, \rho, \overline{\tau}$ .

# Net radiative heating rate, radiative heating function

We now want to derive expressions for the net radiative heating rate,  $\Phi_R$ , and the radiative heating function  $dS/dt|_R$ . With  $F = \int_0^\infty F_V dv$  in Eq. (35) we have

$$\Phi_R = -\frac{dF}{dx} \quad , \tag{76}$$

and

$$\left. \frac{dS}{dt} \right|_R = \frac{\Phi_R}{\rho T} \quad . \tag{77}$$

Integrating Eq. (59) over v we find from Eq. (76) the radiative heating rate

$$\Phi_R = 4\pi \int_0^\infty \kappa_v \left( J_v - S_v \right) dv \quad , \tag{78}$$

and the radiative heating function

$$\left. \frac{dS}{dt} \right|_{R} = \frac{4\pi}{\rho T} \int_{0}^{\infty} \kappa_{\nu} \left( J_{\nu} - S_{\nu} \right) d\nu \quad .$$
<sup>(79)</sup>

### NLTE THERMODYNAMICS AND RADIATION

### Transition rates for lines and continua

In the NLTE situation, as discussed in Section 3.2, one has a well-defined kinetic temperature T which is determined by the *Maxwell velocity distribution*, but the *Boltzmann distribution* and the *Saha equation* (Eqs. 3.11, 3.12) are no longer valid. Chromospheres and coronae are stellar layers where NLTE applies. Conservation of particles requires that for the population  $n_m$  of level m one has

$$\frac{\partial n_m}{\partial t} + \nabla \cdot n_m \mathbf{v} = \sum_{j \neq m} n_j P_{jm} - n_m \sum_{j \neq m} P_{mj} \quad , \tag{80}$$

where the  $P_{jm}$  denote the gain and  $P_{mj}$  the loss processes for electrons in level *m* from/to level *j*. Eq. (80) is called the *time-dependent statistical rate equation*. In static chromospheres and coronae (*v*=0) the population of the atomic energy levels is time-independent, that is, one has *statistical equilibrium*. In statistical equilibrium all processes which populate an energy level are exactly balanced by the processes which depopulate that level and one has the *statistical equilibrium equation*:

$$\sum_{j \neq m} n_j P_{jm} = \sum_{j \neq m} n_j (R_{jm} + C_{jm}) = n_m \sum_{j \neq m} P_{mj} = n_m \sum_{j \neq m} (R_{mj} + C_{mj}) \quad , \qquad (81)$$

where R denotes radiative and C collisional transitions.

#### a. Lines

Between two bound energy levels, a lower level *l* and an upper level *u* (see Fig. 3.5) one has the following *radiative transition rates*  $(cm^{-3}s^{-1})$ 

$$R_{\uparrow} = n_l R_{lu} = n_l B_{lu} \overline{J}_{lu} \quad \text{absorption rate} \quad , \tag{82}$$

$$R_{\downarrow}^{ind} = n_u B_{ul} \overline{J}_{lu} \quad \text{induced emission rate} \quad , \tag{83}$$

$$R_{\downarrow}^{sp} = n_u A_{ul}$$
 spontaneous emission rate , (84)

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where  $A_{ul}$ ,  $B_{lu}$ ,  $B_{ul}$  are the Einstein coefficients (tabulated e.g. by Allen, 1973).  $\overline{J}_{lu}$  is the mean intensity  $J_v$ , averaged over the line. If  $\varphi_v$  is the line profile then

$$\overline{J}_{lu} \equiv \int_{\Delta \mathbf{v}} \varphi_{\mathbf{v}} J_{\mathbf{v}} d\mathbf{v} \quad , \tag{85}$$

where

$$\int_{\Delta v} \varphi_v dv = 1 \quad . \tag{86}$$

Here the frequency integrals extend over the frequency width  $\Delta v$  of the line. In the above equations we have assumed that the emission and absorption profiles of the line are identical, that is, we assume *complete redistribution, CRD*, see Sec. 3.3. The line profile is usually given by the *Voigt profile* (see Appendix D)

$$\varphi_{\nu} = \frac{1}{\sqrt{\pi}\Delta\nu_D}H(a,\nu) \quad , \tag{87}$$

where the *damping parameter a* and the *normalized frequency separation v* are given by

$$a = \frac{\Gamma}{4\pi\Delta\nu_D} \quad , \quad v = \frac{\nu - \nu_0}{\Delta\nu_D} \quad , \tag{88}$$

Here  $v_0$  is the line center frequency,  $\Delta v_D$  the *Doppler width* and  $\Gamma$  the *damping constant*.

In thermal equilibrium (TE), *detailed balancing* is valid, that is, the radiation and collision processes individually balance each other,

$$n_j R_{jm} = n_m R_{mj} ; \quad n_j C_{jm} = n_m C_{mj} \quad , \tag{89}$$

because otherwise a perpeduum mobile of the second kind could be built (e.g. a machine which continuously cools a gas by removing photons). In TE due to detailed balancing for levels *l* and *u* one has with  $J_v = B_v$  and  $\overline{J}_v = B_v$  since the Planck function (3.9) has essentially no frequency variation over the width of the line

$$n_l B_{lu} B_{\nu} = n_u A_{ul} + n_u B_{ul} B_{\nu} \quad . \tag{90}$$

Dividing by  $n_u B_{ul}$ , solving for  $B_v$  and using Eqs. (3.9), (3.11) we obtain

