Can the exoplanetary orbits be explained with planet-planet scattering?

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Abstract

The orbits of the exoplanets discovered so far have eccentricities much higher than those in our own Solar system. Currently, there exist no universally accepted explanation to this. One theory, however, is that the fly-by of a neighbouring star might disturb the planetary orbits enough for the orbits to cross. The planet-planet interaction at one of the crossing points might then lead to the ejection of one of the planets, leaving the other in a tighter, possibly more eccentric orbit. In this project, we make an analytical approach to the problem of planet-planet scattering in order to better understand what happens. A program based on this theory was then written in order to simulate this event for any initial properties of a two-planet system. The results obtained are for certain initial properties able to fit quite well with the eccentricity distribution of the known exoplanets.

1 Introduction

Astronomers have now successfully been looking for exoplanets for thirteen years, resulting in roughly 300 known discoveries, with new ones found almost every week. The orbital properties of the first exoplanet, 51 Peg b, came as a big surprise (Mayor and Queloz [1995]). The planet was Jupiter-sized, but instead of having an orbital period of twelve years like Jupiter, this planet orbited its central star in just over four days. As new exoplanet discoveries was made during the following years, it was found that the close orbit was not an exception only for 51 Peg b. In the same way, a large part of the planets found had oddly high eccentricities, much higher than any of the planets in the Solar system. Both these things were before the discoveries not thought to be possible. When planets form, they do so in disks, and when the planets accumulate the gas and dust is left in the disk, this should mean a correction towards circularity. Why we are observing something so significantly different from circular orbits is still being debated, with no universally accepted theory to explain it.

Another thing astronomers had been certain of was the clear division between terrestrial planets and Jovian planets. Terrestrial planets are the smaller, rocky planets like our own Earth, that should only be possible to form inside the "snow line"', where gas can not condense. The Jovian planets are the gaseous giants, like Jupiter, which are thought to only form outside the snow line (Hayashi [1981], Sasselov and Lecar [2000]). Instead of observing this as a rule among the exoplanets, several planets where found that soon got the name "hot Jupiters", with masses like Jupiter, but with very tight orbits well inside Mercury's. It is worth noting that all methods of finding exoplanets are currently relying on the orbital period of the planet. As of today, it is not possible to find planets with periods above Jupiters, roughly, simply because it takes time for all the necessary data to be collected. It is not impossible at all that we during the following years will find a lot of planet systems just like our own, but the fact that there exist hot Jupiters or highly eccentric orbits at all is still surprising. In Fig (1), the orbital properties of the currently known exoplanets have been plotted, to be compared with the properties of Jupiter and Saturn that are also included.

Today still, thirteen years after the first discovery, the problem with the odd properties of the exoplanets have yet to be answered. Some theories exist to explain these oddities, however. One theory is about planetary migration. The idea is that the heavier Jupiter-massed planets might accrete gas from the disk before it dissipates, and the planet then loses angular momentum and slowly spiral in towards a very tight orbit. This theory can explain the existence of the hot Jupiters, but not the highly eccentric orbits, since the correction from the gas would also mean a certain circularization (Lin et al. [1996], Murray et al. [1998]).

To explain the high eccentricities, it has been suggested that a fly-by of a neighbouring star might disturb one or several of the planetary orbits enough so that they cross at some points. In this scenario, we only look at the interaction between the planets at one of these crossings. The interactions will then lead to changed orbital properties, and because orbits are closed, the two interacting orbits will continue to do so until one of the planets is ejected. The planet left behind would then lose energy and be left in a tighter, likely also more eccentric orbit than before the fly-by (Laughlin and Adams [1998],

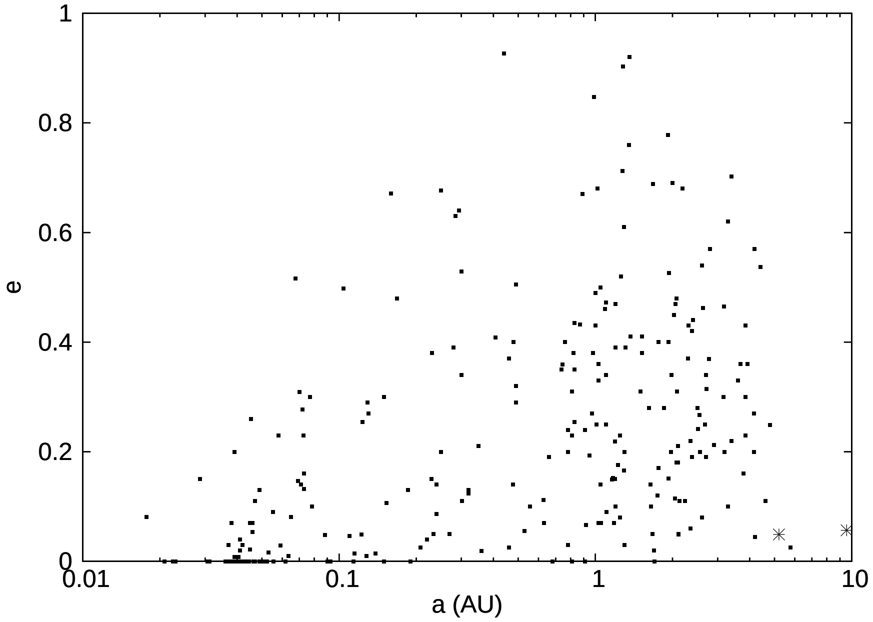

Figure 1: a versus e for all the known exoplanets. Also included for comparison are the properties of Jupiter and Saturn. (Data taken from exoplanet.eu as of 9th of June, 2008.)

Hurley and Shara [2002], Malmberg et al. [2007]).

During this project, I have made an analytical approach to a part of the fly-by scenario. We imagine that a fly-by of a star has disturbed a coplanar two planet system enough for their two orbits to cross. I then investigate what would happen to the system after repeated interactions between the two planets, disregarding any other effects than the actual planet-planet interaction at the crossing. What would the resulting orbital properties be of the planet left behind if the other planet is ejected? Is this work enough to explain the eccentricity distribution of the exoplanets found so far?

The rest of this report is divided into five different sections. In Section 2, we look at the properties of a general eccentric orbit and find an expression for the distance between the planet and the central star as well as the velocity of the planet at any point in the orbit. With these parameters known, we can find the exact crossings between two orbits, which we in Section 3 use to determine what happens during the planet-planet deflection and how it affects the orbital properties. In Section 4, we take a closer look at the programs made during the project and take the more specific problems into account. This is followed by Section 5, where the resulting plots are presented and analyzed, and Section 6, where we discuss how this analytical approach can be developed further.

2 Orbital properties

Before venturing deeper into the theories of planetary deflections and the result these might have on the initial orbits, we will need to know more about the basics of orbital properties. In this first section, our aim is to go through all about the orbits that we will need, so that we can determine the actual crossing points of any two orbits. A lot of the reasoning here is done with the help of Szebehely [1989]. First of all, we begin with one of the facts known from Kepler's laws of planetary motion.

All planets move in eccentric orbits with the central star in one of the foci. Fig (2) shows a generic eccentric orbit with some of its more important properties marked out. An ellipse is a type of conic section that can be described by knowing the size of it's semi-major axis, a, and semi-minor axis, b. Another more commonly used way to describe an eccentric orbit is to use a along with the eccentricity, e. The eccentricity basically tells us how flattened the ellipse is by measuring how far away the foci are from the center of the ellipse, seen in the equation below

$$e = \frac{c}{a} \tag{1}$$

In a circle, the focus lies exactly in the middle, giving us $c = 0$ and therefore $e = 0$. At an $e = 1$, the orbit turns into a parabola and at $e > 1$ we get a hyperbola. These two latter cases will not be discussed further here, but it is important to know that planets with these eccentricities are unbound to the star. Continuing looking at Fig (2), we can see that the point in the orbit closest to the star is called periapsis, and the point farthest away is called apoapsis. An easy way to describe any point in the orbit is by using polar coordinates, r and ϕ, and we chose periapsis to represent the angle $\phi = 0$. This also means that the apoapsis is situated exactly $\phi = \pi$ radians from periapsis. Another important term that we will come back to later is the semi-latus rectum, defined as the distance from a focus to the ellipse a long a line perpendicular to the semi-major axis.

That is about all the basics and the terminology needed for now, and we

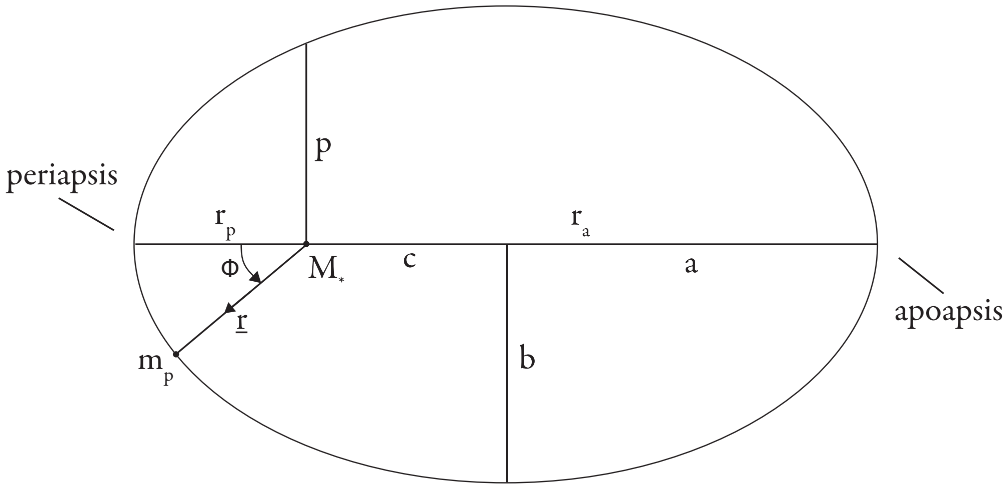

Figure 2: The eccentric orbit with the star in one of the foci. Also marked out are $\underline{r}$, semi-major axis (a), semi-minor axis (b), semi-latus rectum (p), periapsis, apoapsis and ϕ.

can start aiming at answering some of the questions we asked at the end of the introduction. In the end of this section, we want to have derived expressions with which we can determine the points of crossings for any two orbits and the planetary velocities at these points, which is needed to then determine the effect of the planet-planet scattering. An important thing to know is that if two orbits cross, there will always be exactly two crossing points no matter what the orbital properties are. To be able to determine these points, we will first need to find a way to express the distance between each planet and the central star at any point. Knowing this, we will also not be far away from knowing the planetary velocities. To begin somewhere, we know that the angular momentum can be expressed as $\underline{L} = \underline{r} \times \underline{p}$.[1] Angular momentum is an important quantity, but a more handy one for us is the angular momentum per unit mass, $\underline{c}$, that can be written as

$$\underline{c} = \underline{\dot{r}} \times \underline{r} \tag{2}$$

[1] It can here be noted that the underline will be used throughout the article to symbolize vectors.

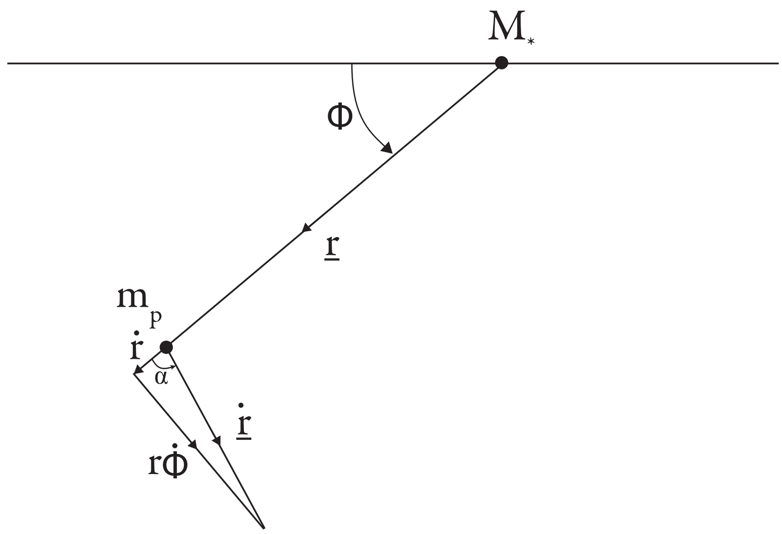

Figure 3: Enlarged figure of a part of the eccentric orbit showing $m_{\rm p}$, M_*, ϕ, $\underline{r}$ along with $\underline{v} = \dot{\underline{r}}$ with the two velocity components $\dot{r}$ and $r\dot{\phi}$ marked out.