$$B_{\nu} = \frac{A_{ul}}{B_{ul}} \frac{1}{\frac{n_l}{n_u} \frac{B_{lu}}{B_{ul}} - 1} = \frac{A_{ul}}{B_{ul}} \frac{1}{\left(\frac{g_l}{g_u} e^{\frac{h\nu_{lu}}{kT}}\right) \frac{B_{lu}}{B_{ul}} - 1} = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad , \tag{91}$$

and because T is arbitrary

$$A_{ul} = \frac{2h\nu^3}{c^2} B_{ul} \quad , \tag{92}$$

$$g_l B_{lu} = g_u B_{ul} \quad . \tag{93}$$

From Eqs. (82) to (84) one sees that  $A_{ul}$  has the dimension  $(s^{-1})$  while  $B_{ul}$ ,  $B_{lu}$  have the dimensions  $(cm^2 sr Hz \ erg^{-1})$ . Eqs. (92), (93) are relations between probabilities which

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involve only atomic parameters and do not depend on atomic level populations, they thus are *also valid in NLTE*. For convenience we introduce an absorption cross section  $(cm^2)$ 

$$\alpha_{lu}(\mathbf{v}) \equiv \frac{\pi e^2}{m_e c_L} f_l \varphi_{\mathbf{v}} \equiv B_{lu} \frac{h v_{lu}}{4\pi} \varphi_{\mathbf{v}} \quad . \tag{94}$$

Here *e* is the electronic charge,  $m_e$  the electron mass and  $f_l$  the oscillator strength. Let us write the Boltzmann distribution (3.11, B.34) in the form

$$\frac{n_u^*}{n_l^*} = \frac{g_u}{g_l} e^{-\frac{hv_{lu}}{kT}} , \qquad (95)$$

where  $g_i = 2i^2$ . Here and in the following equations, populations marked by a  $\star$ , indicates quantities in TE or LTE. From Eqs. (92), (93) we have

$$A_{ul} = B_{lu} \frac{2hv^3}{c^2} \frac{n_l^*}{n_u^*} e^{-\frac{hv}{kT}} \quad , \quad B_{ul} = B_{lu} \frac{n_l^*}{n_u^*} e^{-\frac{hv}{kT}} \quad . \tag{96}$$

With this the radiative transition rates of Eqs. (82) to (84) can be written

$$R_{\uparrow} = n_l \int_{\Delta \nu} \alpha_{lu}(\nu) \frac{4\pi}{h\nu} J_{\nu} d\nu \equiv n_l R_{lu} \text{ radiative absorption rate}, \qquad (97)$$

$$R_{\downarrow}^{ind} = n_u \frac{n_l^*}{n_u^*} \int_{\Delta \nu} \alpha_{lu}(\nu) \frac{4\pi}{h\nu} J_{\nu} e^{-\frac{h\nu}{kT}} d\nu \text{ induced emission rate }, \qquad (98)$$

$$R_{\downarrow}^{sp} = n_u \frac{n_l^{\star}}{n_u^{\star}} \int_{\Delta v} \alpha_{lu}(v) \frac{4\pi}{hv} \frac{2hv^3}{c^2} e^{-\frac{hv}{kT}} dv \text{ spontaneous emission rate }.$$
(99)

Defining

$$R_{ul}^{\dagger} \equiv \int_{\Delta v} \alpha_{lu}(v) \frac{4\pi}{hv} \left(\frac{2hv^3}{c^2} + J_v\right) e^{-\frac{hv}{kT}} dv \quad , \tag{100}$$

and

$$G_{ul} \equiv \frac{g_l}{g_u} \quad , \tag{101}$$

the total radiative deexitation rate is given by

$$R_{\downarrow} = R_{\downarrow}^{sp} + R_{\downarrow}^{ind} \equiv n_u R_{ul} = n_u \int_{\Delta v} \alpha_{lu}(v) G_{ul} \frac{4\pi}{hv} \left(\frac{2hv^3}{c^2} + J_v\right) dv = n_u \frac{n_l^*}{n_u^*} R_{ul}^{\dagger} \quad . \tag{102}$$

For the *collisional transition rates*  $(cm^{-3}s^{-1})$  one has

$$C_{\uparrow} = n_l C_{lu} = n_l n_e \Omega_{lu}(T)$$
 collisional excitation rate , (103)

$$C_{\downarrow} = n_u C_{ul} = n_u n_e \Omega_{ul}(T)$$
 collisional deexcitation rate . (104)

In TE, because of detailed balancing, one finds

$$C_{ul} = \frac{n_l^{\star}}{n_u^{\star}} C_{lu} = \frac{n_l^{\star}}{n_u^{\star}} n_e \Omega_{lu}(T) = \frac{g_l}{g_u} e^{\frac{h v_{ul}}{kT}} n_e \Omega_{lu}(T) \quad .$$
(105)

This relation between the collision cross sections is *also valid in NLTE* because it involves only atomic parameters and the Maxwell velocity distribution. Note that the notation  $R_{ij}$ ,  $C_{ij}$  etc. indicates a transition from level *i* (first index) to level *j* (second index).

### b. Continua

The *radiative transition rates*  $(cm^{-3}s^{-1})$  between a bound level *l* and the continuum *k* (see Fig. 3.5) are given by

$$R_{\uparrow} = n_l R_{lk} = n_l \int_{\nu_l}^{\infty} \alpha_l \left(\nu\right) \frac{4\pi}{h\nu} J_{\nu} d\nu \quad \text{absorption rate} \quad , \tag{106}$$

$$R_{\downarrow}^{ind} = n_k \, n_e \int_{\nu_l}^{\infty} \beta_l(\nu) \, \frac{4\pi}{h\nu} J_{\nu} d\nu \quad \text{induced emission rate} \quad , \tag{107}$$

$$R_{\downarrow}^{sp} = n_k \, n_e \int_{\nu_l}^{\infty} \gamma_l(\nu) \, d\nu \quad \text{spontaneous emission rate} \quad . \tag{108}$$

In TE one has detailed balancing in each frequency interval and finds the statistical equilibrium equation

$$n_l \alpha_l \frac{4\pi}{h\nu} B_{\nu} = n_k n_e \beta_l \frac{4\pi}{h\nu} B_{\nu} + n_k n_e \gamma_l$$

which by multiplying with  $hv/(4\pi n_k n_e)$  can be solved for  $B_v$ 

$$B_{\nu} = \frac{\frac{h\nu}{4\pi}\gamma_l}{\frac{n_l}{n_k n_e}\alpha_l - \beta_l} = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad .$$
(109)

Here the cross-sections  $\gamma_l$ ,  $\beta_l$  are bounded and the temperature is arbitrary. Since for  $T \to 0$ , we obtain with the Saha equation  $n_l/(n_k n_e) \to \infty$ , we must identify

$$\gamma_l = \frac{n_l}{n_k n_e} \alpha_l \frac{4\pi}{h\nu} \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}} \quad , \tag{110}$$

$$\beta_l = \frac{n_l}{n_k n_e} \alpha_l e^{-\frac{hv}{kT}} \quad . \tag{111}$$

The emission rates are therefore given by

$$R_{\downarrow}^{ind} = n_k \frac{n_l^{\star}}{n_k^{\star}} \int_{\nu_l}^{\infty} \alpha_l(\nu) \frac{4\pi}{h\nu} J_{\nu} e^{-\frac{h\nu}{kT}} d\nu \quad , \qquad (112)$$

$$R_{\downarrow}^{sp} = n_k \frac{n_l^{\star}}{n_k^{\star}} \int_{\nu_l}^{\infty} \alpha_l(\nu) \frac{4\pi}{h\nu} \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}} d\nu \quad , \qquad (113)$$

where in TE and LTE the Saha equation (3.12) is written

$$\frac{n_l^{\star}}{n_k^{\star}} = n_e \left(\frac{h^2}{2\pi m_e kT}\right)^{3/2} \frac{g_l}{2u_k} e^{\frac{hv_l}{kT}} \quad , \tag{114}$$

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where  $E_l = hv_l$  is the energy difference between level *l* and the continuum. It can be shown that Eqs. (112) and (113) are also valid in NLTE.  $u_k$  in Eq. (114) is the partition function (B.36)

$$u_k = \sum_{i=1}^{i_{max}} g_i e^{-\frac{h(v_i - v_1)}{kT}} \quad , \tag{115}$$

where the summation is carried out over the  $i_{max}$  bound levels of the next higher ionization stage. Defining

$$R_{kl}^{\dagger} \equiv \int_{\nu_l}^{\infty} \alpha_l(\nu) \frac{4\pi}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_{\nu}\right) e^{-\frac{h\nu}{kT}} d\nu \quad , \tag{116}$$

and

$$G_{kl} \equiv \frac{n_l^*}{n_k^*} e^{-\frac{h\nu}{kT}} \quad , \tag{117}$$

the total radiative recombination rate is given by

$$R_{\downarrow} = R_{\downarrow}^{sp} + R_{\downarrow}^{ind} \equiv n_k R_{kl} = n_k \int_{\nu_l}^{\infty} \alpha_l(\nu) G_{kl} \frac{4\pi}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_{\nu}\right) d\nu = n_k \frac{n_l^{\star}}{n_k^{\star}} R_{kl}^{\dagger}.$$
 (118)

For the *collisional transition rates*  $(cm^{-3}s^{-1})$  one finds

$$C_{\uparrow} = n_l C_{lk} = n_l n_e \Omega_{lk}(T)$$
 collisional ionisation rate , (119)

$$C_{\downarrow} = n_k C_{kl} = n_k n_e^2 \overline{\Omega}_{kl}(T) \equiv n_k n_e \Omega_{kl}(T, ne) \quad \text{collisional recombination rate} \quad . \quad (120)$$

In TE, because of detailed balancing, one has

$$C_{\downarrow} = n_k C_{kl} = n_k \frac{n_l^{\star}}{n_k^{\star}} C_{lk} = n_k \frac{n_l^{\star}}{n_k^{\star}} n_e \Omega_{lk}(T) \quad .$$
(121)

with  $n_l^{\star}/n_k^{\star}$  given by Eq. (114). This relation is also valid in NLTE. Partition functions, absorption and collision cross sections can be found in Allen (1973).

# Line and continuum source functions

When considering transfer of radiation through a stellar gas the source function after (3.20) is defined as the ratio of the emission and absorption coefficients.