The vector product can also be expressed as

$$|\underline{c}| = |\dot{\underline{r}} \times \underline{r}| = |\underline{r}|\,|\dot{\underline{r}}|\sin(\alpha) \tag{3}$$

In Fig (3), a part of the eccentric orbit from Fig (2) have been blown up. Seen are the two bodies at a distance $\underline{r}$. The velocity vector, $\dot{\underline{r}}$, has been divided into two components; $\dot{r}$ which is parallel to $\underline{r}$, and $r\dot{\phi}$ which is perpendicular to $\underline{r}$. Using simple trigonometry from the figure, we see that

$$\sin(\alpha) = \frac{r\dot{\phi}}{|\dot{\underline{r}}|} \quad \Rightarrow \quad |\dot{\underline{r}}|\sin(\alpha) = r\dot{\phi} \tag{4}$$

If we now combine Eq (3) and (4), we can write c as

$$c = r^2\dot{\phi} \tag{5}$$

The only force acting on the planet is the gravitational force between the planet and the central star. We can therefore set the gravitational acceleration equal to the acceleration in the two velocity components $\ddot{r}$ and $r\dot{\phi}^2$, resulting in

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM_*}{r^2} \tag{6}$$

Now, using Eq (5) in (6), we get

$$\ddot{r} - \frac{c^2}{r^3} = -\frac{GM_*}{r^2} \tag{7}$$

To further evolve Eq (7), we can with the help of Eq (5) write the first and second time derivative of r as

$$\frac{dr}{dt} = \frac{dr}{d\phi} \cdot \frac{d\phi}{dt} = \frac{c}{r^2} r' \tag{8}$$

$$\frac{d^2r}{dt^2} = \frac{c^2}{r^4} \left(r'' - 2\frac{r'^2}{r} \right) \tag{9}$$

where $r' = \frac{dr}{d\phi}$ and $r'' = \frac{d^2r}{d\phi^2}$. Now, using these two expressions into Eq (7) and simplifying, we get

$$\frac{c^2}{r^4} \left(r'' - 2r'^{\,2} \right) - \frac{c^2}{r^3} = -\frac{GM_*}{r^2} \tag{10}$$

$$r'' - 2r'^{\,2} - r = -\frac{GM_*}{c^2} r^2 \tag{11}$$

To make this differential equation easier to solve, we introduce the substitution $u = \frac{1}{r}$. That would mean that r' and r'' can be written as

$$r' = -\frac{u'}{u^2} \tag{12}$$

$$r'' = \frac{2u'^2 - uu''}{u^3} \tag{13}$$

Eq (11) can now be written as

$$u'' + u = \frac{GM_*}{c^2} \tag{14}$$

This is a differential equation we can solve, resulting in

$$u = \frac{GM_*}{c^2} + A\cos(\phi) \tag{15}$$

Going back to r as the dependent variable, we finally get an expression for the distance between the planet and the central star:

$$r = \frac{c^2/GM_*}{1 + (Ac^2/GM_*)\cos(\phi)} \tag{16}$$

But as have been discussed earlier, the ellipse is also a sort of conic section. With polar coordinates, we can also write the distance between a focus and any point in the ellipse as

$$r = \frac{p}{1 + e\cos(\phi)} \tag{17}$$

where p is the already discussed semi-latus rectum. Comparing (16) and (17), we can see that p can be written as

$$p = \frac{c^2}{GM_*} \tag{18}$$

Remembering Eq (5), we can use (18) to get

$$r^2\dot{\phi} = \sqrt{p\,GM_*} \tag{19}$$

Taking a step back, remembering Fig (2), we can see that the length of the major axis can be expressed as

$$2a = r_\mathrm{p} + r_\mathrm{a} \tag{20}$$

Using Eq (17) with the figure still in mind, we can see that r_p and r_a can be described as

$$r_\mathrm{p} = \frac{p}{1 + e\cos(0)} = \frac{p}{1+e} \tag{21}$$

$$r_\mathrm{a} = \frac{p}{1 + e\cos(\pi)} = \frac{p}{1-e} \tag{22}$$

Combining these two with Eq (20), we can finally get a useful expression for p:

$$2a = \frac{p}{1+e} + \frac{p}{1-e} \tag{23}$$

$$p = a\left(1 - e^2\right) \tag{24}$$

We now have enough knowledge to also describe the velocity of the planet at any point in the orbit. We begin with describing the velocity component parallel to $\underline{r}$, $\dot{r}$, with the help of Eq (17), (19) and (24).

$$\dot{r} = \frac{dr}{d\phi}\cdot\frac{d\phi}{dt} = \frac{pe\sin(\phi)}{\left(1 + e\cos(\phi)\right)^2}\frac{\sqrt{p\,GM_*}}{r^2} = ...$$
$$= e\sin(\phi)\left(\frac{GM_*}{a\left(1-e^2\right)}\right)^{1/2} \tag{25}$$

Our second component, $r\dot{\phi}$, can then be determined, starting with Eq (19) followed by (17) and (24).

$$\begin{aligned} r\dot{\phi} &= \frac{\sqrt{pGM_*}}{r} = \frac{pGM_*}{p}\left(1 + e\cos(\phi)\right) = ... \\ &= \left(\frac{GM_*}{a\left(1-e^2\right)}\right)^{1/2}\left(1 + e\cos(\phi)\right) \end{aligned} \tag{26}$$

The size of the total velocity can then be calculated from the simple

$$v = \sqrt{\dot{r}^2 + (r\dot{\phi})^2} \tag{27}$$

To verify our velocity equations, we can insert $e = 0$, the eccentricity of a circle, into Eq (25) and (26), resulting in $\dot{r} = 0$ and

$$r\dot{\phi} = \sqrt{\frac{GM_*}{a}} \tag{28}$$

Which, as we wanted, is independent of ϕ and is the well known velocity in a circular orbit.

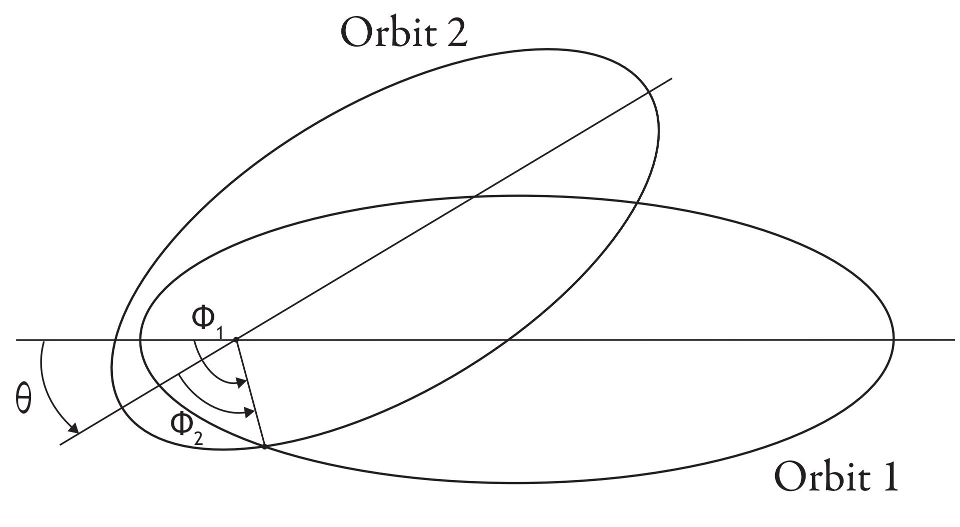

Figure 4: Two eccentric orbits shifted with an angle θ between the two major axes.

We have now discussed the most important properties of a single eccentric orbit and can describe the distance between the planet and the central star at any point with the help of Eq (17) as well as the velocity with Eq (25) and (26). We can therefore introduce a second orbit to our system. Since $M_* >> m_\mathrm{p}$, we can make a good approximation that the two planetary orbits can be treated as two separate two body problems. The distances, r_1 and r_2, can be described as

$$r_1 = \frac{a_1\,(1-e_1^2)}{1+e_1\cos(\phi_1)} \tag{29}$$

$$r_2 = \frac{a_2\,(1-e_2^2)}{1+e_2\cos(\phi_2)} \tag{30}$$

If the two orbits cross, they will always do so at two points, something that has been discussed earlier. The crossings will happen when $r_1(\phi) = r_2(\phi)$. As we see in Fig (4), showing a schematic view of two generic orbits, the orbits will almost always be shifted with an angle θ between the two semi-major axes. According to the figure, we can see that θ can be written as

$$\theta = \phi_1 - \phi_2 \tag{31}$$

With that, we can express ϕ_2 as a function of ϕ_1 and θ, and Eq (30) can subsequently be rewritten as

$$r_2 = \frac{a_2\,(1-e_2^2)}{1+e_2\cos(\phi_1-\theta)} \tag{32}$$

As have previously been mentioned, the two crossings will occur when $r_1(\phi) = r_2(\phi)$. With Eq (29) and (32), we can therefore write our problem as

$$\frac{a_1\,(1-e_1^2)}{1+e_1\cos(\phi_1)} = \frac{a_2\,(1-e_2^2)}{1+e_2\cos(\phi_1-\theta)} \tag{33}$$

To find the crossings, we then have to solve the above equation for ϕ_1. How this have been done in the programs is discussed more thorough in Section 4. With the crossing problem solved, we have a $\phi_{\mathrm{cross},1}$ and $\phi_{\mathrm{cross},2}$ and can then get the two velocity components from Eq (25) and (26). We can then express the relative velocity between the two planets at the point of crossing with

$$\underline{V}(\phi_{\mathrm{cross}}) = \underline{v}_1(\phi_{\mathrm{cross},1}) - \underline{v}_2(\phi_{\mathrm{cross},2}) \tag{34}$$

3 Deflection and its effects

In the previous section, we were able to determine the points of crossing between two orbits and the relative velocities between the two planets at these points. The likelyhood of an interaction have not been examined during this project, but it is safe to say that several of these interactions should occur on an astronomical timescale. This section is dedicated to what will happen to the planets when the event occurs. It is divided into three subsections where we first find the deflection angle between the old and new velocity vector, then look for the change in total velocity of both planets and finally calculate the new orbital properties of both planets.

In the major part of this section, we will disregard the effect of the central star and only focus on the planet-planet interaction. This can be done since the deflection happens so locally that we can approximate the event to happen only right at the point of crossing. The reason why this approximation can be done is based on the following line of reasoning. We first start off with writing the force between the two planets, F_{pp}, and between a planet and the star, F_{ps} as

$$F_{\mathrm{pp}} = G\frac{m_1 m_2}{r_{\mathrm{pp}}^2} \tag{35}$$

$$F_{\mathrm{ps}} = G\frac{m_{\mathrm{p}} M_*}{r_{\mathrm{ps}}^2} \tag{36}$$

To simplify a bit, the masses are set to $m_{\mathrm{p}} = m_1 = m_2$. The point where the two forces are equal is then when the above two equations are equal, and we then evolve it a bit further

$$G\frac{m_1 m_2}{r_{\mathrm{pp}}^2} = G\frac{m_{\mathrm{p}} M_*}{r_{\mathrm{ps}}^2} \tag{37}$$

$$\sqrt{\frac{m_1}{M_*}} = \frac{r_{\mathrm{pp}}}{r_{\mathrm{ps}}} \tag{38}$$

We now imagine two Jupiter-massed planets orbiting the Sun. The masses would be $m_{\mathrm{p}} = 10^{-3} M_{\odot}$ and $M_* = 1 M_{\odot}$. Using the equation above, the distance between the two planets, r_{pp} would then be roughly three per cent of the distance between one of the planets and the star, r_{ps} for the two forces to be equal. There is still quite an approximation in saying that everything happens at one single point, but it will greatly simplify our calculations.

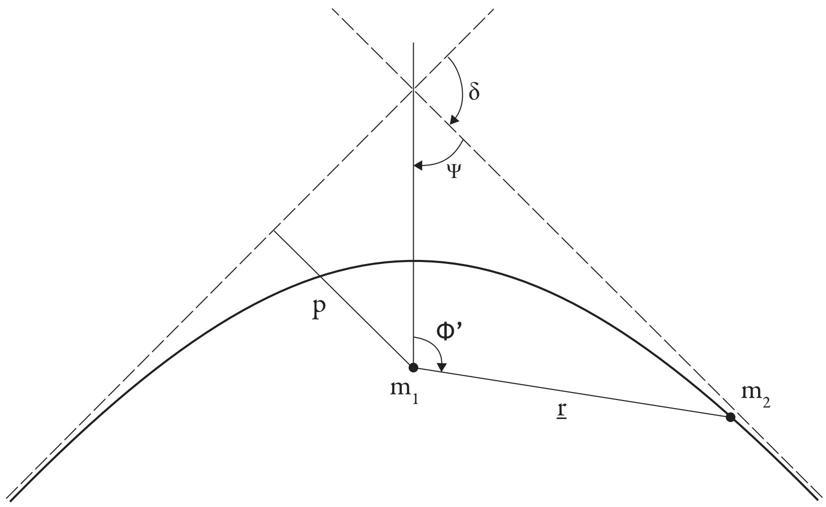

Figure 5: The trajectory of one planet in the rest frame of the other during a deflection. Marked out is the deflection angle, δ, as well as ψ, ϕ' and the impact parameter, p.

3.1 Deflection of planets

In this first subsection, based on Spitzer Jr. [1987], we determine the angle, δ, between the old velocity, $\underline{v}$, and the new velocity after deflection, $\underline{\tilde{v}}$. We do not look for the change in size of the velocity vector at this point, since we will first need δ for that. To determine δ, we switch to the rest frame of planet 1, and then look at the relative orbit planet 2 shows during the deflection. In this frame, we can easily understand why only the relative velocity between the two planets known from Eq (34) is important. A sketch of the relative orbit is given in Fig (5). The dashed lines in the figure are the asymptotes of the relative orbit. The angle ϕ' is measured from the closest distance between the planets in the same way as it was for the single orbit. We also introduce a new property called the impact parameter, p, which is the closest distance between the two planets in the unperturbed orbits. This parameter basically tells us how high the interaction between the two planets will be, and is therefore very important when determining the deflection angle.

From Eq (16) and (17), we can write the inverse distance $1/r$ between the two planets as

$$\frac{1}{r} = \frac{G\,(m_1 + m_2)}{c^2}\,(1 + e\cos(\phi')) \tag{39}$$

The total energy of planet 2 in Fig (5) is taken from the sum of the kinetic and potential energy and can be written as

$$E = \frac{m_2 v^2}{2} - \frac{G m_1 m_2}{r} \tag{40}$$

But since we know that $v^2 = \dot{r}^2 + (r\dot{\phi})^2$, we can with the help of Eq (5) also write the energy per unit mass, E_{um} as

$$2E_{\mathrm{um}} = \left(\frac{dr}{dt}\right)^2 + \frac{c^2}{r^2} - \frac{2G\left(m_1 + m_2\right)}{r} \tag{41}$$

Again with the help of Eq (5), we can replace dt with $r^2 \frac{d\phi}{c}$, and get dr from Eq (39). This yields

$$e^2 = 1 + \frac{p^2 V^4}{G^2\left(m_1 + m_2\right)} \tag{42}$$

where V is the relative velocity at no planetary interaction, E_{um} has been replaced with $V^2/2$, which is the energy per unit mass at infinity, and $c = pV$ has been used, where p is the impact parameter. From Fig (5), we can see that at the angles $\phi' = \pi \pm \psi$, r goes towards infinity, meaning that $\cos(\phi') = 1/e$. From Eq (42), we then get

$$tan(\psi) = \frac{pV^2}{G\left(m_1 + m_2\right)} \tag{43}$$

From Fig (5), we can also see that ψ is related to δ as

$$\delta = \pi - 2\psi \tag{44}$$

And by combining Eq (43) and (44), we finally have a working expression for δ. In the expression for ψ, we can see that the impact parameter, p, logically plays an important role during the deflection of planets. At the same point of crossing between two orbits, the other parameters are constant, but we never have a fixed value on p for every interaction. There is actually a wide range of p:s that are of interest to us, but how those are decided will be discussed further in Section 4.

3.2 The vector triangle plot

Knowing the deflection angle δ from the previous subsection, we are now able to determine the new velocities $\underline{\tilde{v}}_1$ and $\underline{\tilde{v}}_2$. In order to do this, we come up with a trick that will greatly simplify our calculations and the understanding

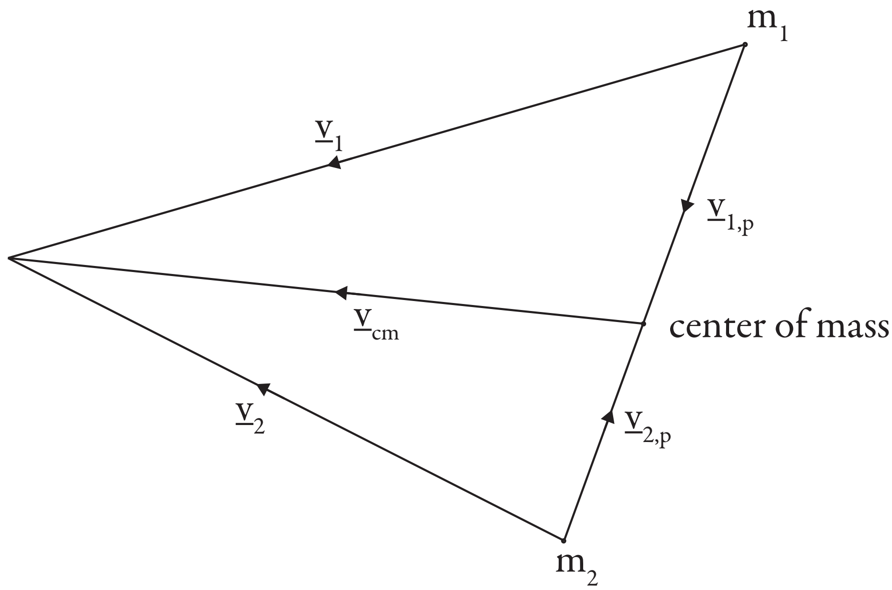

Figure 6: The vector triangle figure for both planets. Marked out is the center of mass, along with $\underline{v}$, $\underline{v}_{\text{cm}}$ and $\underline{v}_{\text{p}}$.

of what happens during the deflection. We switch to vectors related to the center of mass of the two planets according to Fig (6). In this figure, the two planets m_1 and m_2 are situated at a distance $d_1 = v_1 \cdot t$ and $d_2 = v_2 \cdot t$ from the point of crossing between the two orbits. The beauty of this approach will soon be explained, but let us first continue a little bit further. As can be seen in the figure, the total velocity before the deflection can now be described with

$$\underline{v}_1 = \underline{v}_{\text{cm}} + \underline{v}_{1,p} \tag{45}$$

$$\underline{v}_2 = \underline{v}_{\text{cm}} + \underline{v}_{2,p} \tag{46}$$

where $\underline{v}_{\text{cm}}$ is the center of mass velocity, common for both planets according to Fig (6), and $\underline{v}_{\text{p}}$ is the planets velocity relative to the center of mass. The center of mass velocity is basically a mass weighted velocity and can be described by

$$\underline{v}_{\text{cm}} = \frac{m_1 \underline{v}_1 + m_2 \underline{v}_2}{m_1 + m_2} \tag{47}$$

Since v_1 and v_2 at the points of crossings are known from earlier, we can also determine v_{cm}. From Eq (45), we can also get $\underline{v}_{\text{p}}$ for both planets. The fine thing here is that since the center of mass vector is identical for the

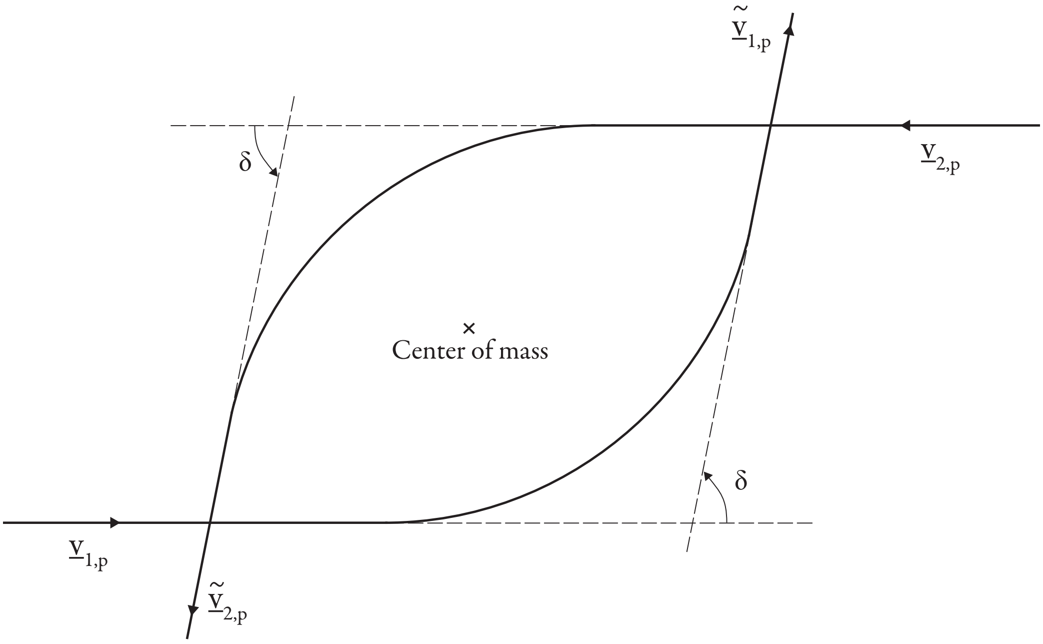

Figure 7: The trajectory of the two planets during the deflection seen from the center of mass frame. Included are δ, $\underline{v}_{1,\mathrm{p}}$ and $\underline{v}_{2,\mathrm{p}}$ along with the resulting velocities $\tilde{\underline{v}}_{1,\mathrm{p}}$ and $\tilde{\underline{v}}_{2,\mathrm{p}}$.

two planets and conserved during deflection, the deflection is only dependent on the $\underline{v}_{\mathrm{p}}$ component. If we stand in the rest frame of the center of mass, the planets movement would only seem to depend on $v_{1,\mathrm{p}}$ and $v_{2,\mathrm{p}}$, which is nicely illustrated in Fig (7). The interaction between the two planets is a pure deflection, meaning that $|\underline{v}_{\mathrm{p}}| = |\tilde{\underline{v}}_{\mathrm{p}}|$. Knowing this, we can write the new velocity, $\tilde{v}$, as

$$\tilde{\underline{v}} = \underline{v}_{\mathrm{cm}} + \tilde{\underline{v}}_p \tag{48}$$

Since the size of the $\underline{v}_{\mathrm{p}}$ component is constant, and the direction of it is directly related to the angle of deflection, the whole range of possible $\tilde{\underline{v}}_{\mathrm{p}}$ will describe a full circle as shown in Fig (8). This approach gives us an opportunity to easily understand several interesting aspects of planetary deflection. We can for example easily see that the largest deflection angle does not equal the largest change in $\underline{v}$, which one might spontaneously think. Instead, we can from the figure easily see that we will get the maximum velocity of $\tilde{\underline{v}}$ when $\tilde{\underline{v}}_p \parallel \underline{v}_{cm}$. In the same way, we get the minimum velocity when $\tilde{\underline{v}}_p$ and $\underline{v}_{cm}$ are oppositely aligned. We can also in an easy way explain why it is more difficult to change direction of a higher mass planet than a lower one. Imag-

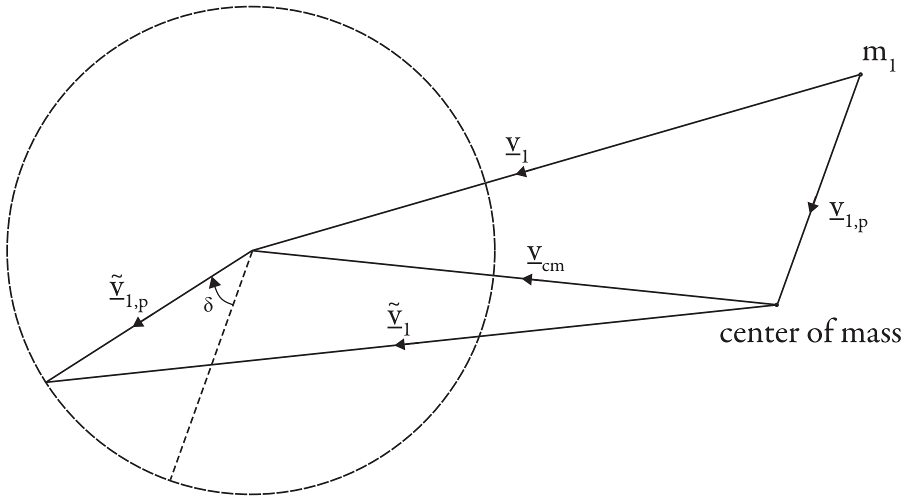

Figure 8: Enlarged vector triangle for one of the planets. The circle is showing all possible $\underline{\tilde{v}}_{1,p}$.

ine a situation where $m_1 >> m_2$. This would mean that the center of mass would lie almost on planet 1, resulting in $v_{1,p} << v_{\rm cm}$ and that $\underline{v}_{\rm cm} \approx \underline{v}_1$. No matter how we rotate $\underline{v}_{1,p}$, the resulting change in velocity would therefore not be very large.

Let us continue with looking at a system where two orbits are crossing, still with the vector triangle in mind. The semi-major axes, as, and the angle shift between these, θ, both seen in Fig (4), are arbitrary. We set $e_1 = 0$ and $0 < e_2 < 1$. The velocity of planet 1 would be constant and given from Eq (28):

$$r\dot{\phi} = \sqrt{\frac{GM_*}{a_1}}$$

Since the two orbits are always crossing at exactly a_1 no matter what θ, the escape speed for both planets at the deflection would always be

$$v_{\rm esc} = \sqrt{\frac{2GM_*}{a_1}} = \sqrt{2}v_1 \tag{49}$$

For any velocity larger than this, the planet would escape the system. Knowing $|\underline{v}_1| = |\underline{\tilde{v}}_\mathrm{p}|$ and Eq (48), we can say that if either of the two following conditions are true, escape is possible after deflection.

$$|\underline{v}_\mathrm{cm}| + |\underline{v}_{1,\mathrm{p}}| \geq \sqrt{2}v_1 \tag{50}$$

$$|\underline{v}_\mathrm{cm}| + |\underline{v}_{2,\mathrm{p}}| \geq \sqrt{2}v_1 \tag{51}$$

This line of reasoning does not, however, directly state how likely ejection would be after a deflection. If either are true, however, given enough deflections, the planet fulfilling the above condition would be ejected. This is is discussed in more detail later on.