#### a. Lines

From (3.19) and (84), noting that the dimension of the transition rates is  $(cm^{-3}s^{-1})$  one finds for the *line emission coefficient* (erg  $cm^{-3}s^{-1}sr^{-1}Hz^{-1}$ )

$$\eta_{\nu}^{line} = \frac{\partial R_{\downarrow}^{sp}}{\partial \nu} \frac{h\nu}{4\pi} = n_u A_{ul} \frac{h\nu}{4\pi} \varphi_{\nu} = \frac{2h\nu^3}{c^2} \alpha_{lu}(\nu) n_u G_{ul}.$$
 (122)

Similarly from (3.14), (82), (83), noting that

$$\frac{\Delta I_{\nu}}{\Delta s} = -I_{\nu}\kappa_{\nu} = -\frac{\partial\left(R_{\uparrow} - R_{\downarrow}^{ind}\right)}{\partial\nu}\frac{h\nu}{4\pi} \quad , \tag{123}$$

one has for the *line opacity*  $(cm^{-1})$ 

$$\kappa_{v}^{line} = (n_{l}B_{lu} - n_{u}B_{ul})\frac{hv}{4\pi}\varphi_{v} = n_{l}B_{lu}\left(1 - \frac{n_{u}g_{l}}{n_{l}g_{u}}\right)\frac{hv}{4\pi}\varphi_{v} = \alpha_{lu}(v)(n_{l} - n_{u}G_{ul}).$$
(124)

The absorption coefficient is defined for a total intensity change (which results from the absorption minus the induced emission). The *line source function* (see Eq. (3.20)) is given by

$$S_{lu}^{line} \equiv \frac{\eta_{v}^{line}}{\kappa_{v}^{line}} = \frac{n_{u}A_{ul}}{n_{l}B_{lu} - n_{u}B_{ul}} = \frac{2hv^{3}}{c^{2}}\frac{1}{\frac{n_{l}g_{u}}{n_{u}g_{l}} - 1} = \frac{2hv^{3}}{c^{2}}\frac{1}{\frac{b_{l}}{b_{u}}e^{\frac{hv}{kT}} - 1} \quad ,$$
(125)

where we have used the Boltzmann distribution. The *departure coefficients*  $b_i$  are defined below in Eq. (130). It is seen that in LTE one has

$$S_{lu}^{line} = B_{v_{lu}}, \text{ with } b_l = b_u = 1$$
 . (126)

In deriving Eqs. (122), (124) we have assumed complete redistribution. Eqs. (126) and (131) below show the validity of Kirchhoff's law (3.21) in LTE as stated in Section 3.4. Note that taking the source function equal to the Planck function is used as primary definition of LTE in some texts. For us here, Eqs. (126), (131) are a consequence of how we defined LTE in Section 3.2.

### b. Continua

From Eqs. (3.19) and (113) we have similarly a continuum emission coefficient (erg  $cm^{-3} s^{-1}sr^{-1}Hz^{-1}$ )

$$\eta_{\nu}^{cont} = \frac{\partial R_{\downarrow}^{sp}}{\partial \nu} \frac{h\nu}{4\pi} = n_k \frac{n_l^{\star}}{n_k^{\star}} \alpha_l(\nu) \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}} = \frac{2h\nu^3}{c^2} \alpha_l(\nu) n_k G_{kl} \quad .$$
(127)

Also, using Eqs. (106), (112), (123) we obtain a *continuum opacity*  $(cm^{-1})$ 

$$\kappa_{v}^{cont} = \alpha_{l}\left(v\right)\left(n_{l} - n_{k}\frac{n_{l}^{\star}}{n_{k}^{\star}}e^{-\frac{hv}{kT}}\right) = \alpha_{l}\left(v\right)\left(n_{l} - n_{k}G_{kl}\right) \quad .$$
(128)

The continuum source function is then given by

$$S_{\nu}^{cont} = \frac{\eta_{\nu}^{cont}}{\kappa_{\nu}^{cont}} = \frac{2h\nu^3}{c^2} \frac{1}{\frac{n_l}{n_k^1} \frac{n_k^*}{n_k} e^{\frac{h\nu}{kT}} - 1} = \frac{2h\nu^3}{c^2} \frac{1}{b_l e^{\frac{h\nu}{kT}} - 1} \quad , \tag{129}$$

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where the *departure from LTE coefficient*  $b_l$  is defined by

$$b_l \equiv \frac{n_l}{n_l^*} \frac{n_k^*}{n_k} \quad . \tag{130}$$

Note that always  $b_k = 1$  and that in LTE one has

$$S_{\nu}^{cont} = B_{\nu}, \quad \text{with} \quad b_l = 1 \quad . \tag{131}$$

# **Computation of the level populations**

We want to compute the number densities and level populations of the different atoms and ions of a stellar gas mixture with a composition given by Tab. 1.1. Let  $n_{r,s,i}$  be the number density  $(cm^{-3})$  of particles in the energy level  $r = 1, \dots, r_{max}$  and the ionisation stage  $s = 1, \dots, s_{max}$  of element *i* (e.g. i = 1 is H, i = 2 is He etc. see Tab. 1.1). Then

$$n_{s,i} = \sum_{r=1}^{r_{max}} n_{r,s,i} \quad , \tag{132}$$

is the total number density of particles of element *i* in ionisation stage *s*, where  $r_{max}$  is the number of bound levels, while

$$n_i = \sum_{s=1}^{s_{max}} n_{s,i} \quad , \tag{133}$$

is the total number density of particles of element *i*, where  $s_{max}$  is the number of ionisation stages. To show the method we first discuss the computation of  $n_{r.s.i}$  in LTE.

#### a. LTE populations

Assume that the temperature *T*, pressure *p* and abundances are known, what are the particle densities  $n_{r,s,i}$  and the electron density  $n_e$  in an LTE situation. Summing Eq. (3.11) over the bound levels *r* we obtain the Boltzmann distribution

$$\frac{n_{r,s,i}}{n_{s,i}} = \frac{g_{r,s,i}}{u_{s,i}} e^{-\frac{E_{r,s,i}}{kT}} , \qquad (134)$$

where the partition function is given by

$$u_{s,i} = \sum_{r=1}^{r_{max}} g_{r,s,i} \ e^{-\frac{(E_{r,s,i} - E_{1,s,i})}{kT}} \quad .$$
(135)

Similarly summing Eq. (3.12) over all bound levels l we find the Saha equation

$$\frac{n_{s+1,i}}{n_{s,i}} = \frac{1}{n_e} \frac{u_{s+1,i}}{u_{s,i}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\frac{E_{s-1,i}-E_{s,i}}{kT}} \quad .$$
(136)

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Now proceed with the following steps

1. Make an estimate of  $n_e$ .

2. Compute the element densities

$$n_i = \left(\frac{p}{kT} - n_e\right) \frac{A_i}{\sum_j A_j} \quad , \tag{137}$$

where the quantity in brackets is the total heavy particle number density.

3. Compute the  $n_{s,i}$ . For this we have  $s_{max} - 1$  Eqs. (136) and Eq. (133). Similarly compute  $n_{r,s,i}$  using the  $r_{max}$  Eqs. (134).

4. Evaluate the new electron density  $n_e$  using the equation of charge conservation

$$n_e = \sum_{i} \sum_{s=2}^{s_{max}} s \, n_{s,i} \quad . \tag{138}$$

In principle by going back to step 1, we could improve  $n_e$  until a converged solution is obtained. One actually uses the much faster Newton-Raphson method by writing Eq. (138) as  $f(n_e) = 0$ . If the true solution is written  $n_e = n_e^0 + \delta n_e$ , where  $n_e^0$  is an estimate, we have  $f(n_e^0) + \frac{df}{dn_e} \delta n_e = 0$ . The derivative can be done analytically. Solving for  $\delta n_e = -f(n_e^0)/\frac{df}{dn_e}$  we get a new estimate  $n_e^0 + \delta n_e$  etc. This Newton-Raphson iteration converges very fast, if the initial estimate  $n_e^0$  is reasonable.

### **b. NLTE populations**

In NLTE the situation is different. A special case occurs when the temperature T, pressure p, abundances and radiation fields  $J_v$  are given. We start again with an estimate of the electron density  $n_e$ . With the starred ratios given by Eqs. (95), (114) we evaluate the radiative and collisional rates and then compute the departure coefficients  $b_{r,s,i}$  using Eqs. (??). We now follow the procedure of the LTE case, using a modified partition function

$$\overline{u}_{s,i} = \sum_{r=1}^{r_{max}} b_{r,s,i} g_{r,s,i} e^{-\frac{(E_{r,s,i}-E_{1,s,i})}{kT}} , \qquad (139)$$

as well as modified Boltzmann distributions and modified Saha equations

$$\frac{n_{r,s,i}}{n_{s,i}} = \frac{b_{r,s,i}g_{r,s,i}}{\overline{u}_{s,i}} e^{-\frac{E_{r,s,i}}{kT}} , \qquad (140)$$

$$\frac{n_{s+1,i}}{n_{s,i}} = \frac{1}{n_e} \frac{\overline{u}_{s+1,i}}{\overline{u}_{s,i}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\frac{E_{s-1,i}-E_{s,i}}{kT}} \quad .$$
(141)

and determine  $n_e$  with a Newton-Raphson iteration. However, the difficulty in the general NLTE situation is, that usually the radiation fields  $J_v$  are not given. How they are consistently evaluated is discussed in the following Subsection.

# Solution of the multilevel transfer and statistical equations

Lit. Avrett, E.H., Loeser, R.: 1992, in: *Cool Stars, Stellar Systems and the Sun*, ASP Conf. Ser. 26, M.S. Giampapa & J.A. Bookbinder Eds., p. 489.

Carlsson, M.: 1986, Uppsala Astron. Obs. Report 33.

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Scharmer, G.B., Carlsson, M.: 1985, J. Comput. Phys. 59, 56.

In a typical NLTE situation where one seeks the level populations  $n_i$  in an atmosphere with a prescribed temperature T and pressure p distribution, the radiation fields  $J_v$  are not given. Here one has to consistently solve the combined multilevel statistical and transfer equations in a multielement mixture. One usually starts with the computation of H to determine the electron density  $n_e$ , which is iterated as discussed above to be consistent with the prescribed pressure distribution. Subsequently the most important chromospheric lines of Mg, Ca, Fe are individually computed, assuming  $n_e$  is given. There are two ways to solve the combined multilevel statistical and transfer equations, the equivalent two-level atom approach and the complete linearization method.

#### The complete linearization method

The complete linearization method exemplified in the computer code MULTI (Scharmer & Carlsson 1985, Carlsson 1986, 1992) is widely used in the astrophysical community as it is an easily available and freely distributed computer code. Here one also starts with a given temperature T and electron density  $n_e$  distribution and an estimate of the level populations  $n_i$ . One then computes the radiation fields  $J_v$  and evaluates the statistical rate equations. In a Newton-Raphson scheme the errors in the statistical rate equations produced by the imperfect estimate are handled as an error vector which is equated to a grand matrix, which can be evaluated analytically, times a vector holding the variations  $\delta n_i$ . Solving for  $\delta n_i$  the updated populations  $n_i + \delta n_i$  are then used for the next iteration.