3.3 The new orbital properties

We have now described what happens during the deflection, and will now again have to consider the force between the star and each planet. In other words; we know the point of interaction, $\underline{r}$, and have just calculated the $\underline{\tilde{v}}$. The only thing left is to find $\tilde{a}$ and $\tilde{e}$ for both planets. To find these, there are two necessary parameters to know, the total energy and angular momentum for each planet, E and L. The angular momentum of a planet is given by

$$\underline{L} = m(r\dot{\phi})\underline{r} \tag{52}$$

where $r\dot{\phi}$ is given by Eq (26). The total energy of the planet is as before the sum of its kinetic and potential energy at the point of crossing:

$$E = T + U = \frac{m_\mathrm{p}\tilde{v}^2}{2} - G\frac{m_\mathrm{p}M_*}{r} \tag{53}$$

But the total energy of an orbit is also given by the expression below, allowing us to determine $\tilde{a}$:

$$E = -\frac{Gm_\mathrm{p}M_*}{2\tilde{a}} \qquad \Rightarrow \qquad \tilde{a} = -\frac{Gm_\mathrm{p}M_*}{2E} \tag{54}$$

Knowing both E and L, we can now also determine $\tilde{e}$ using the following expression (derived in Shapiro and Teukolsky [1983]):

$$\tilde{e}^2 = 1 + \frac{2EL^2}{G^2\mu^3M^2} \approx 1 + \frac{2EL^2}{G^2m_\mathrm{p}^3M_*^2} \tag{55}$$

We now know the new orbital properties for both the planets. However, we also have to consider the new angle between the major axes, $\tilde{\theta}$. We actually already have all the knowledge needed to solve this last question. As have previously been discussed, we know that the old point of crossing will be included in the new orbit. With the help of Eq (17), we can find two points of the new orbit that are both situated at the distance r_{cross}; one where the planet approaches the central star, and one where it is receding it. Only one of these are correct, however, and we will need to figure out which point that is. This can be done with the help of the scalar product $\underline{r} \cdot \underline{\tilde{v}}$. In the same way as using the derivative of a function to know if it is increasing or decreasing, we can from knowing the sign of the scalar product tell if the planet is receding or approaching the central star. Doing this for both planets will then let us determine the point of crossing in ϕ_1 and ϕ_2, and with Eq (31), we can determine $\tilde{\theta}$. We now, finally, have all the theory we need to create the programs made during project.

There is one important thing to take notice of, before continuing, and that is that because the energy of the system is conserved, both planets will after one revolution in their new orbits get back to the exact same point where the two orbits interacted the first time. This means that two orbits that cross will always do so at that exact point no matter what new orbital properties they get as long as both planets are bound. We can therefore keep on using the whole line of reasoning from Sections 2 and 3 until we finally get an ejection.

4 The programs

In order to answer the questions we asked ourselves in the beginning of this project, two programs where written based on the theory of orbits explained above. The first program, called *Orbitcrossing*, is the largest and made to determine the new $\tilde{a_1}$, $\tilde{e_1}$, $\tilde{a_2}$, $\tilde{e_2}$ and $\tilde{\theta}$ from some given set of a_1, e_1, a_2, e_2 and θ, utilizing most things derived in the two earlier sections. The second program, named *Boundtrial*, uses some of the insight we got in the "The vector triangle plot" subsection in order to quickly analyze if ejection of a planet is possible for a wide range of different starting orbits. But let us begin with explaining the *Orbitcrossing* program in more detail.

4.1 Orbitcrossing

The first thing we need to do when writing *Orbitcrossing* is to find a good way to numerically determine the two points of crossing at a satisfactory degree of accuracy. With the help of Eq (29) and (32), we can plot r_1 and

r_2, versus ϕ_1. The orbits cross when $r_1(\phi_1) = r_2(\phi_1 - \theta)$. In the program, we calculate r for discrete values of ϕ equally spaced with $\Delta\phi = 0.1$ degrees for the whole orbit. Because of this, we will almost never see the situation where the two distances r_1 and r_2 are exactly equal. Instead, if we look at the difference $r_1(\phi_1) - r_2(\phi_1 - \theta)$, we know that it will switch sign between two adjacent ϕ when they cross. To get the exact crossing, we can linearly extrapolate the function of r between the two adjacent angles for both orbits. The linear equation for the two orbits would then be

$$r_1 = k_1\phi_1 + m_1 \tag{56}$$

$$r_2 = k_2\phi_1 + m_2 \tag{57}$$

where k is the slope of the line and can be determined from $k = \frac{\Delta r}{\Delta\phi_1}$. Knowing r and its corresponding ϕ, we can then also determine m from $m = r - k\phi$. To get the angle where the two lines cross, we then set $r_1 = r_2$, and get the following expression for the crossing angle:

$$\phi_{\text{cross}} = \frac{m_2 - m_1}{k_1 - k_2} \tag{58}$$

We can now calculate the two velocity components for each planet using Eq (25) and (26) and get the relative velocity from Eq (34). To test if the program so far works as it should, we know for a fact that the two crossings should be π radians apart when the two orbits are identical but shifted with an angle θ. With the same identical orbits, we can also test the relative velocities. At a very slight shift in θ, the two velocities should be almost the same resulting in a relative velocity close to 0. If we instead have a shift of $\theta = \pi$ radians, the two planets would be oppositely aligned resulting in a relative velocity twice the one of an individual planet.

To calculate the deflection at the crossing, we use Eq (43) and (44). At this point, it is important to note that *Orbitcrossing* have been written in two different versions. The first version runs the exact same interaction several times for equally spaced p:s in the whole range of $-p_{\max} \leq p \leq p_{\max}$ and calculates the resulting orbits for each p. The value of $p_{\max}$ will be discussed later. The second version randomly picks one p from the same range as above, then determines the new orbital properties and continues running until one of the planets is ejected. For the random pick, we use the *Luxury* random number generator, giving a value RND ranging from 0 to 1. Because the orbits of the two planets are coplanar, the p range is one dimensional and can be described with the linear relation below

$$p = 2 \cdot \text{RND} \cdot p_{\max} - p_{\max} \tag{59}$$

Determining the value of $p_{\max}$ was done entirely empirical. We were in this project only interested in the deflection where the change in orbital properties was clearly noticeable. To get the right value, we looked at plots acquired from the first version of the program showing δ versus a large range of p. We also wanted the value of $p_{\max}$ to scale with any set of orbits, and found that $p_{\max} = 0.05 \cdot r_{\text{cross}}$ AU worked well, resulting in deflections between around 10 to 350 degrees.

To find the resulting velocities after the deflection, we use Eq (48). When translating the velocity components $\dot{r}$ and $r\dot{\phi}$ to the new system, we need to make an assumption that $\underline{r}_1 \parallel \underline{r}_2$, which is true if we remember the approximation that the interaction happens locally. With the help of Fig (8), we can easily test our code by checking if the maximum and minimum $\tilde{v}$ are situated $\delta = \pi$ radians apart. We can also check the total E and L for the system before and after the deflection and see if they are conserved.

Finally, to get the resulting $\tilde{a}$s, $\tilde{e}$s and $\tilde{\theta}$, we use Eq (52), (53), (54) and (55). A good way of testing this last part of the code is to set $\delta = 0$ and then check if the new orbital properties agree with the old ones. All the tests mentioned in this subsection was performed and confirmed that the program worked as it should.

4.2 Boundtrial

The *Boundtrial* program was written in order to be able to see if ejection of a planet after deflection was possible, using the logics of "The vector triangle plot" subsection, specifically Eq (50) - (51). From these equations, we can see that if the two velocity components, $|\underline{v}_{\text{cm}}|$ and $|\underline{v}_{2,\text{p}}|$ added together are larger than the escape velocity, v_{esc}, escape is possible by aligning the components in the right way. The smaller difference between the $|\underline{v}_{\text{cm}}| + |\underline{v}_{2,\text{p}}|$ and v_{esc}, the smaller the range of possible δs that lead to ejections. With this program, however, we only check if escape is possible or not, and do not bother with the probabilities.

The first orbit is set constant, with $a_1 = 1$ AU and $e_1 = 0$ to keep the program simple. We then look at a large range of a_2s and e_2s and check if ejection is possible or not for every individual case. The program consists of parts of *Orbitcrossing* to determine the crossing points and velocities at these points. We then use Eq (50) - (51) to see if escape is possible. For some arbitrary set of two orbits that cross, we can imagine four different possibilities:

1. Ejection is impossible for both planets

2. Ejection is possible for both planets

a_1 (AU)	e_1	m_1 ($M_\odot$)	a_2 (AU)	e_2	m_2 ($M_\odot$)
1	0	10^{-3}	2	0.8	10^{-3}
1	0	$5 \cdot 10^{-3}$	2	0.8	10^{-3}
1	0	10^{-3}	2	0.8	$5 \cdot 10^{-3}$
1	0	10^{-3}	2	0.6	10^{-3}
1	0	$5 \cdot 10^{-3}$	2	0.6	10^{-3}
1	0	10^{-3}	2	0.6	$5 \cdot 10^{-3}$
1	0	10^{-3}	1	0.6	10^{-3}
1	0	10^{-3}	1	0.2	10^{-3}
2	0.4	10^{-3}	3	0.8	10^{-3}

Table 1: All examined combinations of orbital properties.

3. Ejection is possible for planet 1, but impossible for planet 2
4. Ejection is impossible for planet 1, but possible for planet 2

In the program, with the limitations to orbit 1 that we always have $e_1 = 0$ and $m_1 = m_2$, we will actually only ever see the first two of these possibilities occur. This is related to the conservation of energy and will be discussed more in next Section.

5 Results

With the help of *Orbitcrossing* (attached in Appendix A) and *Boundtrial* (attached in Appendix B), a number of different simulations was run with different initial orbital properties and masses. In order to minimize the amount of factors, only two different masses have been used; 10^{-3} $M_\odot$, equal to around one Jupiter mass, and $5 \cdot 10^{-3}$ $M_\odot$. In most cases, e_1 has been set to 0, both to simplify and to simulate an inner orbit that was not as affected by the fly-by as the outer orbit. In the test runs, all kinds of θs have been used, but for all plots given in "Results", θ have been set to $\pi/2$. This does in most cases not matter, since we are rotation independent if either of the orbits are circular. All runs have been listed in Table 1.

We start off with showing Fig (9), an example plot from *Orbitcrossing* where we have plotted the complete range of equally spaced ps versus the resulting eccentricities of both orbits after deflection, $\tilde{e}$, in order to study the direct effect of different impact parameters. First of all, we can note that both curves seem to go towards the initial properties of $e_1 = 0$ and $e_2 = 0.8$ at high ps, low interactions, which is a good way to see that our program seems to be working correctly. We can also see that the highest resulting eccentricity does not come from the highest possible deflection, which is in line with our discussion in the previous sections. It can also be noted that the diameter of a Jupiter-massed planet would equal around 0.001 AU, effectively excluding a bit of the middle portion of the plot because of planet-planet collision. These collisions have not been directly included in the program, and are discussed more in the Discussion. Both planets also seem to have an $\tilde{e} > 1$ for certain ps, which means that ejection after interaction is possible.

Finally, we can note that at the purely theoretical $p = 0$, which would lead to a collision in reality, the properties of the two orbits seem to swap resulting in $\tilde{e}_1 = 0.8$ and $\tilde{e}_2 = 0.0$. This is once again a feature that can be explained with the help of our nice vector triangle. At $p = 0$, Eq (43) and (44) says that we will get a $\delta = \pi$ radians. Since we also have $m_1 = m_2$, the two velocity components relative to the center of mass will be of equal size. These two facts combined with Eq (45) and (46) results in

$$\underline{\tilde{v}}_1 = \underline{v}_{\text{cm}} + \underline{\tilde{v}}_{1,\text{p}} = \underline{v}_{\text{cm}} + \underline{v}_{2,\text{p}} = \underline{v}_2 \tag{60}$$

$$\underline{\tilde{v}}_2 = \underline{v}_{\text{cm}} + \underline{\tilde{v}}_{2,\text{p}} = \underline{v}_{\text{cm}} + \underline{v}_{1,\text{p}} = \underline{v}_1 \tag{61}$$

Leading to exactly swapped orbits. Interestingly enough, we come to the conclusion that any two crossing orbits with $m_1 = m_2$ can swap orbits in the same way if we don't take collisions into consideration.

We now continue with Fig (10), showing the resulting plot from *Boundtrial* with $m_1 = m_2$. In the plot, we see the range of a_2s and e_2s where ejection is possible for both, although not at the same time, and we also see the range of a_2s and e_2s where ejection is always impossible. According to the figure, when one of the planets have a circular orbit and both planets have the same mass, only the two above situations are possible. If we look back at Fig (9), it shows the same thing, with both planets having spikes above $\tilde{e} \leq 1$. Again, the discussion around this subject will have to wait. In the program, all different combinations of $0 \leq a_2 \leq 8$ and $0 \leq e_2 < 1$ have been gone through, so the white areas show where the two orbits do not even cross. We can also see that both areas are nice and continuous, with no strange oddities.

After a given deflection with some given set of orbits, it would be of interest to identify the different new orbital properties we could get. Imagine running the same planet-planet scattering, but with different ps each time. Would we be able to see some kind of shape or relation between all the new properties, or would they perhaps just be randomly distributed over a larger area? To

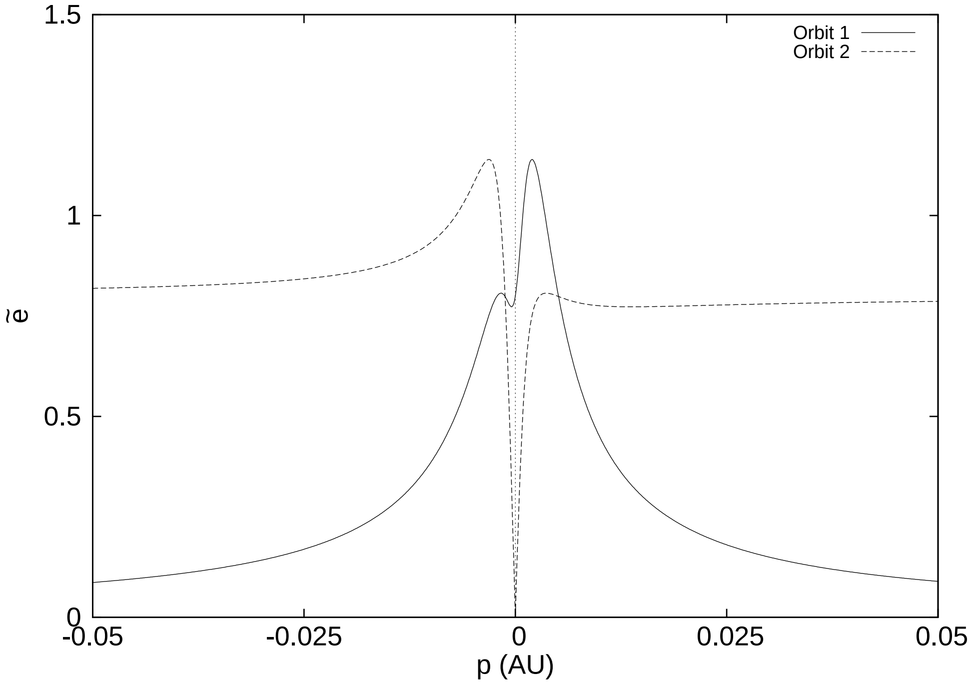

Figure 9: Eccentricity versus impact parameter for orbit 1 and 2 (Initial properties: $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}\ M_\odot$)

answer this question, we let *Orbitcrossing* run over the complete range of ps for one specific encounter. The resulting $\tilde{a}$s and $\tilde{e}$s for each individual p were then plotted and is shown in Figs (11) and (12).