Assume that for the  $n^{th}$  step of the iteration scheme we have the populations  $n_i^{(n)}$  and the transition processes  $P_{ij}^{(n)}$  and that in the next iteration we seek to arrive at the final solution. Then  $n_i^{(n+1)}$  and  $P_{ij}^{(n+1)}$  satisfy Eq. (81) and we have

$$n_i^{(n+1)} = n_i^{(n)} + \delta n_i; \quad P_{ij}^{(n+1)} = P_{ij}^{(n)} + \delta P_{ij} , \qquad (142)$$

where  $\delta n_i$ ,  $\delta P_{ij}$  are hopefully small. From Eq. (81) written for level *i*, with the total number of levels  $N_L = N + 1$ 

$$\left(n_{i}^{(n)} + \delta n_{i}^{(n)}\right) \sum_{j \neq i}^{N_{L}} \left(P_{ij}^{(n)} + \delta P_{ij}^{(n)}\right) - \sum_{j \neq i}^{N_{L}} \left(n_{j}^{(n)} + \delta n_{j}^{(n)}\right) \left(P_{ji}^{(n)} + \delta P_{ji}^{(n)}\right) = 0 \quad .$$
(143)

Expanding and taking only the first order terms we get

$$\delta n_i^{(n)} \sum_{j \neq i}^{N_L} P_{ij}^{(n)} + n_i^{(n)} \sum_{j \neq i}^{N_L} \delta P_{ij}^{(n)} - \sum_{j \neq i}^{N_L} \delta n_j^{(n)} P_{ji}^{(n)} - \sum_{j \neq i}^{N_L} n_j^{(n)} \delta P_{ji}^{(n)} = E_i^{(n)} \quad .$$
(144)

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Our aim is to write  $\delta P_{ij}$ ,  $\delta P_{ji}$  in terms of  $\delta n_i$  and then solve Eq. (144) for  $\delta n_i$ . Clearly  $\delta P_{ij} = \delta R_{ij} + \delta C_{ij}$ , however  $\delta C_{ij} = 0$  as we assume that  $n_e$  and T are given. By the same reason  $\delta R_{ij}^{sp} = 0$  and we have

$$\delta P_{ij} = \delta R_{ij} = \frac{1}{2} \int_{-1\Delta\nu}^{+1} \int \frac{4\pi}{h\nu} \alpha_{ij} G_{ij} \delta I_{\nu\mu} d\nu d\mu , \quad i > j$$
$$= \frac{1}{2} \int_{-1\Delta\nu}^{+1} \int \frac{4\pi}{h\nu} \alpha_{ij} \delta I_{\nu\mu} d\nu d\mu , \quad i < j \quad .$$
(145)

Here  $\alpha_{ij}$  is given by Eq. (92) or by  $\alpha_{ik} = \alpha_i(v)$ , depending whether *j* is a bound or a continuum level. The  $G_{ij}$  are given bei Eqs. (101), (117) and the *v*-integration interval  $\Delta v$  is either over the line width or from  $v_i$  to infinity, depending whether a line or a continuum transition is considered.

The error term  $E_i$  in Eq. (144) is given by

$$E_{i} = -\sum_{j=1}^{i-1} \left( n_{i} P_{ij} - n_{j} P_{ji} \right) - \sum_{j=i+1}^{N_{L}} \left( n_{i} P_{ij} - n_{j} P_{ji} \right) \quad .$$
(146)

For the more detailed form, in which the background continuum is considered and preconditioning is applied to cancel large terms, see Eq. (F.9) in the Appendix F. This takes care of the right hand side of Eq. (144).

To complete the left hand side of this equation the variations of the intensity in Eqs. (145), (146) have to be obtained by a linearization of the transfer equation. This equation relates sources of photons at deeper and higher layers of the star to the intensity variation at the considered depth layer. From Eq. (3.24) the transfer equation is written

$$\mu \frac{dI_{\nu\mu}}{dx} = -\kappa_{\nu\mu}I_{\nu\mu} + \eta_{\nu\mu} \quad . \tag{147}$$

The variation of this equation gives

$$\mu \frac{d\delta I_{\nu\mu}}{dx} = -\kappa_{\nu\mu} \delta I_{\nu\mu} - \delta \kappa_{\nu\mu} I_{\nu\mu} + \delta \eta_{\nu\mu} \quad . \tag{148}$$

Introducing the optical depth  $\tau_{\nu\mu}$  and the source function variation  $\delta S_{\nu\mu}$ 

$$d\tau_{\nu\mu} \equiv -\kappa_{\nu\mu} \frac{dx}{\mu} \quad , \tag{149}$$

$$\delta S_{\nu\mu} \equiv \frac{\delta \kappa_{\nu\mu}}{\kappa_{\nu\mu}} I_{\nu\mu} - \frac{\delta \eta_{\nu\mu}}{\kappa_{\nu\mu}} \quad , \tag{150}$$

we have

$$\frac{d\delta I_{\nu\mu}}{d\tau_{\nu\mu}} = \delta I_{\nu\mu} - \delta S_{\nu\mu} \quad , \tag{151}$$