First of all, we can take note of the region free of points in the region close to the initial orbital properties. These empty regions would contain all points originating from low-deflection interactions belonging to the ps we have excluded by selecting our $p_{\max}$ as we did. We can also, not very surprising after having looked at Fig (9) with the narrow eccentricity peak, see that the density of points is a lot higher close to the initial properties. The most interesting thing we note by comparing Figs (11) and (12) is that that even though the density of points differ between the two, the shapes are actually identical. This only happens when two conditions are met; one of the orbits have to be circular, and the masses have to be the same. Because the masses are the same, we realize from Fig (6) that $|\underline{v}_{1,\mathrm{p}}| = |\underline{v}_{2,\mathrm{p}}|$. Because of this, the resulting circle of possible $\tilde{\underline{v}}$s must be exactly the same for both

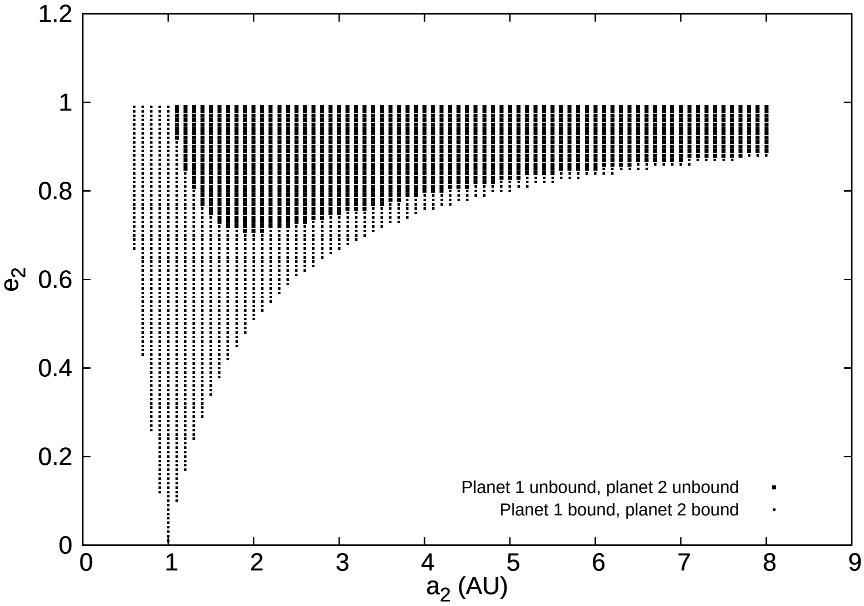

Figure 10: Orbital properties of planets where ejection is possible and where it is not. (Properties of orbit 1: $a_1 = 1$ AU, $e_1 = 0$, $m_1 = m_2 = 10^{-3}\ M_\odot$)

planets. In order to conserve momentum, $\underline{\tilde{v}}_1$ and $\underline{\tilde{v}}_2$ must also always be exactly oppositely aligned for each deflection. This means that the shape for the two planets will be identical, even though the density of points will still differ. This finally explains why only two combinations, both bound or both possible to get unbound, existed in the previous *Boundtrial* run. Since the two shapes are always identical, we can never have a situation where only one planet is possible to get ejected where there the other can not. We can also finally note that the shape is closed, and never exceeds $e = 1$, which means that ejection is impossible for this set of initial orbital properties. Comparing with Fig (10) gives the same results.

Similar to the previous example, we once again run *Orbitcrossing*, now with a different set of initial orbital properties. The previous scattering was not able to result in any ejections, so we now see what happens when we know ejections are possible, as was the case with the system used for Fig (9). Since we still have one orbit circular and both planet masses identical, Fig (13) and (14) shares the same shape. Since the initial properties are the same as those

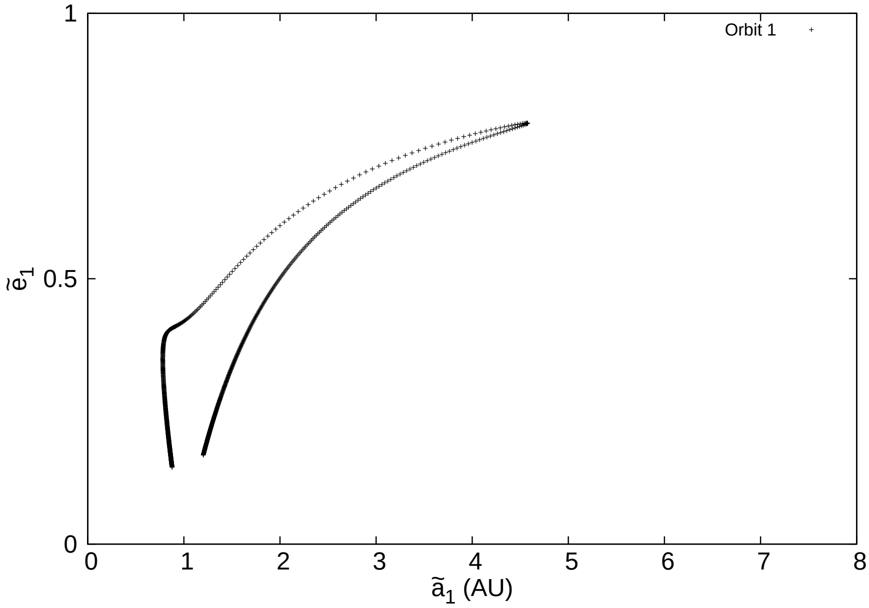

Figure 11: Properties of orbit 1 after deflection with p ranging from $-p_{\max}$ to $p_{\max}$ (Initial properties: $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.6, m_1 = m_2 = 10^{-3}\ M_{\odot}$)

from Fig (9), we also expect to see at least a couple of points above $e = 1$, signaling an unbound planet. However, higher es are associated with larger as, and the few points above $e = 1$ also have huge as, not included in these plots. We can however still see a tendency towards the unbound state in the plot.

We have now discussed the case of one single event of planet-planet scattering. However, to compare our simulations to the reality, we need to run several simulations where we let the scattering event occur, leading to new properties and new scatterings, finally resulting in the ejection of one of the planets. We can then look at the properties of the planet left behind and see what it looks like. In Fig (15), this was done with initial properties $a_1 = 2$ AU, $e_1 = 0.4, a_2 = 3$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}\ M_{\odot}$ and a randomly picked p for every interaction. The loop was then repeated 500 times, giving 500 points of a and e of the planet left behind.

Looking at the figure, we can see that the points are scattered across a wide area around $a = 1$ AU with a quite homogeneous distribution. The interesting feature here is however that see a very sharp line at around $a = 1.2$ AU. Higher as do not seem to exist. This can actually be explained if we express the total energy of orbit 1 and 2 before the ejection with the help of Eq (54), resulting in:

$$E_1 = -G\frac{M_* m_1}{2a_1} \tag{62}$$

$$E_2 = -G\frac{M_* m_2}{2a_2} \tag{63}$$

After the ejection, the energy of the planet left, E_{final}, can be expressed in the same way as above, and the minimum energy of the ejected planet would be obtained if $v_{\text{p,unbound}} = 0$, resulting in

$$E_{\text{final}} = -G\frac{M_* m_{\text{p,bound}}}{2a_{\text{p,bound}}} \tag{64}$$

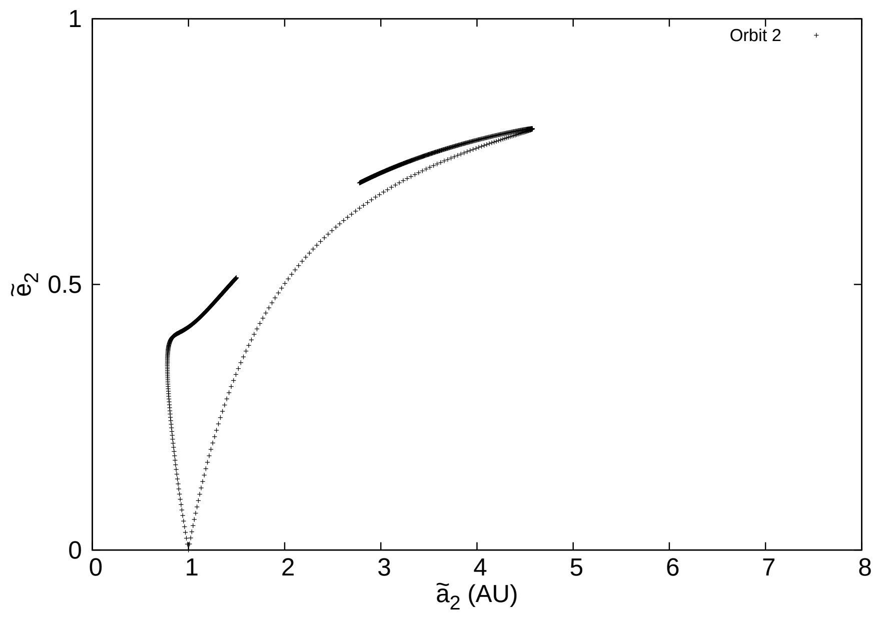

Figure 12: Properties of orbit 2 after deflection with p ranging from $-p_{\max}$ to $p_{\max}$ (Initial properties: $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.6, m_1 = m_2 = 10^{-3}\ M_\odot$)

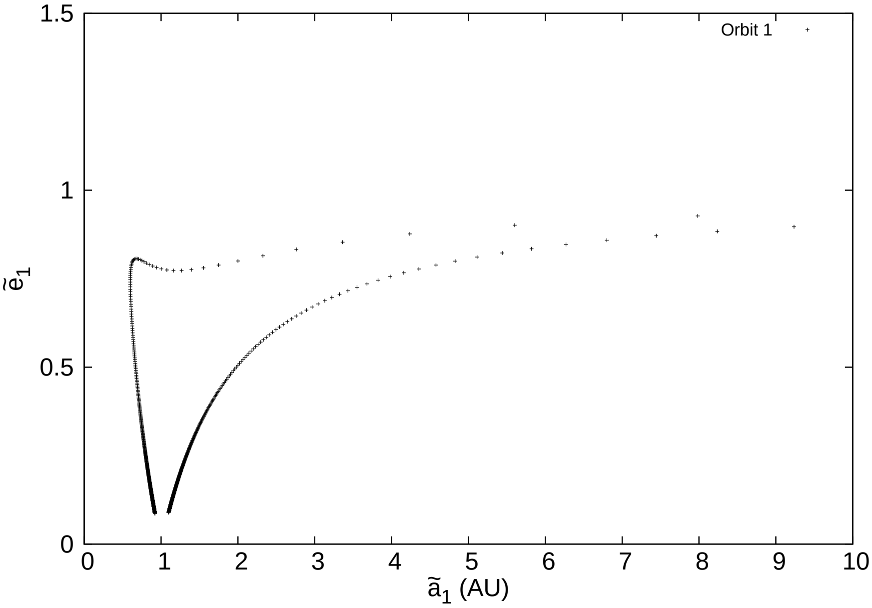

Figure 13: Properties of orbit 1 after deflection with p ranging from $-p_{\max}$ to $p_{\max}$ (Initial properties: $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}\ M_{\odot}$)

$$E_{\text{ejected}} = -G\frac{M_* m_{\text{p,unbound}}}{a_{\text{p,unbound}}} \tag{65}$$

Conservation of energy means that we can put the energies before and after the deflection equal in the following way

$$E_{\text{final}} + E_{\text{ejected}} = E_1 + E_2 \tag{66}$$

The least energy possible of the ejected planet would then be $E_{\text{ejected}} = 0$, meaning that the highest possible energy would be given to the bound orbit and still have an ejection. According to Eq (54), a low binding energy is associated with a large a. The largest possible a is therefore obtained just when $E_{\text{ejected}} = 0$, meaning that we can write our equation as

$$\frac{1}{a_{\text{final}}} \geq \frac{1}{a_1} + \frac{1}{a_2} \tag{67}$$

Inserting $a_1 = 2$ AU and $a_2 = 3$ AU, which is the initial set of properties we are interested in, then results in $a_{final,max} = 1.2$ AU, which is exactly what

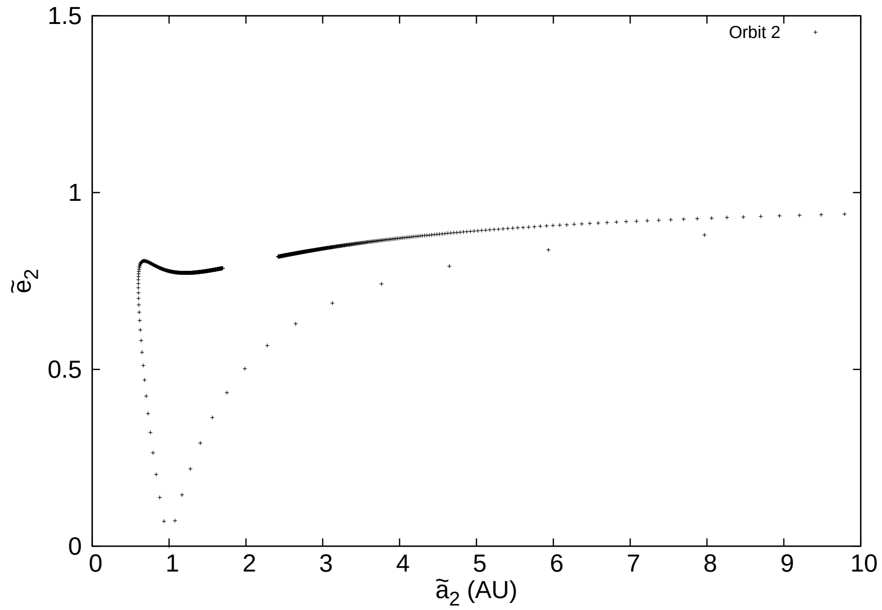

Figure 14: Properties of orbit 2 after deflection with p ranging from $-p_{\max}$ to $p_{\max}$ (Initial properties: $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}$ $M_\odot$)

we see in the plot.