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which can be solved for  $\delta I_{\nu\mu}$  using Eqs. (3.31), (3.32). Varying the opacity and emission coefficient in Eqs. (124), (122), noting that the background contributions are not varied

$$\delta \kappa_{\nu\mu} = \alpha_{ij} \left( \delta n_i - G_{ij} \delta n_j \right) \quad , \quad \delta \eta_{\nu\mu} = \frac{2h\nu^3}{c^2} G_{ij} \alpha_{ij} \delta n_j \quad , \tag{152}$$

we find

$$\delta S_{\nu\mu} = -\frac{\alpha_{ij}}{\kappa_{\nu\mu}} I_{\nu\mu} \delta n_i + \frac{\alpha_{ij} G_{ji}}{\kappa_{\nu\mu}} \left(\frac{2h\nu^3}{c^2} + I_{\nu\mu}\right) \delta n_j \quad . \tag{153}$$

With Eqs. (145), (146), (151), (153) we have now succeeded to write the left hand side of Eq. (144) in terms of  $\delta n$ 's. At this point, however, the reader should note that up to now it had been implicitly assumed that Eqs. (144), (145), (146) are taken at a certain depth  $x_d$  in the atmosphere. The transfer Eq. (151), however, introduces contributions to  $\delta I_{\nu\mu}$  from sources at other depths  $x'_d$  depending on the frequency. Thus the variations  $\delta n_i$ ,  $\delta n_j$  in Eq. (153) are usually at different depths than those in Eq. (144).

To evaluate the outgoing  $\delta I_{\nu\mu}^+$  and ingoing  $\delta I_{\nu\mu}^-$  intensities, the fast, simple and powerful *Barbier-Eddington approximation* is used (Scharmer & Carlsson 1985). This approximation (already used in (Eq. 3.36)) assumes that the source function can be written as a linear function of optical depth  $S = a + b\tau$  which upon integration has the solution

$$I^{+}(\tau) = S(\tau+1) \quad , \quad I^{-}(\tau) = \int_{0}^{\tau} S(t) e^{-(\tau-t)} dt = \left(1 - e^{-\tau}\right) S\left(\frac{\tau}{1 - e^{-\tau}} - 1\right) \quad (154)$$

One thus has

$$\delta I_{\nu\mu}^{+} \approx S_{\nu\mu} \left( \tau_{\nu\mu}^{+} \right) \quad , \quad \delta I_{\nu\mu}^{-} \approx \left( 1 - e^{-\tau_{\nu\mu}} \right) S_{\nu\mu} \left( \tau_{\nu\mu}^{-} \right) \equiv w^{-} S_{\nu\mu} \left( \tau_{\nu\mu}^{-} \right) \quad , \qquad (155)$$

with

$$\tau_{\nu\mu}^{+} = \tau_{\nu\mu} + 1 \quad , \quad \tau_{\nu\mu}^{-} = \frac{\tau_{\nu\mu}}{1 - e^{-\tau_{\nu\mu}}} - 1 \quad .$$
 (156)

>From the height scale *x* the optical depths  $\tau_{\nu\mu}$  at a selected frequency  $\nu$  and angle  $\mu$  is found with Eq. (149). The source functions  $\delta S_{\nu\mu}$  are then taken at depths  $x_{d+}$ ,  $x_{d-}$  corresponding to  $\tau^+_{\nu\mu}$ ,  $\tau^-_{\nu\mu}$ . For the detailed form of the left hand side of Eq. (144), which includes the background continuum and preconditioning, see Eq. (F.11) in Appendix F.



FIGURE 9. Matrix equation in the complete linearization method

For special cases, where not all transitions need to be treated in detail, it often suffices to describe the radiation field with a *radiation temperature*  $T_{Rad}$ 

$$I_{\nu} = B_{\nu} \left( T_{Rad} \right) \quad . \tag{157}$$

For example in a computation with a four level *H*-atom one may treat the  $Ly\alpha$ ,  $Ly\beta$ ,  $H\alpha$  lines and the Lyman continuum in detail, while the Balmer and Paschen continua are given as fixed rates, specified by radiation temperatures. With *fixed rates* one has  $\delta R = 0$  and the left hand side of Eq. (144) simplifies to

$$\delta n_i \left( R_{ji} + C_{ij} \right) - \delta n_j \left( R_{ji} + C_{ji} \right) \quad . \tag{158}$$

With vectors  $\delta \mathbf{n} = \delta n_{id}$ ,  $\mathbf{E} = E_{id}$ , where  $i = 1, \dots, N_L$  is the energy level index and  $d = 1, \dots, N_D$  the depth index, Eq. (144) can be written with a grand matrix  $\overline{w}$ .

$$\overline{w} \, \delta \mathbf{n} = \mathbf{E} \quad . \tag{159}$$

Fig. 4.1 shows this matrix equation in detail.

As an example Fig. 4.2 shows the non-vanishing elements of the grand matrix  $\overline{w}$  for a 5 level + continuum Ca II calculation. The non-zero matrix elements are clustered along the main diagonal showing that the most important radiative interactions are of intermediate range. The off-diagonal matrix elements represent the non-local contributions. This band structure permits an efficient matrix inversion. The little submatrices have the size  $N_L \times N_L$  and there are  $N_D$  submatrices along the main diagonal of matrix  $\overline{w}$ . The first row of the submatrices on the main diagonal is replaced by the equation of conservation of particles. This produces gaps in the off-diagonal submatrices which gives Fig. 4.2 its boxy appearance. The submatrices on the main diagonal moreover contain all the collisional information.