In Fig (16), *Orbitcrossing* have been run in the same way as in the previous case, but with $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}$ $M_\odot$ as initial properties. Where Fig (15) did not have any clear features, this plot shows a clear shape almost resembling a hockey-stick, where the point density is a lot higher than anywhere else. The fact that the density is so much higher actually gives us a little lead to what has happened. The idea is that this system is more keen on early ejections than the previous one, leading to a high amount of prompt ejections. Since the setup of the deflection is the same for so many ejections, we will have a naturally higher density around the same area in the plot. The actual hockey-stick occurs when the p is randomly selected, leading to slight differences between every prompt ejection. If the ejection isn't prompt, on the other hand, the points will smear out and it gets more and more unlikely that any points will be related in the same way to each other. To test this theory, the same setup

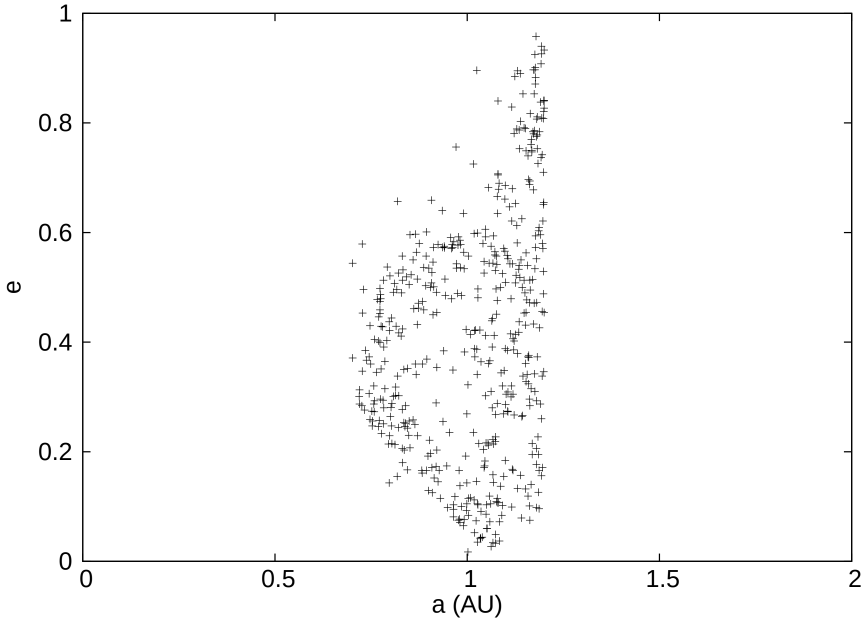

Figure 15: Orbital properties of planet left after ejection (Initial properties: $a_1 = 2$ AU, $e_1 = 0.4, a_2 = 3$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}\ M_{\odot}$)

was run a second time, only allowing one single revolution and therefore only showing the prompt ejections, and the hockey-stick is all we see.

To verify the theory a little further, we make histograms over the number of scatterings needed for each systems to eject one of the planets. Both tested systems ($e_1 = 0$, $e_2 = 0.8$ in Fig (17) and $e_1 = 0.4$, $e_2 = 0.8$ in Fig (18)) are included in the report. Here, we can clearly see that our theory seems to agree with the results. In Fig (17), we can see that the majority of ejects have occurred withing the first four or five scatterings. Roughly ten per cent of the total ejections happen at the first scattering, giving rise to the dense hockey-stick feature. When looking at Fig (18), which didn't have the kind of noticeable feature as the previous case had, there are not one single ejection at the first scattering.

As a final test, we check our programs by comparing a couple of our simulations to the real exoplanets given in Fig (1). With the fly-by theory, we want to explain the large spread of eccentricities for the discovered exoplanets, not the hot Jupiters, and therefore disregard the exoplanets with an $a > 1$

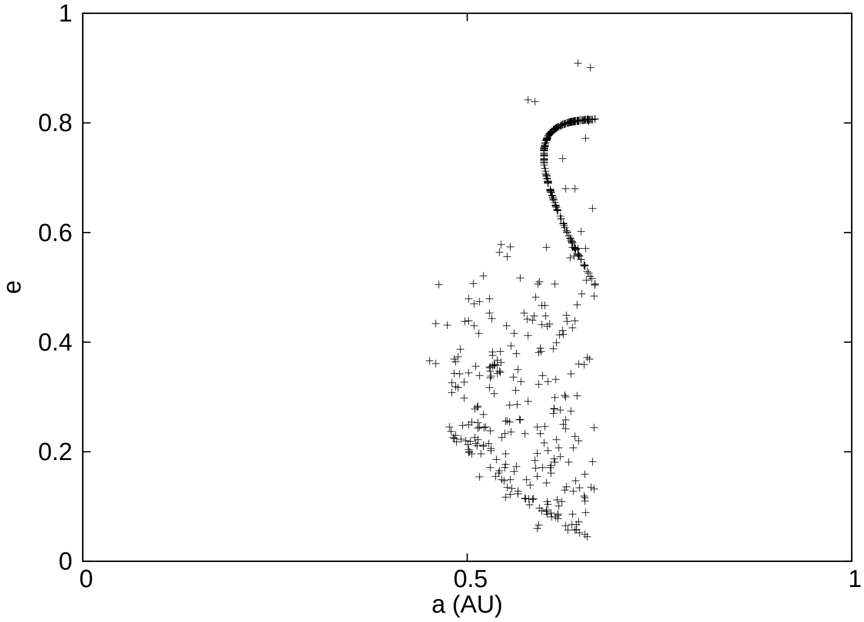

Figure 16: Orbital properties of planet left after ejection (Initial properties: $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}\ M_\odot$)

AU. The comparison is made by looking at the cumulative eccentricity distribution, showing how large fraction of the planets that have a certain e or less. The two different runs are not meant to represent any specific fly-bys, and are just random picks. Interestingly enough, we see that Run 1 actually seem to fit pretty well with reality. There is a slight shift in height between them, but the overall shape agree well. Not so surprisingly, Run 2 fits worse. The steep slope at $e = 0.8$ represents the high density hockey-stick feature seen in Fig (16), which we understood came from the prompt ejections. In reality, we will never see such a hockey-stick, since we will also have a large ratio of smaller interactions that we disregarded in *Orbitcrossing*. Also including these would mean that the hockey-stick wouldn't appear, and Run 2 is therefore just a purely theoretical result of our simulations.

6 Discussion

With the results obtained during this project, we can neither confirm nor discard the theory that the fly-by mechanism is the reason behind the strangely

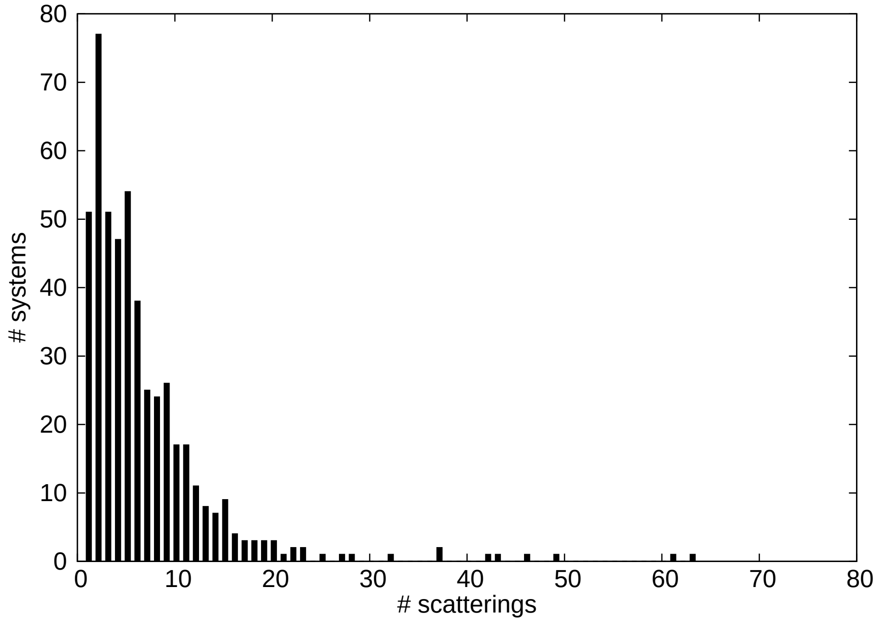

Figure 17: Histogram of the amount of scatterings needed for ejection for 500 runs. (Initial properties: $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}\ M_\odot$)

large spread of orbital eccentricities of the exoplanets discovered so far. Still, with the analytic approach taken, we have come to a better understanding of what happens during planet-planet scatterings. We have also been able to produce highly eccentric orbits after the ejection of one of the planets, and by looking at the results of a some simulations with different initial conditions, we were also able to almost reproduce the cumulative eccentricity distributions of the exoplanets discovered. There has however not been time for any serious investigations that would include statistic weighting of different likely initial conditions. This would definitely be needed for some better comparison between the simulations and reality.

To be able to write the program used here, we've had to do some approximations. A couple of these may be a bit stretchy. Everything done have for example assumed that $M_* >> m_{\rm p}$, meaning the we have completely neglected the effect of the planet on the central star. For Jupiter-massed planets and heavier, this approximation might not be good enough, and at this stage we

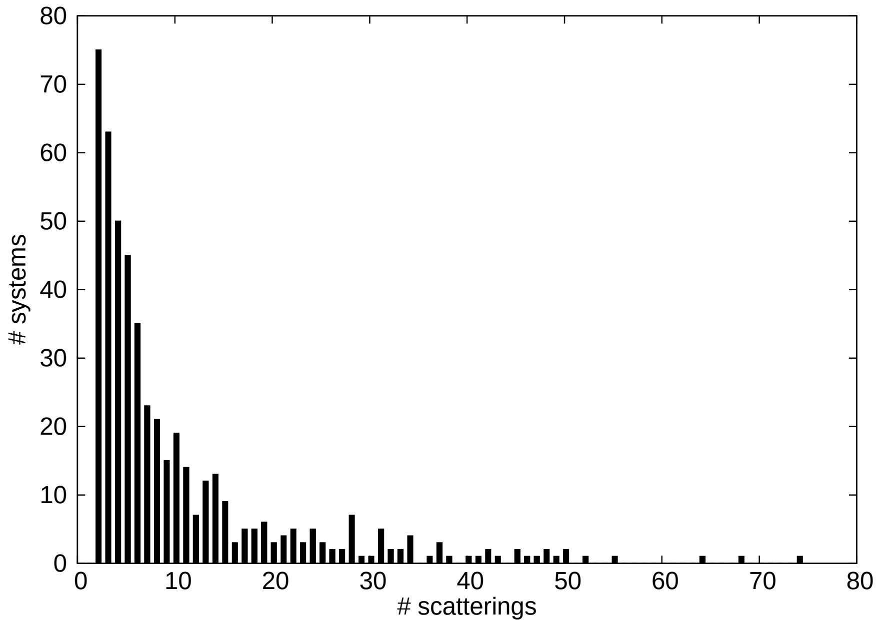

Figure 18: Histogram of the amount of scatterings needed for ejection for 500 runs. (Initial properties: $a_1 = 2$ AU, $e_1 = 0.4, a_2 = 3$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}\ M_\odot$)

simply do not know how this would affect the results. This is not impossible to fix, and to get better precision we could use reduced masses. We have also had to make the approximation that the planet-planet interactions happens locally at one single point in the orbit to avoid having to work with a three-body problem. In reality, we actually have a larger area where the planet-planet interaction occurs. For Jupiter-massed planets, approximating this area to one single point might be a bit too much of an approximation as we saw in Eq (38).