**FIGURE 10.** Grand matrix  $\overline{w}$  for a 5 level + continuum Ca II calculation of the solar VAL81-C model by Scharmer & Carlsson (1985).

### **Chromospheric radiation loss**

From the procedure discussed above the net heating rates  $(erg \ cm^{-3}s^{-1})$  can be determined by  $\Phi_{lu}$ ,  $\Phi_l = hv (R_{\uparrow} - R_{\downarrow})$ :

$$\Phi_{lu} = n_l 4\pi \int_{\Delta \nu} \alpha_{lu}(\nu) J_{\nu} d\nu - n_u \frac{n_l^*}{n_u^*} 4\pi \int_{\Delta \nu} \alpha_{lu}(\nu) \left(\frac{2h\nu^3}{c^2} + J_{\nu}\right) e^{-\frac{h\nu}{kT}} d\nu \quad , \quad (160)$$

$$\Phi_{l} = n_{l} 4\pi \int_{\nu_{l}}^{\infty} \alpha_{l}(\nu) J_{\nu} d\nu - n_{k} \frac{n_{l}^{\star}}{n_{k}^{\star}} 4\pi \int_{\nu_{l}}^{\infty} \alpha_{l}(\nu) \left(\frac{2h\nu^{3}}{c^{2}} + J_{\nu}\right) e^{-\frac{h\nu}{kT}} d\nu \quad , \qquad (161)$$

for lines and continua.

#### **Coronal radiation loss**

Lit. Jain, Narain 1978: Astron. Astrophys. Suppl. **31**, 1. McWhirter RWP, Thonemann P.C., Wilson R.: 1975 A&A 40, 63; 1977 A&A 61, 859 Gehrels N., Williams E.D.: 1993 ApJ 418, L25

In the thin coronal layers the temperature is high and the density is very low. Consider the energy levels of a typical multiply ionised coronal ion (see Fig. 3.5) with its large ionisation  $E_l$  and excitation  $E_{lu}$  energies. Because the photospheric radiation field  $J_V$ does not have photons with enough energy to excite a coronal ion, one has  $R_{\uparrow} \approx 0$ ,  $R_{\downarrow}^{ind} \approx 0$ . In addition, as  $n_e$  is very small in the corona one has  $C_{\downarrow} \approx 0$ . What remains from Eq. (81) is

$$C_{\uparrow} = R_{\downarrow}^{Sp} \quad , \tag{162}$$

which is called the *thin plasma approximation*. From Eqs. (113), (119):

$$n_1 n_e \Omega_1(T) = n_k \frac{n_1^*}{n_k^*} R_{k1}^\dagger = n_k \frac{n_1^*}{n_k^*} \int_{\nu_1}^{\infty} \alpha_1(\nu) \frac{4\pi}{h\nu} \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}} d\nu \equiv n_k n_e f(T) \quad , \qquad (163)$$

where we used the Saha eq. (114),

$$\frac{n_k^{\star}}{n_1^{\star}} = \frac{1}{n_e} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2u_k}{g_1} e^{-\frac{hv_1}{kT}} \quad . \tag{164}$$

From this we obtain

$$\frac{n_k}{n_1} = \frac{\Omega_1(T)}{f(T)} = g(T) \quad . \tag{165}$$

The function g(T) can be tabulated (e.g. Jain and Narain, 1978). Note that as lower bound level we took l = 1 as only the ground level is significantly populated. Eq. (165) states that very differently from the Saha equation (164) in LTE, which depends on  $n_e$ , the coronal ionisation ratio due to the thin plasma approximation depends only



FIGURE 11. Ionisation ratios for Fe in the thin plasma approximation valid for the corona.

on the temperature. As an example, the ionisation ratios  $n_{s,i}/\sum_s n_{s,i}$  in the coronal approximation are shown for Fe in Fig. 4.3. The coronal radiation loss due to lines is

$$-\Phi_{R} = 4\pi\eta - \underbrace{4\pi\kappa J}_{=0} = \sum h\nu \underbrace{n_{u}R_{u1}^{sp}}_{n_{1}C_{1u}} = \sum h\nu n_{Ion}n_{e}\Omega(T) \quad , \quad (166)$$

where the sum is taken over all lines. The total cooling rate includes also other processes and is given by

$$-\Phi_R = n_H n_e P_{Rad} \left( T \right) \quad , \tag{167}$$

where  $P_{Rad}(T)$  is a function shown in Fig. 4.4. Similar functions often used in the literature were computed by Cox and Tucker (1969). For the decending part of the function  $P_{Rad}$ , simple approximations

$$P_{Rad} \approx 5 \cdot 10^{-20} T^{-1/2} \quad , \tag{168}$$

or

$$P_{Rad} \approx 6 \cdot 10^{-17} T^{-1} \quad , \tag{169}$$

are possible. *Note of caution:* When using (167) one has only a *cooling* function. This will cool the gas down to temperatures  $T \rightarrow 0$ . In reality, if T becomes small enough, *radiative heating* must be considered similarly as in  $\Phi_R = 4\pi\kappa (J-B)$ .



**FIGURE 12.** Radiation function  $P_{Rad}$  as function of temperature and various numerical fits of this function after Mc Whirter et al. (1975).