There are a lot of possible follow-ups to this project. With the orbital theory explained in this report, it wouldn't take that much more work to expand *Orbitcrossing* to run multiple-orbit simulations. Each orbit could still be counted as a two-body problem as we did in Section 2, and when looking at the planet-planet scatterings, we would simply need some kind of weighted randomization to work out which crossing to look at if there are several. It is currently believed that most planet systems are formed with more than

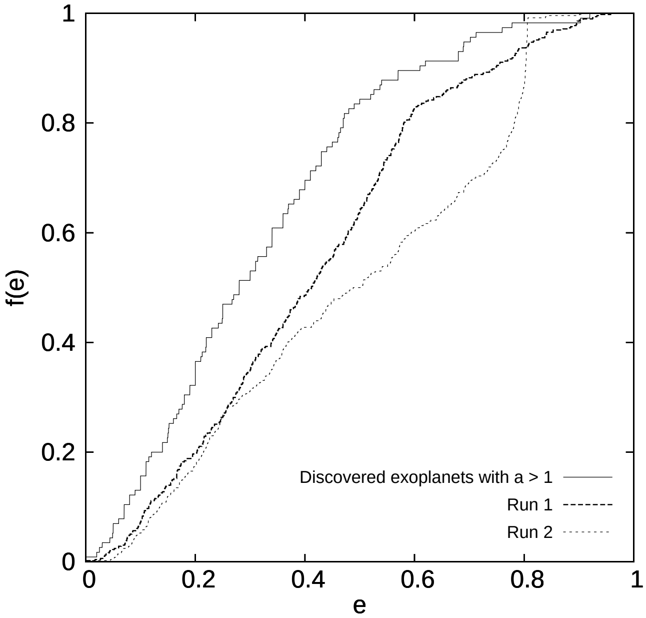

Figure 19: The cumulative eccentricity distribution for the known exoplanets versus the same for two simulations from *Orbitcrossing*. (Run 1: $a_1 = 2$ AU, $e_1 = 0.4, a_2 = 3$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}\ M_\odot$, Run 2: $a_1 = 1$ AU, $e_1 = 0, a_2 = 2$ AU, $e_2 = 0.8, m_1 = m_2 = 10^{-3}$)

two planets, and at this point we do not know how this fact would affect the resulting as and es of the planets left behind, if at all.

Another logical thing to do would be to introduce a third dimension, where we can have planets with an inclination to a reference plane. This would complicate the theoretical approach by quite a bit, but we would gain an even better fit to reality.

One more variable that would be possible to insert into *Orbitcrossing* would be the probability of collisions. These have been completely disregarded in the program, as have been mentioned earlier. Including them, however, would mean that a certain, narrow range of ps equal to the diameter of the planets would be excluded. Since the planets we have been focusing on in this project are Jupiter-massed gas giants, collisions would lead to merging of the two planets rather than elastic collisions, we actually already know what would happen, if we remember Eq (45), (46) and (48). We know that after a merge, we will have $\underline{v}_{1,\mathrm{p}} = \underline{v}_{2,\mathrm{p}}$. Since we also need to abide conservation of momentum, we realize that $\underline{\tilde{v}}$ need to equal v_{cm}. We of course also know $\underline{r}_{\text{cross}}$, which means that we can determine the new orbital properties for the merged planets with the help of Eq (52) - (55). Considering that the excluded range of ps due to collisions will be very small compared to the whole range of ps, collisions will be very unlikely, and almost non-existent for the more stable systems. However, for tighter systems where ejections are more unlikely or perhaps impossible, collisions are probably something that we need to take into consideration. The new type of orbits acquired from the mergers might perhaps be able to explain some of the odder planetary orbits that any other theory can not.

Acknowledgements

Finally, I want to thank Melvyn B. Davies for being a very helpful and enthusiastic supervisor. Thanks also go to Daniel Malmberg for some programming help, and Tobias Albertsson, Nils Håkansson and Åsa Tornborg for putting up with long discussions, as well as for moral support during difficult times.

References

C. Hayashi. Structure of the Solar Nebula, Growth and Decay of Magnetic Fields and Effects of Magnetic and Turbulent Viscosities on the Nebula. *Progress of Theoretical Physics Supplement*, 70:35–53, 1981.

J. R. Hurley and M. M. Shara. Free-floating Planets in Stellar Clusters: Not So Surprising. *ApJ*, 565:1251–1256, February 2002. doi: 10.1086/337921.

G. Laughlin and F. C. Adams. The Modification of Planetary Orbits in Dense Open Clusters. *ApJ*, 508:L171–L174, December 1998. doi: 10.1086/311736.

D. N. C. Lin, P. Bodenheimer, and D. C. Richardson. Orbital migration of the planetary companion of 51 Pegasi to its present location. *Nature*, 380: 606–607, April 1996. doi: 10.1038/380606a0.

D. Malmberg, F. de Angeli, M. B. Davies, R. P. Church, D. Mackey, and M. I. Wilkinson. Close encounters in young stellar clusters: implications for planetary systems in the solar neighbourhood. *MNRAS*, 378:1207–1216, July 2007. doi: 10.1111/j.1365-2966.2007.11885.x.

M. Mayor and D. Queloz. A Jupiter-Mass Companion to a Solar-Type Star. *Nature*, 378:355–+, November 1995. doi: 10.1038/378355a0.

N. Murray, B. Hansen, M. Holman, and S. Tremaine. Migrating Planets. *Science*, 279:69–+, January 1998.

D. D. Sasselov and M. Lecar. On the Snow Line in Dusty Protoplanetary Disks. *ApJ*, 528:995–998, January 2000. doi: 10.1086/308209.

S. L. Shapiro and S. A. Teukolsky. *Black holes, white dwarfs, and neutron stars: The physics of compact objects*. Research supported by the National Science Foundation. New York, Wiley-Interscience, 1983, 663 p., 1983.

L. Spitzer Jr. *Dynamical evolution of globular clusters*. Princeton University Press, 1987.

V. G. Szebehely. *Adventures in Celestial Mechanics*. University of Texas Press, 1989.

A Appendix: Orbitcrossing

```
PROGRAM Orbitcrossing

USE LUXURY

IMPLICIT NONE

!!! Determines the points of crossing between two orbits, finds
!!! the velocities at these points, calculates the deflection
!!! and the resulting new orbital properties

DOUBLE PRECISION,DIMENSION(3600) :: dist1,dist2,diffdist
DOUBLE PRECISION,DIMENSION(3600,2) :: r1,r2
DOUBLE PRECISION,DIMENSION(2,2) :: k,m
INTEGER,DIMENSION(2,2) :: crossing
DOUBLE PRECISION,DIMENSION(3600) :: phid
DOUBLE PRECISION, DIMENSION(2,2) :: vrad,vtan,vtot,vconst
DOUBLE PRECISION :: ecc1,ecc2,a1,a2,m11,m12
DOUBLE PRECISION :: mu1,mu2,G,M1,M2,PI,phir1,phir2,theta
DOUBLE PRECISION, DIMENSION(2) :: phicross
DOUBLE PRECISION, DIMENSION(2,1) ::  vrel1,vrel2
INTEGER :: n,phi,i
DOUBLE PRECISION :: psi,p,distcross
DOUBLE PRECISION, DIMENSION(2) :: rcross,v1,v2,v1p,v2p,vcm
DOUBLE PRECISION, DIMENSION(2) :: vn1p,vn2p,vn1px,vn1py,vn2px,vn2py
DOUBLE PRECISION :: E1,E2,J1,J2
DOUBLE PRECISION :: E1t,E2t,J1t,J2t,an1t,an2t
DOUBLE PRECISION :: pmax,pmin,delta
DOUBLE PRECISION, DIMENSION(2) :: vn1,vn2
DOUBLE PRECISION :: an1,an2,eccn1,eccn2,ptab,vn1size,vn2size
INTEGER :: loop_number,loops
DOUBLE PRECISION, DIMENSION(2) :: nphicross1,nphicross2
DOUBLE PRECISION :: rv1,rv2
DOUBLE PRECISION :: E1test,E2test,an1test,an2test,ecc1t,ecc2t

REAL :: randVec(1)
INTEGER :: IV(8), myseed

G = 1.0
```

```
pi = 3.141592
vconst = 29.79

m11 = 1.D-3
m12 = 1.D-3
m2 = 1.
mu1 = G*(m11 + m2)
mu2 = G*(m12 + m2)

pmin = 1.D-3

open(unit=9,file="ejectionstats.txt")

!!! Get random seed for Luxury RNG

CALL DATE_AND_TIME(VALUES=IV)
myseed =  (IV(1)*IV(2)+IV(3))*IV(7)+(IV(5)*IV(6)+IV(3))*IV(8)+2.
CALL RLUXGO(4,myseed,0,0)

outest_loop: do loops = 1,500

ecc1 = 0.
ecc2 = 0.8
a1 = 1.
a2 = 2.
theta = 90.
loop_number = 1

outer_loop: do

!!! Finds distances between orbits and central star ranging
!!! from 1-360 degrees phi
!!! Writes distances for all phi and the 2D orbits to files

do phi=1,3600

   phir1 = 0.1*phi*PI/180.0
   phir2 = 0.1*(phi-10*theta)*PI/180.0

   phid(phi)= 0.1*phi
```

```
   dist1(phi)= a1*(1.-ecc1**2.)/(1.+ecc1*cos(phir1))
   r1(phi,1) = -dist1(phi)*cos(phir1)
   r1(phi,2) = -dist1(phi)*sin(phir1)

   dist2(phi)= a2*(1.-ecc2**2)/(1.+ecc2*cos(phir2))
   r2(phi,1) = -dist2(phi)*cos(phir1)
   r2(phi,2) = -dist2(phi)*sin(phir1)

end do

diffdist = dist1-dist2

open(unit=1,file="orbitdist.txt")
do i=1,3600
   write(1,10),phid(i), dist1(i), dist2(i),diffdist(i)
end do
10 format(f10.5,f10.5,f10.5,f10.5)
close(1)

open(unit=2,file="2D-orbits.txt")
do i=1,3600
  write(2,20),phid(i), r1(i,1), r1(i,2), r2(i,1), r2(i,2)
end do
20 format(f10.5, f10.5, f10.5, f10.5, f10.5)
close(2)


!!! See if the orbits ever cross

if (a1 > a2) then
   if (a1*(1.-ecc1) > a2*(1.+ecc2)) then
      print *, "The two orbits do not cross (1)"
      print *, 'ecc1 = ', ecc1
      print *, 'ecc2 = ', ecc2
      exit outer_loop
   end if
else if (a2 > a1) then
   if (a2*(1.-ecc2) > a1*(1.+ecc1)) then
      print *, "The two orbits do not cross (2)"
      print *, 'ecc1 = ', ecc1
      print *, 'ecc2 = ', ecc2
```

```
      exit outer_loop
   end if
end if


!!! Finds the approximate crossings

n=1
do phi=1,3599
   if (diffdist(phi) == 0) then
      crossing(n,1)=phi
      crossing(n,2)=phi
      n=n+1
   elseif ((diffdist(phi) > 0 .AND. diffdist(phi+1) < 0) .OR.
          (diffdist(phi) < 0 .AND. diffdist(phi+1) > 0)) then
      crossing(n,1)=phi
      crossing(n,2)=phi+1
      n=n+1
   end if
end do

if (n /= 3) then
      print *, "The two orbits do not cross (3)"
      exit outer_loop
end if


!!! Finds the more exact crossings by approximating short
!!! interval linearity

do n=1,2
   if (crossing(n,1) == crossing(n,2)) then
      phicross(n) = 0.1*crossing(n,1)
   else
      k(n,1) = (dist1(crossing(n,2)) - dist1(crossing(n,1))) /
       (crossing(n,2)-crossing(n,1))
      m(n,1) = dist1(crossing(n,1)) - k(n,1)*crossing(n,1)
      k(n,2) = (dist2(crossing(n,2)) - dist2(crossing(n,1))) /
       (crossing(n,2)-crossing(n,1))
      m(n,2) = dist2(crossing(n,1)) - k(n,2)*crossing(n,1)
      phicross(n) = 0.1*(m(n,2)-m(n,1))/(k(n,1)-k(n,2))
```

```
   end if
end do

rcross(1) = a1* (1.-ecc1**2.) / (1.+ecc1*cos(phicross(1)*PI/180.0))
rcross(2) = a1* (1.-ecc1**2.) / (1.+ecc1*cos(phicross(2)*PI/180.0))


!!! Calculate the velocities at interaction points

do n=1,2
   phir1 = phicross(n)*PI/180.0
   phir2 = (phicross(n)-theta)*PI/180.0

   vrad(n,1)= ecc1*sin(phir1) *sqrt( m2/ (a1*(1.-ecc1**2)) )
   vrad(n,2)= ecc2*sin(phir2) *sqrt( m2/ (a2*(1.-ecc2**2)) )

   vtan(n,1)= ( 1.+ecc1*cos(phir1) ) *sqrt( m2/(a1*(1.-ecc1**2)) )
   vtan(n,2)= ( 1.+ecc2*cos(phir2) ) *sqrt( m2/(a2*(1.-ecc2**2)) )

   vtot(n,1)= sqrt( vrad(n,1)**2 + vtan(n,1)**2 )
   vtot(n,2)= sqrt( vrad(n,2)**2 + vtan(n,2)**2 )
end do

E1test = m11*vtot(1,1)**2/2. - G*m11*m2 / rcross(1)
E2test = m12*vtot(1,2)**2/2. - G*m12*m2 / rcross(1)
an1test = -G * m11*m2 / (2.*E1test)
an2test = -G * m12*m2 / (2.*E2test)


!!! Find the relative velocities between the two planets at
!!! the interactions

vrel1(1,1) = vrad(1,1)-vrad(1,2)
vrel1(2,1) = vtan(1,1)-vtan(1,2)
vrel2(1,1) = vrad(2,1)-vrad(2,2)
vrel2(2,1) = vtan(2,1)-vtan(2,2)

open(unit=3,file="crossingvel.txt")

   write(3,30),'crossing1 = ',phicross(1), 'crossing2 = ',phicross(2)
   write(3,30),
```

```
   write(3,30),'(First number is orbit, second number interaction)'
   write(3,30),'vrad11 = ', vrad(1,1)*vconst, 'vrad21 = ',
    vrad(1,2)*vconst
   write(3,30),'vrad12 = ', vrad(2,1)*vconst, 'vrad22 = ',
    vrad(2,2)*vconst
   write(3,30),
   write(3,30),'vtan11 = ', vtan(1,1)*vconst, 'vtan21 = ',
    vtan(1,2)*vconst
   write(3,30),'vtan12 = ', vtan(2,1)*vconst, 'vtan22 = ',
    vtan(2,2)*vconst
   write(3,30),
   write(3,30),'vtot11 = ', vtot(1,1)*vconst, 'vtot21 = ',
    vtot(1,2)*vconst
   write(3,30),'vtot12 = ', vtot(2,1)*vconst, 'vtot22 = ',
    vtot(2,2)*vconst
   write(3,30),
   write(3,30),'The resulting relative velocities at the two
    interaction points:'
   write(3,30),'vrelrad1 = ', vrel1(1,1)*vconst, 'vreltan1 = ',
    vrel1(2,1)*vconst
   write(3,30),
   write(3,30),'vrelrad2 = ', vrel2(1,1)*vconst, 'vreltan2 = ',
    vrel2(2,1)*vconst
   write(3,30),
   write(3,35),'myseed = ', myseed

30 format(a,f10.5)
35 format(a,i10)
close(3)


!!! Planetary deflection

sevp_loop: do n = 1,1      ! Loop only used when we want several p
 ! for same orbital properties

!!! Randomly pick a p ranging from -pmax to pmax with the help of
!!! the Luxury RNG

   pmax = 0.05 * rcross(1)
```

```
   CALL RANLUX(randVec,1)          ! To be used when using the RNG

!   randVec(1) = real(n) / 1000.   ! To be used when not using the RNG

   p = 2.0*randVec(1)*pmax-pmax

!   if (abs(p) <= abs(pmin)) then  ! To be used when we want to
                                   ! consider collisions

      ! Skip head on collisions

!   else

   psi = atan( p * (vrel1(1,1)**2.+vrel1(2,1)**2.) / (G*(m11+m12)) )
   delta = PI-2.*psi


!!! Calculate the new velocity vectors

   v1(1) = vrad(1,1)
   v1(2) = vtan(1,1)

   v2(1) = vrad(1,2)
   v2(2) = vtan(1,2)

   vcm = (m11*v1+m12*v2)/(m11+m12)
   v1p = v1 - vcm
   v2p = v2 - vcm

   vn1px  = cos(delta)*v1p
   vn1py(1) = sin(delta)*v1p(2)
   vn1py(2) = sin(delta)*(-v1p(1))
   vn1p = vn1px + vn1py
   vn1(1) = vn1p(1) + vcm(1)
   vn1(2) = vn1p(2) + vcm(2)
   vn1size = sqrt(vn1(1)**2.+vn1(2)**2.)

   vn2px = cos(-delta)*v2p
   vn2py(1) = sin(-delta)*(-v2p(2))
   vn2py(2) = sin(-delta)*v2p(1)
   vn2p = vn2px + vn2py
```

```
   vn2(1) = vn2p(1) + vcm(1)
   vn2(2) = vn2p(2) + vcm(2)
   vn2size = sqrt(vn2(1)**2.+vn2(2)**2.)


!!! Get the new orbital properties

   E1 = m11*( vn1(1)**2 + vn1(2)**2 ) /2.0 - G*m11*m2/rcross(1)
   E2 = m12*( vn2(1)**2 + vn2(2)**2 ) /2.0 - G*m12*m2/rcross(1)

   an1 = -G * m11*m2 / (2.*E1)
   an2 = -G * m12*m2 / (2.*E2)

   J1 = m11 * vn1(2) * rcross(1)
   J2 = m12 * vn2(2) * rcross(1)

   eccn1 = sqrt( 1.+2.*E1*J1**2 / ( G**2*( m11 )**3*(m2)**2) )
   eccn2 = sqrt( 1.+2.*E2*J2**2 / ( G**2*( m12 )**3*(m2)**2) )

   open(unit=4,file="newecca.txt")

      write(4,40), p, delta, eccn1, an1, eccn2,an2,vn1size*vconst,
       vn2size*vconst

40    format(f12.7, f12.5, f12.5, f12.5, f12.5, f12.5, f12.5, f12.5)

end do sevp_loop
close(4)


!!! Check if any of the two planets become unbound

ecc1 = eccn1
ecc2 = eccn2
a1 = an1
a2 = an2

   if (ecc1 >= 1.) then
      write(9,99), a2, ecc2, loop_number
      exit outer_loop
   else if (ecc2 >= 1.) then
```

```
      write(9,99), a1, ecc1, loop_number
      exit outer_loop
   end if


!!! Find new theta between the semi major axes

nphicross1(1) = acos(an1*(1.-eccn1**2.) /(eccn1*rcross(1))-1./eccn1)
nphicross1(2) = 2.*pi - nphicross1(1)

nphicross2(1) = acos(an2*(1.-eccn2**2.) /(eccn2*rcross(1))-1./eccn2)
nphicross2(2) = 2.*pi - nphicross2(1)

rv1 = rcross(1)*vn1(1)
rv2 = rcross(1)*vn2(1)

if (rv1 >= 0. .and. rv2 >= 0.) then
   theta = nphicross1(1)-nphicross2(1)
else if (rv1 >= 0. .and. rv2 <= 0.) then
    theta = nphicross1(1)-nphicross2(2)
else if (rv1 <= 0. .and. rv2 >= 0.) then
   theta = nphicross1(2)-nphicross2(1)
else if (rv1 <= 0. .and. rv2 <= 0.) then
   theta = nphicross1(2)-nphicross2(2)
end if

theta = theta*180.0 / pi

if (loop_number == 100) then
   exit outer_loop
end if

loop_number = loop_number+1

end do outer_loop

end do outest_loop
99 format(f8.3, f8.3, i4)
close(9)

END PROGRAM Orbitcrossing
```

B Appendix: Boundtrial

```
PROGRAM Boundtrial

IMPLICIT NONE

!!! Determines the points of crossing between two orbits, and
!!! checks if the velocities at these points are high enough
!!! to lead to ejections during the deflection

DOUBLE PRECISION,DIMENSION(3600) :: dist1,dist2,diffdist
DOUBLE PRECISION,DIMENSION(3600,2) :: r1,r2
DOUBLE PRECISION,DIMENSION(2,2) :: k,m
INTEGER,DIMENSION(2,2) :: crossing
DOUBLE PRECISION,DIMENSION(3600) :: phid
DOUBLE PRECISION, DIMENSION(2,2) :: vrad,vtan,vtot,vconst
DOUBLE PRECISION :: ecc1,ecc2,a1,a2,m11,m12
DOUBLE PRECISION :: mu1,mu2, G,M1,M2,PI,phir1,phir2,theta
DOUBLE PRECISION, DIMENSION(2) :: phicross
DOUBLE PRECISION, DIMENSION(2,1) ::  vrel1,vrel2
INTEGER :: n,phi,i
DOUBLE PRECISION :: psi,p,distcross
DOUBLE PRECISION, DIMENSION(2) :: rcross,v1,v2,v1p,v2p,vcm
DOUBLE PRECISION, DIMENSION(2) :: nphicross1,nphicross2
DOUBLE PRECISION :: rv1,rv2
INTEGER :: ecc2_loop,a2_loop,loop_number
DOUBLE PRECISION :: vcmsize,v1psize,v2psize,vescsize

G = 1.0
PI = 3.141592
vconst = 29.79

m11 = 1.D-3
m12 = 1.D-3
m2 = 1.
mu1 = G*(m11 + m2)
mu2 = G*(m12 + m2)

open(unit=1,file="1u2u.txt")
open(unit=2,file="1u2b.txt")
open(unit=3,file="1b2u.txt")
```

```
open(unit=4,file="1b2b.txt")

ecc1 = 0.
a1 = 1.
theta = 90.

loop_number = 1

ecc_loop: do ecc2_loop = 1,99

   ecc2 = real(ecc2_loop) / 100.

   a_loop: do a2_loop = 10,800,10

      a2 = real(a2_loop) / 100.

!!! See if the orbits ever cross

if ((a1 > a2) .and. (a1*(1.-ecc1) > a2*(1.+ecc2))) then
      ! Skip if orbits don't cross

else if ((a2 > a1) .and. (a2*(1.-ecc2) > a1*(1.+ecc1))) then
      ! Skip if orbits don't cross

else

      ! Proceed with the rest of the program


!!! Finds distances between orbits and central star ranging
!!! from 1-360 degrees phi
!!! Writes distances for all phi and the 2D orbits to files

do phi=1,3600

   phir1 = 0.1*phi*PI/180.0
   phir2 = 0.1*(phi-10*theta)*PI/180.0

   phid(phi)= 0.1*phi

   dist1(phi)= a1*(1.-ecc1**2.)/(1.+ecc1*cos(phir1))
```

```
   r1(phi,1) = -dist1(phi)*cos(phir1)
   r1(phi,2) = -dist1(phi)*sin(phir1)

   dist2(phi)= a2*(1.-ecc2**2)/(1.+ecc2*cos(phir2))
   r2(phi,1) = -dist2(phi)*cos(phir1)
   r2(phi,2) = -dist2(phi)*sin(phir1)

end do

diffdist = dist1-dist2


!!! Finds the approximate crossings

n=1
do phi=1,3599
   if (diffdist(phi) == 0) then
      crossing(n,1)=phi
      crossing(n,2)=phi
      n=n+1
   elseif ((diffdist(phi) > 0 .AND. diffdist(phi+1) < 0) .OR.
    (diffdist(phi) < 0 .AND. diffdist(phi+1) > 0)) then
      crossing(n,1)=phi
      crossing(n,2)=phi+1
      n=n+1
   end if
end do

if (n /= 3) then

  ! Skip if orbits don't cross

else

!!! Finds the more exact crossings by approximating short
!!! interval linearity

do n=1,2
   if (crossing(n,1) == crossing(n,2)) then
      phicross(n) = 0.1*crossing(n,1)
   else
```

```
      k(n,1) = (dist1(crossing(n,2)) - dist1(crossing(n,1))) /
       (crossing(n,2)-crossing(n,1))
      m(n,1) = dist1(crossing(n,1)) - k(n,1)*crossing(n,1)
      k(n,2) = (dist2(crossing(n,2)) - dist2(crossing(n,1))) /
       (crossing(n,2)-crossing(n,1))
      m(n,2) = dist2(crossing(n,1)) - k(n,2)*crossing(n,1)
      phicross(n) = 0.1*(m(n,2)-m(n,1))/(k(n,1)-k(n,2))
   end if
end do

rcross(1) = a1* (1.-ecc1**2.) / (1.+ecc1*cos(phicross(1)*PI/180.0))
rcross(2) = a1* (1.-ecc1**2.) / (1.+ecc1*cos(phicross(2)*PI/180.0))


!!! Calculate the velocities at interaction points

do n=1,2
   phir1 = phicross(n)*PI/180.0
   phir2 = (phicross(n)-theta)*PI/180.0

   vrad(n,1)= ecc1*sin(phir1) * sqrt( m2/ (a1*(1.-ecc1**2)) )
   vrad(n,2)= ecc2*sin(phir2) * sqrt( m2/ (a2*(1.-ecc2**2)) )

   vtan(n,1)= (1.+ecc1*cos(phir1)) * sqrt( m2/(a1*(1.-ecc1**2)) )
   vtan(n,2)= (1.+ecc2*cos(phir2)) * sqrt( m2/(a2*(1.-ecc2**2)) )

   vtot(n,1)= sqrt( vrad(n,1)**2 + vtan(n,1)**2 )
   vtot(n,2)= sqrt( vrad(n,2)**2 + vtan(n,2)**2 )
end do

   v1(1) = vrad(1,1)
   v1(2) = vtan(1,1)

   v2(1) = vrad(1,2)
   v2(2) = vtan(1,2)

   vcm = (m11*v1+m12*v2)/(m11+m12)
   v1p = v1 - vcm
   v2p = v2 - vcm

   vescsize = sqrt(2.)*sqrt( v1(1)**2+v1(2)**2)
```

```
   vcmsize = sqrt( vcm(1)**2 + vcm(2)**2)
   v1psize = sqrt( v1p(1)**2 + v1p(2)**2)
   v2psize = sqrt( v2p(1)**2 + v2p(2)**2)


!!! Check if escape is possible

   if ( (v1psize + vcmsize >= vescsize) .and.
      (v2psize + vcmsize >= vescsize)) then
      write(1,10), a2, ecc2

   else if ( (v1psize + vcmsize >= vescsize) .and.
           (v2psize + vcmsize <= vescsize)) then
      write(2,10), a2, ecc2

   else if ( (v1psize + vcmsize <= vescsize) .and.
           (v2psize + vcmsize >= vescsize)) then
      write(3,10), a2, ecc2

   else if ( (v1psize + vcmsize <= vescsize) .and.
           (v2psize + vcmsize <= vescsize)) then
      write(4,10), a2, ecc2

   else
      print *, "Something odd happened at e2 = ",
       ecc2, " and a2 = ", a2
   end if


end if

end if

loop_number = loop_number +1

end do a_loop

end do ecc_loop

10  format(f8.2, f8.2)
```

```
close(1)
close(2)
close(3)
close(4)

print *, 'loop_number = ', loop_number

END PROGRAM Boundtrial
```